Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Dorina Mitrea
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Sylvie Monniaux

Groupoid Metrization Theory

With Applications to Analysis on Quasi-Metric Spaces and Functional Analysis

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic. Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-theart *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, timefrequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Biomedical signal processing
Compressive sensing
Communications applications
Data mining/machine learning
Digital signal processing
Fast algorithms
Gabor theory and applications
Image processing
Numerical partial differential equations
Prediction theory
Radar applications
Sampling theory
Spectral estimation
Speech processing
Time-frequency and time-scale analysis
Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function". Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantors set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, e.g., by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in timefrequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

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Preface

At its core, this research monograph deals with the following basic issue in analysis: identifying a setting as general as reasonably possible that permits construction of a metric that is compatible – quantitatively, topologically, or algebraically – with a given setting. Some classical results in this spirit are as follows:

- the Alexandroff–Urysohn metrization theorem (point-set topology),
- the Macías–Segovia metrization theorem (harmonic analysis),
- the Aoki–Rolewicz normability theorem (functional analysis),
- the Birkhoff–Kakutani metrization theorem (topological group theory).

The point of view promoted in the current book is that these are all particular manifestations of a more general phenomenon. The unifying language that permits such an extension is that of groupoids (a significantly weaker notion than that of groups), and the theory developed here in this context turns out to be sharp both from an analytic and an algebraic perspective. Such a level of generality is desirable since it is precisely this that makes the theory applicable to a large range of contexts, which traditionally have been regarded as essentially unrelated. This claim is substantiated by the numerous specific applications discussed in the body of the monograph, ranging from harmonic analysis (with particular emphasis on aspects pertaining to analysis on spaces of homogeneous type) to functional analysis (open mapping and closed graph theorems, nonlocally convex topological vector spaces).

Our treatment is largely self-contained, with special care taken to fully define all relevant notions and, at the same time, with significant effort directed at avoiding heavy jargon associated with one area or another. This makes the book accessible to both specialists and graduate students with a basic background in analysis. In particular, the monograph may be used in a number of topic courses targeted at any of the following themes:

- *Harmonic Analysis*: a sharp metrization theory for quasimetric spaces and applications;
- *Functional Analysis I*: quantitative aspects of nonlocally convex topological vector spaces;

- *Functional Analysis II*: a new generation of open mapping and closed graph theorems;
- *Noncommutative Algebra*: an introduction to the theory of semigroupoids and groupoids.

In relation to the topic of analysis of metric spaces (which is much more prevalent in the literature), the current monograph addresses the need to understand the mechanics of manufacturing (often custom-designed) metrics and, as such, complements this body of results in a manner that leads to a deeper understanding and appreciation of this type of quantitative analysis.

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Contents

Chapter 1
Introduction

This monograph has two distinct, yet closely interrelated, parts. In the first part (consisting of Chaps. 1–3) we develop a metrization theory in the abstract setting of groupoids that, among other things, contains as particular cases the Aoki–Rolewicz theorem for locally bounded topological vector spaces and a sharpened version of the Macías–Segovia metrization theorem for quasimetric spaces. We also indicate how this theory can be used to provide a conceptually natural proof of the Alexandroff–Urysohn metrization theorem for uniform topological spaces. For this portion of our work, the methods employed are predominantly functional-analytic/algebraic, and the bulk of our results actually hold in the more general context of semigroupoids.

In the second part (comprised of Chaps. 4–6) we present a multitude of applications of our metrization theory to the area of analysis on quasimetric spaces (with a special emphasis on the structure and role of Hölder functions in such a setting), function space theory (covering topics such as completeness, embeddings, pointwise convergence, and separability of certain large classes of function spaces equipped with locally bounded, yet nonlocally convex, topologies), as well as classical functional analysis, dealing with open mapping and closed-graph-type theorems, and uniform boundedness principles, among other things, in settings where the notions of vector space and norm are significantly weakened. While precise details will be given later, here we wish to note that all our results hold in the class of quasi-Banach spaces. Regarding the significance of this category of spaces, in [67] N. Kalton writes that "*There are sound reasons to want to develop understanding of these spaces, but the absence of one of the fundamental tools of functional analysis, the Hahn–Banach theorem, has proved a very significant stumbling block. However, there has been some progress in the non-convex theory and arguably it has contributed to our appreciation of Banach space theory.*"

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1.1 Overview

In general, a topological space is said to be `metrizable` provided it is homeomorphic to a metric space. More transparently, a topological space (X, τ) is said to be metrizable if there exists a metric d on X with the property that the topology induced by d on X coincides with τ. As such, metrization theorems are results that give sufficient conditions for a topological space to be metrizable.

Consider three fundamental metrization results in various branches of mathematics:

- The Alexandroff–Urysohn metrization theorem for uniform spaces (topology),
- The Macías–Segovia metrization theorem for quasimetric spaces (harmonic analysis), and
- The Aoki–Rolewicz theorem for quasinormed vector spaces (functional analysis).

The formal statement of the first theorem above is as follows (the reader is referred to future chapters for definitions clarifying the terminology employed here).

Theorem 1.1 (Alexandroff–Urysohn). *Let X be a topological space. Then X is metrizable if and only if X is Hausdorff and the topology on X is induced by a uniform structure on X that has a countable fundamental system of entourages.*

A related version of this theorem states that a uniform space is pseudometrizable (i.e., its topology is induced by a pseudometric) if and only if its uniformity has a countable base. See the discussion in J. Kelley's book [71, Note 14, p. 186], where it is indicated that Theorem 1.1 originates in [4] (cf. also the discussion in Comment 2.82 in the last part of Sect. 2.2, which further underscores the prominent role of this classical result).

While the fact that the topology induced by a given quasidistance on a quasimetric space is metrizable is readily implied[1] by Theorem 1.1 (something that was known long before Macías and Segovia's work in [79]), Macías and Segovia's main contribution was to bring to prominence the quantitative aspects of this result (in the setting of quasimetric spaces). This is apparent from an inspection of the statement of their theorem, which plays a basic role in the area of analysis on spaces of homogeneous type and which we recall below (as a slight reformulation of [79, Theorem 2, p. 259]).

Theorem 1.2 (Macías-Segovia). *Let (X, ρ) be a quasimetric space, that is, X is a nonempty set and $\rho : X \times X \to [0, +\infty)$ is a quasidistance, i.e., a function that, for every $x, y, z \in X$, satisfies*[2]

[1] Any quasimetric space (X, ρ) may be canonically viewed as a uniform space whose uniformity has a countable fundamental system of entourages, say, $\{(x, y) \in X \times X : \rho(x, y) < n^{-1}\}$, $n \in \mathbb{N}$.

[2] The interested reader is referred to [99] for historical references pertaining to quasinormed spaces.

$$\rho(x, y) = 0 \Longleftrightarrow x = y, \quad \rho(x, y) = \rho(y, x), \quad \rho(x, y) \le c(\rho(x, z) + \rho(z, y)), \tag{1.1}$$

for some fixed finite constant $c \geq 1$. *Then there exists another quasidistance* ρ_* *on* X *that is equivalent to* ρ *(in the sense that each is dominated by a fixed multiple of the other) and satisfies the following additional properties. If*

$$\alpha := \frac{1}{\log_2 [c(2c+1)]} \in (0, 1), \tag{1.2}$$

then the following assertions hold:

(1) The function $(\rho_*)^\alpha : X \times X \to [0, +\infty)$ *satisfies*

$$\rho_*(x, y)^\alpha \le \rho_*(x, z)^\alpha + \rho_*(z, y)^\alpha, \qquad \forall\, x, y, z \in X. \tag{1.3}$$

Hence, $(\rho_*)^\alpha$ *is a distance on* X *that induces the same topology on* X *as the original quasidistance* ρ*. In particular, this topology is metrizable.*

(2) The function ρ_* *satisfies the following Hölder-type regularity condition of order* α*:*

$$|\rho_*(x, z) - \rho_*(y, z)| \le \tfrac{1}{\alpha} \max\big\{\rho_*(x, z)^{1-\alpha}, \rho_*(y, z)^{1-\alpha}\big\}\rho_*(x, y)^\alpha, \quad \forall\, x, y, z \in X. \tag{1.4}$$

Ever since its original inception, Theorem 1.2 has played a pivotal role in analysis on spaces of homogeneous type since the natural setting for analysis in this context is that of quasimetric spaces. As noted earlier, the latter spaces are in fact metrizable, but it is a rather subtle matter to associate metrics, inducing the same topology, in a way that brings out the quantitative features of the quasimetric space in question in an optimal manner. The seminal work on this topic done by R. Macías and C. Segovia has been very influential in the study of spaces of homogeneous type. In particular, Theorem 1.2 is a popular result that has been widely cited; see, e.g., the discussion in the monographs [32] by Christ, [114] by Stein, [123] by Triebel, [59] by Heinonen, [56] by Han and Sawyer, [38] by David and Semmes, and [39] by Deng and Han, to name a few. Strictly speaking, Macías and Segovia's original statement of this theorem has $3c^2$ in place of $c(2c+1)$ in (1.2) but, as indicated in the discussion in Comment 2.83 at the end of Sect. 2.2, the number $c(2c+1)$ is the smallest constant for which their approach works as initially intended.

On to a different topic. It is well known that there are many function spaces of basic importance in partial differential equations that are not Banach but merely quasi-Banach. Indeed, this is the case for significant portions of the following familiar scales of spaces: Lebesgue spaces, weak Lebesgue spaces, Lorentz spaces, Hardy spaces, weak Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel–Lizorkin spaces, and weighted versions of these spaces (among many others).

In the context of quasinormed spaces, the Aoki–Rolewicz theorem reads as follows (the original references are [7, 103], and an excellent, timely exposition may be found in [69]).

Theorem 1.3 (Aoki–Rolewicz). *Let X be a vector space equipped with a quasi-norm $\|\cdot\|$, i.e., a nonnegative function defined on X that satisfies for each $x, y \in X$ and each $\lambda \in \mathbb{R}$*

$$\|x\| = 0 \Longleftrightarrow x = 0, \quad \|\lambda x\| = |\lambda|\|x\|, \quad \|x + y\| \leq c(\|x\| + \|y\|), \tag{1.5}$$

for some fixed finite constant $c \geq 1$. Then there exists a quasinorm $\|\cdot\|_*$ on X that is equivalent to $\|\cdot\|$ and is a p-norm for some $p \in (0, 1]$, i.e., it satisfies

$$\|x + y\|_*^p \leq \|x\|_*^p + \|y\|_*^p \quad \textit{for all} \quad x, y \in X. \tag{1.6}$$

In particular, the topology induced by $\|\cdot\|$ on X is metrizable since it coincides with the topology induced by the (left- and right-invariant) distance $d(x, y) := \|x - y\|_*^p$ for all $x, y \in X$, on X.

Even though, strictly speaking, Theorems 1.2 and 1.3 are distinct results, it is inescapable that, at least formally, they share some basic characteristics (e.g., in [3, p. 319], the Macías–Segovia result is referred to as "an analogue" of the Aoki–Rolewicz theorem). This becomes even more apparent if Theorem 1.3 is reformulated in a (slightly more general, as it turns out) manner that places more emphasis on the quantitative aspects of the phenomenon at hand. For the reader's convenience, we will first momentarily digress for the purpose of recalling some basic facts and terminology from the theory of topological vector spaces.

A topological vector space is said to be `locally bounded` provided there exists a (topologically) bounded neighborhood of the zero vector. Recall that, in this context, being bounded means that the set in question is absorbed by each neighborhood of zero (and *not* that it is geometrically bounded, in the sense of having a finite diameter). Specifically, E is (topologically) bounded if and only if for every neighborhood V of the zero vector there exists a real number $\lambda_* > 0$ such that $E \subseteq \lambda V$ for every scalar $\lambda > \lambda_*$. Also (cf., e.g., [74, (1) p. 159]), a topological vector space X is locally bounded if and only if there exists a quasinorm $\|\cdot\|$ on X that yields the same topology on X as the original one (which amounts to the condition that the balls $\{y \in X : \|y\| < r\}$, $r \in (0, +\infty)$, constitute a fundamental system of neighborhoods for the zero vector). Moreover, a set in a quasinormed space is bounded in the topology induced by the quasinorm if and only if it has finite diameter with respect to the quasinorm. Hence, a quasi-Banach space is a complete, locally bounded topological vector space. Let us also recall that a set E in a vector space is said to be `balanced` if $\lambda E \subseteq E$ for every scalar λ with $|\lambda| \leq 1$. Remarkably, a quasinormed vector space is `locally convex` if and only if it is linearly isomorphic to a normed vector space (local convexity signifies the existence of a fundamental system of neighborhoods for the zero vector consisting of absorbing, balanced, and convex sets; one convenient description of the fact that a subset E of a vector space is convex is that $(\lambda + \eta)E = \lambda E + \eta E$ for all scalars $\lambda, \eta > 0$). Compare, for example, [69] for more details.

Returning to the mainstream discussion, we record the following more precise version of the Aoki–Rolewicz theorem (again, see the informative discussion in [69]).

Theorem 1.4 (Aoki–Rolewicz). *Let X be a Hausdorff, locally bounded, topological vector space. In particular, there exists a bounded and balanced neighborhood B of the zero vector in X. Let $c \in (1, +\infty)$ be such that $B + B \subseteq cB$, and define*

$$p := \frac{1}{\log_2 c} \in (0, +\infty). \tag{1.7}$$

Finally, for each $x \in X$, set

$$\|x\| := \inf\Big\{\Big(\sum_{i=1}^{N} |\!|\!| x_i |\!|\!|_B^p\Big)^{\frac{1}{p}} : N \in \mathbb{N},\ x_1, \dots, x_N \in X \ \text{ such that } \sum_{i=1}^{N} x_i = x\Big\}, \tag{1.8}$$

where $|\!|\!| \cdot |\!|\!|_B$ is the Minkowski gauge function associated with B, i.e.,

$$|\!|\!| x |\!|\!|_B := \inf\{\lambda > 0 : \lambda^{-1}x \in B\}, \qquad \forall\, x \in X. \tag{1.9}$$

Then $\|\cdot\|$ defined in (1.8) is a p-norm on the vector space X, which is equivalent to the quasinorm $|\!|\!| \cdot |\!|\!|_B$ (and, hence, induces the same topology on X as the original one). As a consequence, X is a locally p-convex vector space whenever $c > 2$ and a locally convex vector space whenever $c \leq 2$, and the topology on X is metrizable via a two-sided invariant distance.

On the face of the evidence presented so far, an optimistic observer would hope that the formal analogies between the statements of Theorems 1.2 and 1.4 would indicate that there is a more general phenomenon at work here encompassing the named results as particular manifestations. In this vein, it is worth recalling a popular dictum of E.H. Moore to the effect that whenever there are parallel theories, typically there is one that subsumes them all.

One of the goals of the present monograph is to shed light on this issue by proving a metrization theorem that contains both Theorem 1.2 and Theorem 1.4 (hence also Theorem 1.3) in a canonical fashion and that may also be used to provide a conceptually natural proof of Theorem 1.1. We manage to accomplish this without compromising the sharpness of the quantitative aspects of the results in question (for example, even when specialized to the particular case of quasimetric spaces our results yield a significant improvement of Theorem 1.2) and, also, are able to work under minimal algebraic assumptions, which ensures a desirable degree of versatility for our result. The latter aspect is particularly important for applications, as will become apparent from the discussion in Chaps. 4–6, where the impact of this metrization theory on other branches of mathematics is brought to light.

The unifying language that permits such a generalization is that of groupoids. Recall that the concept of groupoid was originally introduced by H. Brandt[3] in 1926 as an algebraic structure generalizing the notion of group by allowing the multiplication to be just partially defined (for more on this topic see the discussion in Sect. 2).

1.2 First Look at the Groupoid Metrization Theorem

A sample of the metrization results proved here in the context of groupoids is as follows (the body of the monograph contains stronger results in the sense that they indicate what can be achieved with weaker, or fewer, assumptions; see also Theorem 3.26 for a substantially expanded version of this result).

Theorem 1.5. *Let $(G, *, (\cdot)^{-1})$ be a groupoid, with partial multiplication $*$ and inverse operation $(\cdot)^{-1}$. For each $N \in \mathbb{N}$ denote by $G^{(N)}$ the set of all ordered N-tuples of elements in G whose product (in the given order) is meaningfully defined. Furthermore, denote by $G^{(0)}$ the unit space of G, and introduce $\mathcal{G}^{\mathrm{R}} := \{(a,b) \in G \times G : (a, b^{-1}) \in G^{(2)}\}$.*

Next, assume that $\psi : G \to [0, +\infty)$ is a function for which there exist two finite constants $C_0 \geq 0$ and $C_1 \geq 1$ such that the following properties hold:

- *quasisubadditivity:* $$\psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\}, \quad \textit{for all} \ \ (a,b) \in G^{(2)}, \tag{1.10}$$
- *quasisymmetry:* $$\psi(a^{-1}) \leq C_0\, \psi(a), \qquad \textit{for every} \ \ a \in G, \tag{1.11}$$
- *nondegeneracy:* $$a \in G \textit{ and } \psi(a) = 0 \Leftrightarrow a \in G^{(0)}, \textit{ i.e., } \psi^{-1}(\{0\}) = G^{(0)}. \tag{1.12}$$

Denote by τ_ψ^{R} the right topology induced by ψ on G, defined as the largest topology on G with the property that for any element $a \in G$ a fundamental system of neighborhoods is given by $\{B_\psi^{\mathrm{R}}(a,r)\}_{r>0}$, where for each $r \in (0, +\infty)$,

$$B_\psi^{\mathrm{R}}(a,r) := \big\{ b \in G : (a,b) \in \mathcal{G}^{\mathrm{R}} \textit{ and } \psi(a * b^{-1}) < r \big\}. \tag{1.13}$$

Also, with $C_1 \geq 1$ as in (1.10), let

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty]. \tag{1.14}$$

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[3] Strictly speaking, in [21] Brandt introduced a smaller class of groupoids, i.e., what is nowadays referred to as transitive groupoids.

Finally, introduce a symmetrized version of ψ *by setting*

$$\psi_{\rm sym}(a) := \max\{\psi(a), \psi(a^{-1})\}, \quad \forall\, a \in G, \tag{1.15}$$

and define the canonical regularization $\psi_{\rm reg} : G \to [0, +\infty)$ *of* ψ *by considering, for each* $a \in G$,

$$\psi_{\rm reg}(a) := \inf \left\{ \left(\sum_{i=1}^{N} \psi_{\rm sym}(a_i)^{\alpha} \right)^{\frac{1}{\alpha}} : N \in \mathbb{N}, \right.$$
$$\left. (a_1, \dots, a_N) \in G^{(N)},\ a = a_1 * \cdots * a_N \right\} \tag{1.16}$$

(with a natural alteration in the case when $\alpha = +\infty$*).*

Then the following conclusions hold.

(1) The function $\psi_{\rm reg}$ *is symmetric, in the sense that*

$$\psi_{\rm reg}(a^{-1}) = \psi_{\rm reg}(a) \quad \textit{for every} \quad a \in G, \tag{1.17}$$

and $\psi_{\rm reg}$ *is quasisubadditive, in the precise sense that, with* C_1 *denoting the same constant as in* (1.10)*, one has*

$$\psi_{\rm reg}(a * b) \leq C_1 \max\{\psi_{\rm reg}(a), \psi_{\rm reg}(b)\} \quad \textit{for all} \quad (a, b) \in G^{(2)}. \tag{1.18}$$

(2) With C_0 *and* C_1 *as in* (1.10) *and* (1.11)*, there holds*

$$C_1^{-2}\psi \leq \psi_{\rm reg} \leq \max\{1, C_0\}\,\psi \quad \textit{on} \quad G. \tag{1.19}$$

In particular, $\psi_{\rm reg}^{-1}(\{0\}) = G^{(0)}$.

(3) For each $\beta \in (0, \alpha]$ *the function* $\psi_{\rm reg}$ *is* β*-subadditive in the sense that one has (with a natural interpretation when* $\beta = \alpha = +\infty$*)*

$$\psi_{\rm reg}(a * b) \leq \Big(\psi_{\rm reg}(a)^{\beta} + \psi_{\rm reg}(b)^{\beta} \Big)^{\frac{1}{\beta}}, \qquad \forall\, (a, b) \in G^{(2)}. \tag{1.20}$$

(4) For each finite number $\beta \in (0, \alpha]$ *the function* $\psi_{\rm reg}$ *satisfies the following Hölder-type regularity condition of order* β*:*

$$\big|\psi_{\rm reg}(a) - \psi_{\rm reg}(b)\big| \leq \tfrac{1}{\beta} \max\big\{\psi_{\rm reg}(a)^{1-\beta}, \psi_{\rm reg}(b)^{1-\beta}\big\}\big[\psi_{\rm reg}(a * b^{-1})\big]^{\beta} \tag{1.21}$$

whenever $(a, b) \in \mathcal{G}^{\rm R}$ *(with the understanding that when* $\beta \geq 1$*, one also imposes the condition that* $a, b \notin G^{(0)}$*). Furthermore, the upper bound* (1.14) *for the exponent* β *appearing in this Hölder-type regularity result is sharp.*

(5) *The function* $\psi_{\rm reg} : (G, \tau_\psi^{\rm R}) \to [0, +\infty)$ *is continuous, and for every* $a \in G$ *and* $r > 0$ *the right* $\psi_{\rm reg}$*-ball* $B_{\psi_{\rm reg}}^{\rm R}(a, r) := \{b \in G : (a, b) \in \mathcal{G}^{\rm R}$ *and* $\psi_{\rm reg}(a * b^{-1}) < r\}$ *is open in the topology* $\tau_\psi^{\rm R}$.

(6) *For each finite number* $\beta \in (0, \alpha]$ *define the function*

$$d_{\psi,\beta}^{\rm R} : \mathcal{G}^{\rm R} \to [0, +\infty), \qquad d_{\psi,\beta}^{\rm R}(a, b) := \big[\psi_{\rm reg}(a * b^{-1})\big]^\beta, \quad \forall\, (a, b) \in \mathcal{G}^{\rm R}. \tag{1.22}$$

Then $d_{\psi,\beta}^{\rm R}$ *is a partially defined distance on* G *with domain* $\mathcal{G}^{\rm R}$*, i.e., it satisfies the following conditions:*

$$\begin{aligned} &\text{for any } (a, b) \in \mathcal{G}^{\rm R}, \text{ one has } d_{\psi,\beta}^{\rm R}(a, b) = 0 \text{ if and only if } a = b, \\ &d_{\psi,\beta}^{\rm R}(a, b) = d_{\psi,\beta}^{\rm R}(b, a) \text{ for every } (a, b) \in \mathcal{G}^{\rm R}, \\ &d_{\psi,\beta}^{\rm R}(a, b) \le d_{\psi,\beta}^{\rm R}(a, c) + d_{\psi,\beta}^{\rm R}(c, b) \text{ for all } (a, c), (c, b) \in \mathcal{G}^{\rm R}. \end{aligned} \tag{1.23}$$

Moreover, the topology induced by the partially defined distance $d_{\psi,\beta}^{\rm R}$ *on* G *is* $\tau_\psi^{\rm R}$.

(7) *The partially defined distance* $d_{\psi,\beta}^{\rm R}$ *introduced in* (1.22) *is right-invariant, in the sense that*

$$\begin{aligned} &(a, b) \in \mathcal{G}^{\rm R} \text{ and } c \in G \text{ such that } (a, c), (b, c) \in G^{(2)} \\ \Longrightarrow\ &(a * c, b * c) \in \mathcal{G}^{\rm R} \text{ and } d_{\psi,\beta}^{\rm R}(a * c, b * c) = d_{\psi,\beta}^{\rm R}(a, b). \end{aligned} \tag{1.24}$$

In the particular case when G *is a group,* $\mathcal{G}^{\rm R} = G \times G$ *and, hence, the function* $d_{\psi,\beta}^{\rm R}$ *is a genuine right-invariant distance on* G.

Theorem 1.5 contains Macías and Segovia's metrization result formulated in Theorem 1.2 when specialized to the particular case when the groupoid G is the so-called pair groupoid $X \times X$ associated with the ambient set X (as described in Example 2.31) of a quasimetric space. Moreover, Theorem 1.5 subsumes Aoki and Rolewicz's metrization result stated in Theorem 1.4 in the scenario in which the groupoid G is the underlying (Abelian) additive group of a given vector space X (cf. Example 2.29). The interplay between these results is studied in more detail in the body of the monograph; see the discussion in Sect. 3.2.3 in this regard. In particular, here we also elaborate on the manner in which Theorem 1.5 contains the Alexandroff–Urysohn metrization theorem (formulated in Theorem 1.1).

We wish to stress that the actual optimal value of the Hölder regularity exponent α (playing the role of upper bound of βs for which (1.21) holds) is not an issue of mere curiosity since this number plays a most fundamental role in the theory of function spaces that can be developed on spaces of homogeneous type. For example, the issue of identifying the sharp value of the Hölder regularity exponent α from (1.4) is raised explicitly in Remark 5.3 on p. 133 of [62], where the reader may find more details pertaining to the case of Hardy spaces. Here we wish to note that, when combined with the work in [80], our results lead to a satisfactory theory for Hardy spaces $H^p(X)$ whenever the quasimetric space (X, ρ) is equipped with an Ahlfors–David regular measure μ of order $d > 0$ and

$$\frac{d}{d + \min\{d, [\log_2 C_\rho]^{-1}\}} < p \leq 1, \tag{1.25}$$

where C_ρ is the optimal constant in the inequality $\rho(x, y) \leq C\ \max\{\rho(x, z), \rho(z, y)\}$ for all $x, y, z \in X$ (see Theorem 4.102 for details). It is worth remarking that this range for p is in the nature of best possible since from (1.25) we recover the familiar condition $\frac{n}{n+1} < p \leq 1$ (associated with atomic Hardy spaces for atoms satisfying one vanishing moment condition) in the case when $X := \mathbb{R}^n$, $n \in \mathbb{N}$, equipped with the Euclidean distance and the n-dimensional Lebesgue measure. In fact, similar considerations apply to the case of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type, as discussed in [39, 56] and others.

Another perspective that highlights the usefulness of a sharp Hölder regularity exponent α (in the context of (1.4)) is as follows. On the one hand, one naturally expects to have $\alpha = 1$ in the case when (X, ρ) is actually a metric space, since a distance function is Lipschitz in each of its variables. On the other hand, in the setting of Theorem 1.2, the condition that ensures that (X, ρ) is a metric space is $c = 1$, and, according to (1.2)–(1.4), this only yields the generally unsatisfactory result that a distance function is Hölder continuous of order $1/\log_2 3$. By way of contrast, the value of α in (1.14) becomes, as expected, 1 when $C_1 = 2$.

Our approach builds on and extends the work of Peetre and Sparr [97] (in the setting of normed Abelian groups), Gustavsson [53] (where a metrization theorem for semigroupoids is proved for a nonoptimal exponent α, namely $\alpha = (2\log_2 C_1)^{-1}$, i.e., half the value of α in (3.190)), and the classical work of Frink [49]. For a more in-depth discussion elaborating on the connections between Theorems 1.5 and 1.1–1.4, which also provides further motivational examples and background, the reader is referred to Sects. 3.2.3 and 3.2.4.

The organization of the monograph is as follows. The material in Sects. 2.1.1 and 2.1.2 amounts to a concise (yet self-contained) introduction to the theory of semigroupoids and groupoids, and in Sect. 2.2 we review topics of a topological flavor. The bulk of the work pertaining to quantitative metrization results is concentrated in Chap. 3. In particular, the regularization results for quasisubadditive mappings established in Sect. 3.1 greatly facilitate the presentation of our main groupoid metrization theorem. The latter is stated in Sect. 3.2.1 and proved in Sect. 3.2.2, and its various connections with Macías–Segovia, Aoki–Rolewicz, and Alexandroff–Urysohn theorems are highlighted in Sect. 3.2.3. The scope of this result is further expanded in Sect. 3.3.1 to the setting of semigroupoids. Several applications of this semigroupoid metrization theory are subsequently discussed in Sects. 3.3.2 and 3.3.3. Next, in Sect. 3.4, we state and prove a sharpened version of the Macías–Segovia result; cf. Theorem 3.46.

Moving on, in Chap. 4, we present a significant number of applications of our metrization theorems to analysis on quasimetric spaces. Without going into detail, the list of topics considered in this chapter includes extensions of Hölder functions, separation, density and embedding properties of Hölder functions, the regularized distance function to a set, Whitney-like partitions of unity via Hölder functions, the smoothness indexes of a quasimetric space, distribution theory on quasimetric spaces, Hardy spaces on Ahlfors-regular quasimetric spaces, approximation to the identity on Ahlfors-regular quasimetric spaces, bi-Lipschitz Euclidean embeddings of quasimetric spaces, the quasimetric version of Kuratowski's and Fréchet's embedding theorems, the Pompeiu–Hausdorff quasidistance on quasimetric spaces, and the Gromov–Pompeiu–Hausdorff distance between quasimetric spaces.

Chapter 5 is devoted to presenting applications of the metrization theory developed in Chap. 3 to function space theory, with a special emphasis on topics such as completeness, embeddings, pointwise convergence, and separability of certain inclusive classes of function spaces endowed with locally bounded, yet nonlocally convex, topologies. Finally, in Chap. 6 we revisit some of the cornerstones of classical functional analysis (including open mapping and closed-graph-type theorems, as well as uniform boundedness principles) in settings where the traditional context of a normed vector space is significantly relaxed. Once again, our metrization theory developed in the earlier chapters plays a key role in this endeavor.

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Chapter 2
Semigroupoids and Groupoids

This chapter is devoted to surveying the algebraic fundamentals of semigroupoids and groupoids and to reviewing the properties exhibited by these structures which play a crucial role in our subsequent work. While various results pertaining to the basic theory of semigroupoids and groupoids are scattered in the literature, we found it difficult to identify a few comprehensive, readily accessible accounts which cover all aspects dealt with here.

That being said, two useful references we wish to single out are the monographs [86, 101]. These contain brief chapters on the basics of groupoid theory, though the overall focus is on the role played by groupoids in indexing representations of operator algebras. The interested reader is also referred to the expository paper [23], which, among other things, gives a flavor of the role played by groupoids in A. Grothendieck's work in algebraic geometry, G.W. Mackey's work in ergodic theory, A. Connes' work in noncommutative geometry, and which contains a wealth of references to earlier articles.

It is primarily for this reason that we decided to make the presentation of the material in this chapter as self-contained, free of excessive jargon, and accessible to the nonexpert, as realistically possible. In the process, we contribute to the existing theory by clarifying certain aspects and by proving several new results of independent interest.

2.1 Algebraic Considerations

This section amounts to a concise, self-contained introduction to the algebraic theory of semigroupoids (Sect. 2.1.1) and groupoids (Sect. 2.1.2).

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2.1.1 Semigroupoids

We start with the following definition.

Definition 2.1. Given a nonempty set G, a `partially defined binary operation` on G is a function $*: \mathcal{G} \to G$, where $\mathcal{G}$ is a subset of $G \times G$ called the domain of $*$. If $a, b \in G$, then call $a * b$ `meaningfully defined` if $(a, b) \in \mathcal{G}$.

Next, we recall the concept of semigroupoid.

Definition 2.2. A `semigroupoid` is a nonempty set G equipped with a partially defined binary operation $*$ on G that is associative in the following precise sense.

Let $a, b, c \in G$. If $a * b$ and $b * c$ are meaningfully defined, then $(a * b) * c$ and $a * (b * c)$ are meaningfully defined and equal. Moreover, if either of these last two expressions is meaningfully defined, then so is the other and, again, they are equal.

The reader is alerted to the fact that other terms are occasionally used in the literature in place of *semigroupoid*, most notably *half-groupoid* and *incomplete groupoid*. Given a semigroupoid $(G, *)$, the binary operation $*: \mathcal{G} \to G$ can be thought of as a partially defined multiplication. In this setting, it is customary to introduce

$$G^{(2)} := \{(a, b) \in G \times G : a * b \text{ is meaningfully defined}\} \tag{2.1}$$

and refer to the latter as the `set of composable pairs` of the semigroupoid G. Several relevant examples of semigroupoids follow.

Example 2.3 (Semigroups). Any semigroup can naturally be regarded as a semigroupoid.

For instance, if (X, τ) is a topological space and $\mathrm{Comp}(X)$ denotes the collection of all compact subsets of X, then $(\mathrm{Comp}(X), \cup)$ may be naturally regarded as a semigroupoid. This example has already appeared in the proof of Corollary 3.36, where it plays a basic role.

Example 2.4 (Markov semigroupoids). Assume that X is an arbitrary, nonempty set and that $\Lambda : X \times X \to \{0, 1\}$ is an arbitrary function. Consider the collection of admissible words

$$G := \{\alpha = (x_1, \dots, x_n) : n \in \mathbb{N},\ x_i \in X,\ 1 \le i \le n, \text{ and}$$
$$\Lambda(x_i, x_{i+1}) = 1,\ 1 \le i \le n-1\}, \tag{2.2}$$

and for each $\alpha = (x_1, \dots, x_n)$, $\beta = (y_1, \dots, y_m) \in G$ satisfying $\Lambda(x_n, y_1) = 1$ define $\alpha * \beta := (x_1, \dots, x_n, y_1, \dots, y_m)$. Then $(G, *)$ is a semigroupoid.

Example 2.5 (Path Semigroupoids). Assume that X is an arbitrary set, and let Γ_X be the set of all functions $\gamma : [0, 1] \to X$. Call $\gamma_1, \gamma_2 \in \Gamma_X$ compatible if $\gamma_1(1) = \gamma_2(0)$ and, for each such compatible pair, define $\gamma_1 * \gamma_2 \in \Gamma_X$, the concatenation of γ_1 and γ_2, by setting

$$(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2], \\ \gamma_2(2t-1) & \text{if } t \in [1/2, 1], \end{cases} \qquad \forall\, t \in [0, 1]. \tag{2.3}$$

Next, consider a subset G of Γ_X and, given $\gamma_1, \gamma_2 \in G$, call $\gamma_1 * \gamma_2$ meaningfully defined if γ_1, γ_2 are compatible and $\gamma_1 * \gamma_2 \in G$. Then $(G, *)$ is a semigroupoid.

Example 2.6 (Direct sums of semigroupoids). If $(G, *)$ and $(H, \circ)$ are two given semigroupoids, then $(G \times H, \cdot)$ where $(a_1, b_1) \cdot (a_2, b_2) := (a_1 * a_2, b_1 \circ b_2)$ for every pair $((a_1, b_1), (a_2, b_2)) \in (G \times H)^{(2)}$, where

$$(G \times H)^{(2)} := \{((a_1, b_1), (a_2, b_2)) \in G \times H : (a_1, a_2) \in G^{(2)}, \\ (b_1, b_2) \in H^{(2)}\} \tag{2.4}$$

becomes a semigroupoid, which we denote by $G \oplus H$.

Example 2.7 (Subset semigroupoids). Let X be an arbitrary set, and denote by $\mathscr{P}(X)$ the collection of all subsets of X. Then any subset $\mathscr{X}$ of $\mathscr{P}(X)$ satisfying the condition

$$\text{whenever } \ A, B, C \in \mathscr{X} \ \text{ are such that } \ A \cup B \in \mathscr{X}, \ \text{ then} \\ B \cup C \in \mathscr{X} \iff A \cup B \cup C \in \mathscr{X} \tag{2.5}$$

has a natural semigroupoid structure by taking $\mathscr{X}^{(2)} := \{(A, B) \in \mathscr{X} \times \mathscr{X} : A \cup B \in \mathscr{X}\}$ and adopting the union of sets as the semigroupoid operation.

Let $(G, *)$ be a semigroupoid. In analogy with (2.1), let us set $G^{(1)} := G$, then, inductively, define for each $N \in \mathbb{N}$, $N \geq 2$,

$$G^{(N)} := \big\{(a_1, \dots, a_N) \in G \times \cdots \times G : (a_1, \dots, a_{N-1}) \in G^{(N-1)} \\ \text{and } \big((\cdots((a_1 * a_2) * \cdots) * a_{N-1}), a_N\big) \in G^{(2)}\big\}. \tag{2.6}$$

Based on the associativity axiom and induction, it can be proved that for each $N \in \mathbb{N}$, $N \geq 2$, and each N-tuple $(a_1, \dots, a_N) \in G^{(N)}$ the product $a_1 * a_2 * \cdots * a_{N-1} * a_N$ can be unambiguously defined. Furthermore, the same type of reasoning shows that

$$G^{(N)} = \big\{(a_1, \dots, a_N) \in G \times \cdots \times G : \\ (a_j, a_{j+1}) \in G^{(2)} \ \ \forall\, j \in \{1, \dots, N-1\}\big\}, \tag{2.7}$$

and, for every $M, N \in \mathbb{N}$,

$$\begin{aligned}(a_1, \dots, a_N) \in G^{(N)} \text{ and } (a_N, \dots, a_{N+M-1}) \in G^{(M)} \\ \Longrightarrow (a_1, \dots, a_{N+M-1}) \in G^{(N+M-1)}.\end{aligned} \tag{2.8}$$

Let us also note here that, for any semigroupoid $(G, *)$,

$$(G, *) \text{ is a semigroup} \iff G^{(2)} = G \times G. \tag{2.9}$$

In the notation introduced in (2.1), the associativity axiom in Definition 2.2 reads as follows. For every $a, b, c \in G$, the following three implications hold:

$$(a, b),\ (b, c) \in G^{(2)} \Longrightarrow \begin{cases} (a * b, c) \in G^{(2)},\ (a, b * c) \in G^{(2)} \\ \text{and } (a * b) * c = a * (b * c), \end{cases} \tag{2.10}$$

$$(a, b) \in G^{(2)},\ (a * b, c) \in G^{(2)} \Longrightarrow \begin{cases} (b, c) \in G^{(2)},\ (a, b * c) \in G^{(2)} \\ \text{and } (a * b) * c = a * (b * c), \end{cases} \tag{2.11}$$

$$(b, c) \in G^{(2)},\ (a, b * c) \in G^{(2)} \Longrightarrow \begin{cases} (a, b) \in G^{(2)},\ (a * b, c) \in G^{(2)} \\ \text{and } (a * b) * c = a * (b * c). \end{cases} \tag{2.12}$$

The set consisting of all composable pairs of a given semigroupoid carries a natural structure of semigroupoid, as explained below.

Example 2.8 (The semigroupoid of composable pairs). Given a semigroupoid $(G, *)$, call $(a, b), (c, d) \in G^{(2)}$ composable if $c = a * b$, and define a partial multiplication $\circ$ on $G^{(2)}$ by setting $(a, b) \circ (c, d) := (a, b * d)$ if $(a, b), (c, d) \in G^{(2)}$ are such that $c = a * b$. Then $(G^{(2)}, \circ)$ becomes a semigroupoid.

Moving on, for any two arbitrary subsets A, B of G, we will use the notation

$$A * B := \{a * b : (a, b) \in G^{(2)} \cap (A \times B)\}. \tag{2.13}$$

In particular, for $A \subseteq G$ and $a \in G$ abbreviate

$$A * a := A * \{a\}, \quad a * A := \{a\} * A, \tag{2.14}$$

$$A^n := \underbrace{A * A * \cdots * A}_{n \text{ factors}}, \quad \forall\, n \in \mathbb{N}. \tag{2.15}$$

Also, define $G^{(0)}$ as the collection of idempotent elements in the semigroupoid G, i.e.,

$$G^{(0)} := \{a \in G : (a, a) \in G^{(2)} \text{ and } a * a = a\}. \tag{2.16}$$

Definition 2.9. Let $(G,*)$ and $(H,\circ)$ be semigroupoids. A function $\phi : G \to H$ is called a (semigroupoid) `homomorphism` provided

$$\forall\,(a,b) \in G^{(2)} \implies \big(\phi(a),\phi(b)\big) \in H^{(2)} \text{ and } \phi(a*b) = \phi(a)\circ\phi(b). \quad (2.17)$$

Denote by $\mathrm{Hom}\,(G,H)$ the collection of all (semigroupoid) homomorphisms mapping G into H. Finally, a (semigroupoid) homomorphism $\phi : G \to H$ is called a (semigroupoid) `isomorphism` provided ϕ is bijective and $\phi^{-1} : H \to G$ is also a (semigroupoid) homomorphism.

Of course, any two isomorphic semigroupoids are abstractly identical. It is also clear that, given two semigroupoids $(G,*)$ and $(H,\circ)$, a function $\phi \in \mathrm{Hom}\,(G,H)$ is a semigroupoid isomorphism if and only if there exists $\psi \in \mathrm{Hom}\,(H,G)$ such that $\phi(\psi(b)) = b$ for each $b \in H$ and $\psi(\phi(a)) = a$ for each $a \in G$. In addition, the class of semigroupoid homomorphisms is stable under composition in the sense described below.

Remark 2.10. (i) If G_1, G_2, G_3 are semigroupoids, then $\phi_2 \circ \phi_1 \in \mathrm{Hom}\,(G_1,G_3)$ whenever $\phi_1 \in \mathrm{Hom}\,(G_1,G_2)$ and $\phi_2 \in \mathrm{Hom}\,(G_2,G_3)$. Furthermore, if ϕ_1, ϕ_2 are actually isomorphisms, then so is $\phi_2 \circ \phi_1$.

(ii) Assume that G_1, G_2 and G_1', G_2' are two given pairs of semigroupoids. A function $\Phi = (\phi,\phi') : G_1 \times G_2 \to G_1' \times G_2'$ belongs to $\mathrm{Hom}\,(G_1 \times G_2, G_1' \times G_2')$ if and only if $\phi \in \mathrm{Hom}\,(G_1, G_1')$ and $\phi' \in \mathrm{Hom}\,(G_2, G_2')$.

(iii) The definition of the notion of semigroupoid isomorphism might, at first sight, appear somewhat peculiar when compared to that of semigroup isomorphism. Recall that, given two semigroups $(G,*)$ and $(H,\circ)$, a function $\phi : G \to H$ is a semigroup isomorphism provided ϕ is a bijection and it satisfies $\phi(a*b) = \phi(a)\circ\phi(b)$ for all $a,b \in G$. By way of contrast, there exist two semigroupoids $(G,*)$, $(H,\circ)$ and a function $\phi \in \mathrm{Hom}\,(G,H)$ that is bijective, and yet $\phi^{-1} \notin \mathrm{Hom}\,(G,H)$. To see this, suppose the semigroupoid $(G,*)$ is such that $G := \mathbb{N}$, $G^{(2)} := \{(a,b) \in \mathbb{N}\times\mathbb{N} : a,b \text{ even}\}$, and $a * b := a + b$ for each $(a,b) \in G^{(2)}$. Also, take the semigroup $(\mathbb{N},+)$ to play the role of the semigroupoid $(H,\circ)$ and consider the function $\phi : G \to H$ given by $\phi(a) := a$ for each $a \in G$. Then ϕ is a bijective semigroupoid homomorphism, but the fact that $(G,*)$ is not a semigroup prevents ϕ^{-1} itself from being a semigroupoid homomorphism.

Some other basic properties of semigroupoid homomorphisms are discussed in Propositions 2.11 and 2.12 below.

Proposition 2.11. *Suppose that $(G,*)$ and $(H,\circ)$ are two semigroupoids, and assume that $\phi \in \mathrm{Hom}\,(G,H)$. Then $\phi(G^{(0)}) \subseteq H^{(0)}$. Furthermore, if ϕ is actually an isomorphism, then $\phi(G^{(0)}) = H^{(0)}$.*

Proof. If $G^{(0)}$ is empty, then there is nothing to prove. On the other hand, if $a \in G^{(0)}$, then $a = a * a$ so $\phi(a) = \phi(a*a) = \phi(a)\circ\phi(a)$. Thus $\phi(a) \in H^{(0)}$, proving $\phi(G^{(0)}) \subseteq H^{(0)}$. In the case when ϕ is an isomorphism, what we have just shown gives $\phi^{-1}(H^{(0)}) \subseteq G^{(0)}$. It follows that $H^{(0)} \subseteq \phi(G^{(0)})$, and hence $\phi(G^{(0)}) = H^{(0)}$ in this situation. □

Proposition 2.12. *Given semigroupoids G, H, a function $\phi : G \to H$ is a (semigroupoid) homomorphism if and only if*

$$\Phi : G^{(2)} \longrightarrow H^{(2)}, \qquad \Phi(a,b) := (\phi(a), \phi(b)) \quad \forall\,(a,b) \in G^{(2)}, \tag{2.18}$$

is a well-defined mapping that is a (semigroupoid) homomorphism when $G^{(2)}$ and $H^{(2)}$ are regarded as semigroupoids in the sense explained in Example 2.8.

Proof. This is a straightforward consequence of definitions. □

We conclude our discussion of semigroupoids by proving a general structure theorem (cf. Theorem 2.14 below). As a preamble, we revisit Example 2.7 and generalize its main underlying principle as described in the following lemma.

Lemma 2.13. *Let $(S, *)$ be a semigroup, and assume that G is a subset of S. Then the following two statements are equivalent:*

(1) The pair $(G, *)$ becomes a semigroupoid by considering the domain of $*$ to be the set $G^{(2)} := \{(a,b) \in G \times G : a * b \in G\}$.
(2) The following "chain conditions" are satisfied for every $a, b, c \in G$:

$$\begin{aligned} a * b \in G \ \text{ and } \ b * c \in G &\Longrightarrow a * b * c \in G, \\ a * b \in G \ \text{ and } \ a * b * c \in G &\Longrightarrow b * c \in G, \\ b * c \in G \ \text{ and } \ a * b * c \in G &\Longrightarrow a * b \in G. \end{aligned} \tag{2.19}$$

Proof. This follows in a straightforward manner by unraveling definitions (cf. (2.10)–(2.12)). □

The type of semigroupoid described in Lemma 2.13 is actually typical, as the following theorem shows. This theorem reinforces the intuition that semigroupoids are (up to isomorphisms) "incomplete" semigroups, i.e., semigroups from which a number of elements are discarded (without violating the chain conditions for the remaining set).

Theorem 2.14 (The structure of semigroupoids). *Given a semigroupoid $(G, *)$, there exist a semigroup $(S, \circ)$ and a set $H \subseteq S$ with the following properties:*

(1) The pair $(H, \circ)$ is a semigroupoid, considering $H^{(2)} := \{(u,v) \in H \times H : u \circ v \in H\}$ (i.e., H satisfies the chain conditions described in (2.19)).
(2) The semigroupoid $(H, \circ)$ is isomorphic to $(G, *)$ (in the sense of Definition 2.9).

In the proof of Theorem 2.14, presented below, we will need the following lemma.

Lemma 2.15. *Assume that* $(G, *)$ *is a semigroupoid. Then for any* $(a, b), (c, d) \in G^{(2)}$ *there holds*

$$(a * b, c * d) \in G^{(2)} \iff (b, c) \in G^{(2)}. \tag{2.20}$$

Proof. Suppose $a, b, c, d \in G$ are such that $(a, b), (c, d) \in G^{(2)}$. If $(a * b, c * d) \in G^{(2)}$, then (2.11) ensures that $(b, c * d) \in G^{(2)}$; hence, further, $(b, c) \in G^{(2)}$ thanks to (2.12). This establishes the right-pointing implication in (2.20). Conversely, if $(b, c) \in G^{(2)}$, then, first, $(a*b, c) \in G^{(2)}$ by (2.11), then, second, $(a*b, c*d) \in G^{(2)}$ once again by (2.11). □

We are now prepared to present the

Proof of Theorem 2.14 To set the stage, define the collection of "words" associated with the set G (viewed as "alphabet") by

$$W := \{\langle a_1, \dots, a_N\rangle : N \in \mathbb{N},\ a_i \in G \text{ for all } 1 \le i \le N\}. \tag{2.21}$$

For each word $w = \langle a_1, \dots, a_N\rangle \in W$, we will refer to N as its length and write $\ell(w) := N$. Naturally, call two words equal provided they have the same length and identical letters (taking into account the order in which they are listed). When equipped with the binary operation of concatenation of words, i.e.,

$$\begin{aligned} \langle a_1, \dots, a_N\rangle \otimes \langle b_1, \dots, b_M\rangle &:= \langle a_1, \dots, a_N, b_1, \dots, b_M\rangle, \\ \forall\, \langle a_1, \dots, a_N\rangle,\ &\langle b_1, \dots, b_M\rangle \in W, \end{aligned} \tag{2.22}$$

the set W becomes a semigroup. Call a word $w = \langle a_1, \dots, a_N\rangle \in W$ *reducible* (modulo the semigroupoid multiplication $*$) provided $N \ge 2$ and there exists $i \in \{1, \dots, N-1\}$ such that $(a_i, a_{i+1}) \in G^{(2)}$. In such a scenario, we will say that the word w may be *contracted* to $w' := \langle a_1, \dots, a_{i-1}, a_i * a_{i+1}, a_{i+2}, \dots, a_N\rangle \in W$. Alternatively, we will refer to w' as being *a contraction of* w. Note that when a word is contracted, in the manner just described, its length decreases by one unit. Finally, call a word *irreducible* provided it is not reducible and introduce $W_{\text{irr}} := \{w \in W : w \text{ is an irreducible word}\}$. Let us note that, tautologically,

$$\langle a\rangle \in W_{\text{irr}}, \qquad \forall\, a \in G. \tag{2.23}$$

Next, for each $w \in W$ consider $\mathcal{R}(w) \subseteq W$ consisting of w as well as all of its successive contractions. Specifically, for each $w \in W$, set

$$\begin{aligned} \mathcal{R}(w) := \Big\{\widetilde{w} \in W :\ &\text{either } \widetilde{w} = w \text{ or } \exists\, w_1, \dots, w_k \in W, k \ge 2, \\ &\text{such that } w_1 = w, w_k = \widetilde{w}, \\ &\text{and } w_{i+1} \text{ is a contraction of } w_i \text{ for } 1 \le i \le k-1\Big\}. \end{aligned} \tag{2.24}$$

Then, clearly, for each $w \in W$ we have

$$w \in \mathcal{R}(w) \quad \text{and} \quad \mathcal{R}(w) = \{w\} \iff w \in W_{\mathrm{irr}}, \tag{2.25}$$

$$\mathcal{R}(\widetilde{w}) \subseteq \mathcal{R}(w), \qquad \text{for every} \quad \widetilde{w} \in \mathcal{R}(w). \tag{2.26}$$

Furthermore, for any $w_1, w_2 \in W$ we have

$$\widetilde{w}_1 \otimes \widetilde{w}_2 \in \mathcal{R}(w_1 \otimes w_2), \qquad \forall\, \widetilde{w}_1 \in \mathcal{R}(w_1), \ \forall\, \widetilde{w}_2 \in \mathcal{R}(w_2). \tag{2.27}$$

Let us now assume that an arbitrary $w = \langle a_1, \ldots, a_N \rangle \in W$ has been fixed. Regarding the structure of words in the set (2.24), an induction argument (carried backward) on the length of the word shows that for any $\widetilde{w} \in \mathcal{R}(w)$ there exist integers $M \in \mathbb{N}$ and $i_1, \ldots, i_{M+1} \in \mathbb{N}$ satisfying the following properties:

$$\begin{gathered} 1 = i_1 < i_2 < \cdots < i_k < i_{k+1} < \cdots < i_{M+1} = N + 1, \\ (a_{i_k}, \ldots, a_{i_{k+1}-1}) \in G^{(i_{k+1}-i_k)}, \quad \forall\, k \in \{1, \ldots, M\}, \quad \text{and} \\ \widetilde{w} = \langle \underbrace{a_1 * \cdots * a_{i_2-1}}_{\text{1st letter}}, \ldots, \underbrace{a_{i_k} * \cdots * a_{i_{k+1}-1}}_{k\text{th letter}}, \ldots, \underbrace{a_{i_M} * \cdots * a_N}_{M\text{th letter}} \rangle. \end{gathered} \tag{2.28}$$

In addition, Lemma 2.15 and (2.12)–(2.11) show that, in this context,

$$\widetilde{w} \in W_{\mathrm{irr}} \iff (a_{i_{k+1}-1}, a_{i_{k+1}}) \notin G^{(2)}, \quad \forall\, k \in \{1, \ldots, M-1\}. \tag{2.29}$$

Next, since $\emptyset \neq \{\ell(\widetilde{w}) : \widetilde{w} \in \mathcal{R}(w)\} \subseteq \mathbb{N}$, it follows that $K := \min\{\ell(\widetilde{w}) : \widetilde{w} \in \mathcal{R}(w)\}$ is a well-defined natural number. Moreover, there exists some $w_{\min} \in \mathcal{R}(w)$ with the property that $\ell(w_{\min}) = K$. Since $w_{\min}$ has minimal length among all words in $\mathcal{R}(w)$, it follows that $w_{\min}$ can no longer be contracted, so we have $w_{\min} \in W_{\mathrm{irr}}$. Consequently, from (2.28) and (2.29) we deduce that there exist integers $j_1, \ldots, j_{K+1} \in \mathbb{N}$ such that

$$\begin{gathered} 1 = j_1 < j_2 < \cdots < j_k < j_{k+1} < \cdots < j_{K+1} = N + 1, \\ (a_{j_k}, \ldots, a_{j_{k+1}-1}) \in G^{(j_{k+1}-j_k)}, \quad \forall\, k \in \{1, \ldots, K\}, \\ (a_{j_{k+1}-1}, a_{j_{k+1}}) \notin G^{(2)}, \quad \forall\, k \in \{1, \ldots, K-1\}, \quad \text{and} \\ w_{\min} = \langle \underbrace{a_1 * \cdots * a_{j_2-1}}_{\text{1st letter}}, \ldots, \underbrace{a_{j_k} * \cdots * a_{j_{k+1}-1}}_{k\text{th letter}}, \ldots, \underbrace{a_{j_K} * \cdots * a_N}_{K\text{th letter}} \rangle. \end{gathered} \tag{2.30}$$

At this stage we make the claim that

$$\mathcal{R}(w) \cap W_{\mathrm{irr}} = \{w_{\min}\}. \tag{2.31}$$

The fact that $w_{\min} \in \mathcal{R}(w) \cap W_{\text{irr}}$ has already been noted. As such, it remains to show that if $\widetilde{w} \in \mathcal{R}(w) \cap W_{\text{irr}}$, then necessarily $\widetilde{w} = w_{\min}$. To this end, assume that some $\widetilde{w} \in \mathcal{R}(w) \cap W_{\text{irr}}$ has been given. Thus, we may assume that $\widetilde{w}$ is as in (2.28) and that $(a_{i_{k+1}-1}, a_{i_{k+1}}) \notin G^{(2)}$ for each $k \in \{1, \ldots, M-1\}$. Also, by definition, $M \geq K$. Suppose that $i_2 < j_2$. Then, on the one hand, we have that $(a_{i_2-1}, a_{i_2}) \notin G^{(2)}$. On the other hand, the second line in (2.30) implies $(a_{i_2-1}, a_{i_2}) \in G^{(2)}$, given that $i_2 \leq j_2 - 1$. Thus, necessarily, $i_2 \geq j_2$. In a similar fashion, we also obtain that $j_2 \geq i_2$, hence, ultimately, $i_2 = j_2$. Continuing this reason in an inductive fashion we arrive at the conclusion that $i_k = j_k$ for each index $k \in \{2, \ldots, K\}$. Having established this, we go on to observe that if $K < M$, then $K \leq M - 1$, which, by our assumptions on $\widetilde{w}$ (with $k := K$), further implies that $(a_{i_{K+1}-1}, a_{i_{K+1}}) \notin G^{(2)}$. On the other hand, from the second line in (2.30) and the fact that $j_K = i_K$, we obtain $(a_{i_K}, \ldots, a_N) \in G^{(N+1-i_K)}$. This, of course, implies $(a_{i_{K+1}-1}, a_{i_{K+1}}) \in G^{(2)}$, contradicting our earlier conclusion. This proves that, necessarily, $M = K$. Upon recalling that, by design, $i_1 = 1 = j_1$ and $i_{K+1} = N + 1 = j_{M+1}$, it follows that $\widetilde{w} = w_{\min}$, completing the proof of (2.31).

As a corollary of (2.31), we see that $w \in W$ uniquely determines $w_{\min}$. Hence, we may unequivocally talk about the contraction of minimal length of any given word. Also, we know that this contraction of minimal length is irreducible. Moreover, given an arbitrary $w \in W$ and $\widetilde{w} \in \mathcal{R}(w)$, based on (2.26) and (2.31), we may write that $\{\widetilde{w}_{\min}\} = \mathcal{R}(\widetilde{w}) \cap W_{\text{irr}} \subseteq \mathcal{R}(w) \cap W_{\text{irr}} = \{w_{\min}\}$. Hence, $\widetilde{w}_{\min} = w_{\min}$, which further entails

$$w'_{\min} = w''_{\min}, \qquad \forall\, w \in W \ \text{ and } \ \forall\, w', w'' \in \mathcal{R}(w). \tag{2.32}$$

In particular, since $w, w_{\min} \in \mathcal{R}(w)$ for any $w \in W$, we see from (2.32) that

$$\big(w_{\min}\big)_{\min} = w_{\min}, \qquad \forall\, w \in W. \tag{2.33}$$

For further use, let us also remark here that

$$w = w_{\min} \iff w \in W_{\text{irr}}, \qquad \forall\, w \in W, \tag{2.34}$$

$$\text{and } \ \langle a \rangle_{\min} = \langle a \rangle, \qquad \forall\, a \in G. \tag{2.35}$$

Indeed, (2.34) is a direct consequence of (2.31), while (2.35) follows from (2.34) and (2.23). In addition, since for any $w', w'' \in W$ we have $w'_{\min} \otimes w''_{\min} \in \mathcal{R}(w' \otimes w'')$ thanks to (2.31) and (2.27), we deduce, with the help of (2.32), that

$$\big(w' \otimes w''\big)_{\min} = \big(w'_{\min} \otimes w''_{\min}\big)_{\min}, \qquad \forall\, w', w'' \in W. \tag{2.36}$$

Moving on, given two arbitrary words $w', w'' \in W$, define

$$w' \sim w'' \overset{\text{def}}{\iff} w'_{\min} = w''_{\min}. \tag{2.37}$$

Obviously, $\sim$ is an equivalence relation on W and we let $W/\sim$ stand for the collection of all equivalence classes of words in W modulo $\sim$. Also, if $[w]$ denotes the equivalence class of a generic word $w \in W$, then (2.33) gives

$$[w] = \big[w_{\min}\big], \qquad \forall\, w \in W. \tag{2.38}$$

Let $S := W/\sim$ stand for the collection of all equivalence classes induced by $\sim$ on W, and consider the binary operation $\circ$ defined on S by the formula

$$[w'] \circ [w''] := [w' \otimes w''], \qquad \forall\, [w'],\ [w''] \in S. \tag{2.39}$$

Note that $\circ$ is unambiguously defined since whenever $w', w'', \widetilde{w'}, \widetilde{w''} \in W$ are such that $w' \sim \widetilde{w'}$ and $w'' \sim \widetilde{w''}$, it follows from (2.37) that $w'_{\min} = \widetilde{w'}_{\min}$ and $w''_{\min} = \widetilde{w''}_{\min}$. Hence, based on (2.38) and (2.36), we may write

$$\begin{aligned}
[w' \otimes w''] &= \big[(w' \otimes w'')_{\min}\big] = \big[(w'_{\min} \otimes w''_{\min})_{\min}\big] = \big[(\widetilde{w'}_{\min} \otimes \widetilde{w''}_{\min})_{\min}\big] \\
&= \big[(\widetilde{w'} \otimes \widetilde{w''})_{\min}\big] = [\widetilde{w'} \otimes \widetilde{w''}],
\end{aligned} \tag{2.40}$$

as desired. Granted what we have just proved, it follows that $(S, \circ)$ is a semigroup.

The next step is to consider the family of equivalence classes of singletons in W, i.e.,

$$H := \big\{[\langle a\rangle] : a \in G\big\} \subseteq S. \tag{2.41}$$

We claim that the set H satisfies the chain conditions formulated in (2.19). To see that this is the case, consider $a, b, c \in G$ with the property that $[\langle a\rangle] \circ [\langle b\rangle] \in H$ and $[\langle b\rangle] \circ [\langle c\rangle] \in H$. Then $[\langle a, b\rangle] \in H$ and $[\langle b, c\rangle] \in H$, and hence there exist $d, e \in G$ such that $[\langle a, b\rangle] = [\langle d\rangle]$ and $[\langle b, c\rangle] = [\langle e\rangle]$. In turn, since $\langle d\rangle, \langle e\rangle \in W$ are irreducible (cf. (2.23)), this forces $(a, b), (b, c) \in G^{(2)}$. Consequently, $(a * b, c) \in G^{(2)}$ by (2.10), so that

$$[\langle a\rangle] \circ [\langle b\rangle] \circ [\langle c\rangle] = [\langle a, b, c\rangle] = [\langle a * b, c\rangle] = [\langle a * b * c\rangle] \in H, \tag{2.42}$$

as desired. Next, consider the case when $a, b, c \in G$ are such that $[\langle a\rangle] \circ [\langle b\rangle] \in H$ and $[\langle a\rangle] \circ [\langle b\rangle] \circ [\langle c\rangle] \in H$. Hence, $[\langle a, b\rangle] \in H$ and $[\langle a, b, c\rangle] \in H$, so there exist $d, e \in G$ such that $[\langle a, b\rangle] = [\langle d\rangle]$ and $[\langle a, b, c\rangle] = [\langle e\rangle]$. As before, since $\langle d\rangle \in W$ is irreducible, we conclude that $(a, b) \in G^{(2)}$. In turn, this shows that $[\langle e\rangle] = [\langle a, b, c\rangle] = [\langle a * b, c\rangle]$. Then the same pattern of reasoning gives that $(a * b, c) \in G^{(2)}$, hence, ultimately, $(b, c) \in G^{(2)}$ by (2.11). With this in hand, we may then write

$$[\langle b\rangle] \circ [\langle c\rangle] = [\langle b, c\rangle] = [\langle b * c\rangle] \in H, \tag{2.43}$$

which shows that the analog of the second line in (2.19) holds in the current situation. Finally, the analog of the third line in (2.19) is verified in a similar manner, and this completes the proof of the fact that the set H satisfies the chain conditions. Thus, in light of Lemma 2.13, considering

$$\begin{aligned} H^{(2)} &:= \big\{([\langle a\rangle],[\langle b\rangle]) :\ [\langle a\rangle],[\langle b\rangle] \in H \text{ such that } [\langle a\rangle]\circ[\langle b\rangle] \in H\big\} \\ &= \big\{([\langle a\rangle],[\langle b\rangle]) :\ (a,b) \in G^{(2)}\big\} \end{aligned} \tag{2.44}$$

turns $(H, \circ)$ into a semigroupoid. There remains to check that the mapping

$$\phi : G \to H, \qquad \phi(a) := [\langle a\rangle] \quad \forall\, a \in G, \tag{2.45}$$

is a semigroupoid isomorphism (in the sense of Definition 2.9). To this end, observe that whenever $(a,b) \in G^{(2)}$, we have $(\phi(a),\phi(b)) = ([\langle a\rangle],[\langle b\rangle]) \in H^{(2)}$ by (2.44), and

$$\phi(a)\circ\phi(b) = [\langle a\rangle]\circ[\langle b\rangle] = [\langle a,b\rangle] = [\langle a*b\rangle] = \phi(a*b). \tag{2.46}$$

Hence, ϕ is a semigroupoid homomorphism. By design, ϕ is surjective. In addition, if $a,b \in G$ are such that $\phi(a) = \phi(b)$, then $[\langle a\rangle] = [\langle b\rangle]$, and hence $\langle a\rangle = \langle a\rangle_{\min} = \langle b\rangle_{\min} = \langle b\rangle$ by (2.35) and (2.37). In turn, this forces $a = b$, which shows that ϕ is also injective. Moreover, this reasoning proves that the inverse $\phi^{-1} : H \to G$ of the bijective function ϕ is meaningfully given by $\phi^{-1}([\langle a\rangle]) := a$ for any $[\langle a\rangle] \in H$. Much as in the case of ϕ, the function ϕ^{-1} is also a semigroupoid homomorphism. All in all, the mapping ϕ in (2.45) is a (semigroupoid) isomorphism, and this completes the proof of the theorem. □

It is also worth recording a version of Theorem 2.14 that avoids involving a semigroupoid isomorphism and that may be viewed as a completion procedure turning semigroupoids into semigroups in a canonical manner.

Theorem 2.16 (Completing semigroupoids to semigroups). *For any given semigroupoid $(G,*)$ there exists a semigroup $(\widetilde{G},\widetilde{*})$ with the property that*

$$G \subseteq \widetilde{G} \ \text{ and } \ a\,\widetilde{*}\,b = a*b \ \text{ whenever } \ (a,b) \in G^{(2)}, \tag{2.47}$$

i.e., the inclusion map ι of G into $\widetilde{G}$ is well-defined and belongs to Hom $(G,\widetilde{G})$.

Furthermore, a completion procedure associating to any semigroupoid a semigroup related to it as in (2.47) may be devised so that the following properties are satisfied:

(i) A semigroupoid $(G,*)$ is a semigroup if and only if $G = \widetilde{G}$.
(ii) Given any two semigroupoids $(G,*)$ and $(H,\star)$, any semigroupoid homomorphism $\phi : G \to H$ may be extended to a semigroup homomorphism $\widetilde{\phi} : \widetilde{G} \to \widetilde{H}$.
(iii) For any three semigroupoids G_1, G_2, G_3 and any two semigroupoid homomorphism $\phi : G_1 \to G_2$, $\psi : G_2 \to G_3$, it follows that $\widetilde{\psi\circ\phi} = \widetilde{\psi}\circ\widetilde{\phi}$.
(iv) In the context of (ii), if ϕ is a semigroupoid isomorphism, then $\widetilde{\phi}$ is a semigroup isomorphism.

Proof. Utilizing notation employed in the proof of Theorem 2.14, observe first that $\{\langle a\rangle : a \in G\} \subseteq W_{\rm irr}$, and that $\{[w] : w \in W_{\rm irr}\}$ is a listing, without repetitions, of all elements in S. Then the idea is to define $\widetilde{G}$ as the set obtained from this description of S by relabeling $[\langle a\rangle]$ simply as a for any $a \in G$, i.e.,

$$\widetilde{G} := \{[w] : w \in W_{\rm irr} \text{ and } \ell(w) \geq 2\} \cup G, \tag{2.48}$$

disjoint union. This definition guarantees that $G \subseteq \widetilde{G}$ and that the function

$$\psi : \widetilde{G} \longrightarrow S, \qquad \psi(u) := \begin{cases} u & \text{if } u \in \widetilde{G} \setminus G, \\ [\langle u\rangle] & \text{if } u \in G, \end{cases} \qquad \forall\, u \in \widetilde{G}, \tag{2.49}$$

is a bijection. For this portion of the proof, let $\circ$ stand for the semigroup multiplication on S; cf. (2.39). We then define the binary operation $\widetilde{*}$ on $\widetilde{G}$ according to

$$u \,\widetilde{*}\, v := \psi^{-1}\big(\psi(u) \circ \psi(v)\big), \qquad \forall\, u, v \in \widetilde{G}. \tag{2.50}$$

The pair $(\widetilde{G}, \widetilde{*})$ is then a semigroup and, for any $(a, b) \in G^{(2)}$, there holds

$$\begin{aligned} a \,\widetilde{*}\, b &= \psi^{-1}\big(\psi(a) \circ \psi(b)\big) = \psi^{-1}\big([\langle a\rangle] \circ [\langle b\rangle]\big) = \psi^{-1}\big([\langle a, b\rangle]\big) \\ &= \psi^{-1}\big([\langle a * b\rangle]\big) = a * b, \end{aligned} \tag{2.51}$$

as desired. This justifies the claim made in the first part of the statement of the theorem.

Let us consider now the properties of the completion scheme described previously for the semigroupoid $(G, *)$ [cf. (2.48)] in greater detail. In doing so, whenever convenient, we will not distinguish between $\widetilde{G}$ and S (i.e., agree to identify $[\langle a\rangle]$ with a, for each $a \in G$). Obviously, if $G = \widetilde{G}$, then $(G, *)$ is a semigroup. Conversely, if $(G, *)$ is a semigroup, then $G^{(2)} = G \times G$ and, hence, $W_{\rm irr} = \{\langle a\rangle : a \in G\}$. Then (2.48) shows that $G = \widetilde{G}$. This completes the proof of the claim in item (i).

To set the stage for dealing with the claim in item (ii), let $(G, *)$ be a semigroupoid and, in a first stage, assume that $(H, \star)$ is a semigroup. Given a semigroupoid homomorphism $\phi : G \to H$, we note that for any $a_1, \dots, a_N \in G$ and for any integers $M \in \mathbb{N}$ and $i_1, \dots, i_{M+1} \in \mathbb{N}$ satisfying

$$\begin{aligned} &1 = i_1 < i_2 < \cdots < i_k < i_{k+1} < \cdots < i_{M+1} = N + 1, \\ &\text{and } \ (a_{i_k}, \dots, a_{i_{k+1}-1}) \in G^{(i_{k+1} - i_k)}, \quad \forall\, k \in \{1, \dots, M\}, \end{aligned} \tag{2.52}$$

we have

$$\phi(a_1) \star \cdots \star \phi(a_N) = \prod_{k=1}^{M} \phi\big(a_{i_k} * \cdots * a_{i_{k+1}-1}\big), \tag{2.53}$$

where the product on the right-hand side is taken in $(H, \star)$. In concert with the structure of words in $\mathcal{R}(w)$, for a generic $w \in W$, described in (2.28), this allows us to conclude that

$$\begin{aligned} \phi(a_1) \star \cdots \star \phi(a_N) = \phi(b_1) \star \cdots \star \phi(b_M) \\ \text{whenever } \langle b_1, \ldots, b_M\rangle \in \mathcal{R}\big(\langle a_1, \ldots, a_N\rangle\big). \end{aligned} \tag{2.54}$$

In turn, this further implies that

$$\begin{aligned} \phi(a_1) \star \cdots \star \phi(a_N) = \phi(b_1) \star \cdots \star \phi(b_M) \\ \text{whenever } \langle b_1, \ldots, b_M\rangle \sim \langle a_1, \ldots, a_N\rangle \end{aligned} \tag{2.55}$$

since $\langle b_1, \ldots, b_M\rangle_{\min} = \langle a_1, \ldots, a_N\rangle_{\min} \in \mathcal{R}\big(\langle a_1, \ldots, a_N\rangle\big) \cap \mathcal{R}\big(\langle b_1, \ldots, b_M\rangle\big)$ in such a case.

Property (2.55) permits us to unambiguously define $\widetilde{\phi} : \widetilde{G} \to H$ by setting

$$\widetilde{\phi}\big([\langle a_1, \ldots, a_N\rangle]\big) := \phi(a_1) \star \cdots \star \phi(a_N), \qquad \forall\, a_1, \ldots, a_N \in G. \tag{2.56}$$

It may then be readily verified that $\widetilde{\phi}$ is a semigroup homomorphism, which extends ϕ. This proves the claim in item *(ii)* in the particular case when $(H, \star)$ is a semigroup.

Having dealt with this special case facilitates the treatment of *(ii)* in its full generality, i.e., when $(H, \star)$ is merely a semigroupoid. Indeed, if $(\widetilde{H}, \widetilde{\star})$ is the semigroup associated with $(H, \star)$ as in the first part of the proof, then, as already noted, $\iota : H \hookrightarrow \widetilde{H}$ is a semigroupoid homomorphism. Then, thanks to item *(i)* in Remark 2.10, $\iota \circ \phi : G \to \widetilde{H}$ is a semigroupoid homomorphism and, since $(\widetilde{H}, \widetilde{\star})$ is a semigroup, what we have just proved in the previous paragraph allows us to extend $\iota \circ \phi$ to a semigroup homomorphism $\widetilde{\iota \circ \phi} : \widetilde{G} \to \widetilde{H}$. Since this continues to be an extension of the original ϕ, the desired conclusion follows. This completes the proof of the claim made in item (ii), though we wish to make one last remark in connection with (ii). Specifically, since we agreed to make the identifications $[\langle a\rangle] \equiv a$ for each $a \in G$ and $[\langle b\rangle] \equiv b$ for each $b \in H$, for the extension $\widetilde{\phi}$ of ϕ constructed previously (cf. (2.56)) we may write

$$\begin{aligned} \widetilde{\phi}\big([\langle a_1, \ldots, a_N\rangle]\big) &= \widetilde{\phi}\big([\langle a_1\rangle] \circ \cdots \circ [\langle a_N\rangle]\big) = \widetilde{\phi}\big([\langle a_1\rangle]\big) \circ \cdots \circ \widetilde{\phi}\big([\langle a_N\rangle]\big) \\ &\equiv \phi(a_1) \star \cdots \star \phi(a_N) \equiv [\langle \phi(a_1)\rangle] \circ \cdots \circ [\langle \phi(a_N)\rangle] \\ &= [\langle \phi(a_1), \ldots, \phi(a_N)\rangle] \end{aligned} \tag{2.57}$$

for any $a_1, \ldots, a_N \in G$.

Finally, the claim made in item *(iii)* is a direct consequence of (2.57), whereas the claim made in item *(iv)* is a simple consequence of *(iii)* and *(ii)* applied both to the function ϕ and to the function ϕ^{-1}. □

2.1.2 Groupoids

The concept of groupoid, generalizing that of group, is defined as follows.

Definition 2.17. A `groupoid` is a triplet $(G, *, (\cdot)^{-1})$ with the property that $(G, *)$ is a semigroupoid and $(\cdot)^{-1} : G \to G$ is a function subject to the following two axioms:

(i) For every $a \in G$ the products $a^{-1} * a$ and $a * a^{-1}$ are meaningfully defined.
(ii) If $a, b \in G$ are such that $a * b$ is meaningfully defined, then $a * b * b^{-1} = a$ and $a^{-1} * a * b = b$.

While a variety of examples of groupoids will be presented shortly, for the time being we record an intuitively simple example that mimics, in the abstract, the groupoid structure associated with the collection of all geometric vectors in $\mathbb{R}^n$, regarded as physical arrows.

Example 2.18 (Vector groupoids). Let V be the collection of "vectors" $\vec{v}$, informally thought of as arrows, uniquely determined by a source, denoted by $s(\vec{v})$, and a target, denoted by $t(\vec{v})$. Consider a partially defined binary operation "+" such that two vectors $\vec{v}, \vec{w} \in V$ can be added if and only if $t(\vec{v}) = s(\vec{w})$, in which case $\vec{v} + \vec{w}$ is the arrow with source $s(\vec{v})$ and target $t(\vec{w})$ (assumed to be in V). Also, define $-\vec{v}$, the inverse of a vector $\vec{v} \in V$, as being the arrow with source $t(\vec{v})$ and target $s(\vec{v})$ (assumed to be in V). Then $(V, +, -)$ is a groupoid, referred to as a `vector groupoid`.

An equivalent way of expressing axioms (i) and (ii) in Definition 2.17, which is more reminiscent of the manner in which the usual group axioms are postulated, is as follows:

(i′) One has

$$a \in G \implies (a^{-1}, a),\ (a, a^{-1}) \in G^{(2)}. \tag{2.58}$$

(ii′) For every $a, b \in G$

$$(a, b) \in G^{(2)} \implies a * b * b^{-1} = a \quad \text{and} \quad a^{-1} * a * b = b. \tag{2.59}$$

For an arbitrary subset A of a groupoid $(G, *, (\cdot)^{-1})$ define

$$A^{-1} := \{a^{-1} : a \in A\}, \tag{2.60}$$

and call $A \subseteq G$ `symmetric` if $A = A^{-1}$. Several useful identities valid in a groupoid $(G, *, (\cdot)^{-1})$ that are consequences of axioms (i′) and (ii′) are collected in the next lemma.

Lemma 2.19. *Let $(G, *, (\cdot)^{-1})$ be a groupoid. Then the following relations hold:*

$$(a, b) \in G^{(2)} \iff a, b \in G \text{ and } a^{-1} * a = b * b^{-1}, \tag{2.61}$$

$$(a, c), (b, c) \in G^{(2)} \text{ and } a * c = b * c \implies a = b, \tag{2.62}$$

$$(c, a), (c, b) \in G^{(2)} \textit{ and } c * a = c * b \implies a = b, \tag{2.63}$$

$$(a^{-1})^{-1} = a, \quad \forall\, a \in G, \tag{2.64}$$

$$(a, b) \in G^{(2)} \implies (b^{-1}, a^{-1}) \in G^{(2)} \textit{ and } (a * b)^{-1} = b^{-1} * a^{-1}, \tag{2.65}$$

$$(a, b^{-1}) \in G^{(2)} \implies a * b^{-1} * b = a \textit{ and } a * a^{-1} * b = b, \tag{2.66}$$

$$(a, c),\ (c^{-1}, b) \in G^{(2)} \implies (a, b),\ (a * c, c^{-1} * b) \in G^{(2)} \textit{ and }$$
$$a * b = (a * c) * (c^{-1} * b). \tag{2.67}$$

Proof. Implications (2.62) and (2.63) follow directly from the cancellation properties in (2.59). To justify (2.64), write the first identity in (2.59) with $b = a^{-1}$ to obtain $a * a^{-1} * (a^{-1})^{-1} = a$. Multiplying both sides of this identity to the left by a^{-1} and using the second identity in (2.59) written for $b = a^{-1}$ we obtain that $a^{-1} * (a^{-1})^{-1} = a^{-1} * a$. This and (2.63) allow us to conclude that (2.64) holds.

To justify (2.65) for any $(a, b) \in G^{(2)}$, we write, based on the cancellation property (2.59), that $(a * b)^{-1} * a = \big((a * b)^{-1} * a\big) * (b * b^{-1}) = \big((a * b)^{-1} * (a * b)\big) * b^{-1} = b^{-1}$. This shows that $(b^{-1}, a^{-1}) \in G^{(2)}$, and, once again appealing to (2.59), we obtain $(a * b)^{-1} = (a * b)^{-1} * a * a^{-1} = b^{-1} * a^{-1}$, as desired. Going further, (2.66) is a consequence of (2.59) and (2.64).

Let us now check the equivalence in (2.61). In one direction, if $(a, b) \in G^{(2)}$, then $c := a * b \in G$ is well defined and $a = c * b^{-1}$ by (2.59). Based on this, (2.64), and the cancellation property (2.59), we may then write $a^{-1} * a = (b^{-1})^{-1} * c^{-1} * c * b^{-1} = b * b^{-1}$, as desired. Concerning the opposite direction, assume that $a, b \in G$ are such that $a^{-1} * a = b * b^{-1}$. Since, by (2.10), $(a, a^{-1} * a) \in G^{(2)}$, it follows that $(a, b * b^{-1}) \in G^{(2)}$. Given that $(b * b^{-1}, b) \in G^{(2)}$ and $b * b^{-1} * b = b$ by (2.66), this further implies that $(a, b) \in G^{(2)}$, by virtue of (2.10).

Finally, (2.67) follows from (2.59) and the associativity of the partial multiplication on G. □

An equivalent way of introducing the concept of groupoid that makes no reference to the notion of semigroupoid is presented below.

Remark 2.20. A groupoid is a triplet $(G, *, (\cdot)^{-1})$, where G is a nonempty set, $*$ is a partially defined binary operation on G, and $(\cdot)^{-1} : G \to G$ is a function such that, with $G^{(2)} := \big\{(a, b) \in G \times G : a * b \text{ is meaningfully defined}\big\}$, the following axioms are valid:

(i) `[Associativity]` If (a, b) and (b, c) are in $G^{(2)}$, then so are $(a * b, c)$ and $(a, b * c)$, and $(a * b) * c = a * (b * c)$.
(ii) `[Cancellation]` For every $a \in G$ it follows that $(a, a^{-1}), (a^{-1}, a) \in G^{(2)}$, and if $(a, b) \in G^{(2)}$, then $a^{-1} * (a * b) = b$ and $(a * b) * b^{-1} = a$.

Moving on, given an arbitrary nonempty set X, call a subset $\mathscr{R}$ of $X \times X$ the `graph of an equivalence relation` on X provided

$$\operatorname{diag}(X) := \{(x,x) : x \in X\} \subseteq \mathscr{R}, \quad (x,y) \in \mathscr{R} \Rightarrow (y,x) \in \mathscr{R}, \\ \text{and} \quad (x,z),(z,y) \in \mathscr{R} \Rightarrow (x,y) \in \mathscr{R}. \tag{2.68}$$

Hence, if $(G, *, (\cdot)^{-1})$ is a groupoid and $\mathcal{G}^{\mathrm{R}}, \mathcal{G}^{\mathrm{L}} \subseteq G \times G$ are defined as

$$\mathcal{G}^{\mathrm{R}} := \{(a,b) \in G \times G : (a, b^{-1}) \in G^{(2)}\}, \tag{2.69}$$

$$\mathcal{G}^{\mathrm{L}} := \{(a,b) \in G \times G : (a^{-1}, b) \in G^{(2)}\}, \tag{2.70}$$

then it follows from (2.64)–(2.67) that

$$\text{each of the sets } \mathcal{G}^{\mathrm{R}} \text{ and } \mathcal{G}^{\mathrm{L}} \text{ is the graph of an equivalence relation on } G. \tag{2.71}$$

Finally, define the `unit space` of the groupoid G as the collection of idempotent elements in G, i.e., the set $G^{(0)}$ from (2.16). Unlike groups, the unit space can be quite large, given that the groupoid multiplication is only partly defined. In fact, a groupoid G is actually a group if and only if its unit space $G^{(0)}$ is a singleton. Combining this with identity (6) in Proposition 2.21, it follows that

$$\text{if } G \text{ is a groupoid, then} \quad G^{(2)} = G \times G \iff G \text{ is a group.} \tag{2.72}$$

Some of the properties of $G^{(0)}$ that are of relevance for our work are singled out next.

Proposition 2.21. *Let $(G, *, (\cdot)^{-1})$ be a groupoid, and let $G^{(0)}$ be as in (2.16). Then the following properties hold.*

(1) $G^{(0)} := \{a * a^{-1} : a \in G\} = \{a^{-1} * a : a \in G\}$.
(2) If $a \in G^{(0)}$, then $a^{-1} = a$.
(3) If $(a,b) \in G^{(2)}$, then $a * b = a$ if and only if $b \in G^{(0)}$.
(4) If $(a,b) \in G^{(2)}$, then $a * b = b$ if and only if $a \in G^{(0)}$.
(5) For any $a, b \in G$ we have

$$(a,b) \in \mathcal{G}^{\mathrm{R}} \text{ and } a * b^{-1} \in G^{(0)} \Leftrightarrow a = b \Leftrightarrow (a,b) \in \mathcal{G}^{\mathrm{L}} \text{ and } a^{-1} * b \in G^{(0)}. \tag{2.73}$$

(6) $(G^{(0)} \times G^{(0)}) \cap G^{(2)} = \operatorname{diag}(G^{(0)})$.

Proof. If $a \in G^{(0)}$, then $(a,a) \in G^{(2)}$ and $a * a = a$. Hence, if we use (2.59), then we may write $a = a * a * a^{-1} = a * a^{-1}$ and $a = a^{-1} * a * a = a^{-1} * a$. This proves that $G^{(0)}$ is included in the two sets described in (1). Suppose now that $a \in G$ is arbitrary, and let $b := a * a^{-1}$. Then (2.67) implies $(b,b) \in G^{(2)}$ and $b * b = a * a^{-1} = b$, thus $b \in G^{(0)}$. Similarly, if $c := a^{-1} * a$, then $c \in G^{(0)}$. This completes the proof of (1). The statements in (2)–(5) can be easily seen from (1) and definitions. One inclusion corresponding to the equality in (6) is immediate from the definition of $G^{(0)}$, while the other follows from (3) and (4). □

Remark 2.22. Let $(G, *)$ be a groupoid. By naturally regarding this as a semigroupoid, we may associate to it the semigroup $(S, \circ)$ as in the proof of Theorem 2.14. Employing notation introduced on that occasion, we may then define

$$\big[\langle a_1, \dots, a_N\rangle\big]^{-1} := \big[\langle a_N^{-1}, \dots, a_1^{-1}\rangle\big], \qquad \forall\, a_1, \dots, a_N \in G. \tag{2.74}$$

Thanks to (2.66), it follows that this definition is unambiguous. Also,

$$\big([w]^{-1}\big)^{-1} = [w] \ \text{ and } \ \big([w'] \circ [w'']\big)^{-1} = [w'']^{-1} \circ [w']^{-1}, \qquad \forall\, [w], [w'], [w''] \in S, \tag{2.75}$$

by Lemma 2.19. Next, call $[w'], [w''] \in S$ `linkable` provided the pair consisting of the last letter in the word w' and the first letter in the word w'' is composable in $(G, *)$. Relying on Lemma 2.15 it is not difficult to see that this definition is unambiguous. Recall H from (2.41). Then, in addition to (2.75), the operation defined in (2.74) satisfies, for any $[w], [w'] \in S$,

$$[w]^{-1} \circ [w] \circ [w'] = [w'] \iff [w] \in H \ \text{ and } \ [w], [w'] \text{ are linkable.} \tag{2.76}$$

We continue by adopting the following convention.

Convention 2.23. *Given that any groupoid is a semigroupoid, it is agreed that the notions of homomorphism and isomorphism for groupoids will retain the same significance as in the previous setting (cf. Definition 2.9).*

An example of a groupoid homomorphism that will play a significant role later is described in the next remark.

Remark 2.24. Let $(G, *, (\cdot)^{-1})$ be a groupoid, and set $V := \{\vec{v}_a : a \in G\}$, where for each $a \in G$ the object $\vec{v}_a$ is considered a vector with source $s(\vec{v}_a) := a * a^{-1}$ and target $t(\vec{v}_a) := a^{-1} * a$. It is then possible to organize this set as a vector groupoid $(V, +, -)$ as in Example 2.18. If we now introduce the mapping

$$G \ni a \mapsto \vec{v}_a \in V, \tag{2.77}$$

then, based on (2.61), we may write

$$(a, b) \in G^{(2)} \ \Leftrightarrow\ a^{-1} * a = b * b^{-1} \ \Leftrightarrow\ t(\vec{v}_a) = s(\vec{v}_b) \ \Leftrightarrow\ (\vec{v}_a, \vec{v}_b) \in V^{(2)} \tag{2.78}$$

and

$$\vec{v}_{a*b} = (a * a^{-1}, b^{-1} * b) = \vec{v}_a + \vec{v}_b, \quad \forall\, (a, b) \in G^{(2)}. \tag{2.79}$$

As a consequence, the mapping (2.77) is a groupoid homomorphism.

Remark 2.25. Let $(G, *, (\cdot)^{-1})$ be a groupoid. Remark 2.24 suggests that it is natural to consider the `source` and `target` mappings s, t defined by

$$s, t : G \longrightarrow G, \quad s(a) := a * a^{-1}, \quad t(a) := a^{-1} * a, \quad \forall\, a \in G. \tag{2.80}$$

Then some of the basic algebraic properties enjoyed by the groupoid G naturally translate into properties of the source and target mappings just defined. More precisely, the following statements are true.

(1) For every $a, b \in G$ we have $(a, b) \in G^{(2)}$ if and only if $t(a) = s(b)$.
(2) If $(a, b) \in G^{(2)}$, then $s(a * b) = s(a)$ and $t(a * b) = t(b)$.
(3) If $a \in G$, then $s(a) = t(a^{-1})$ and $t(a) = s(a^{-1})$.
(4) If $a \in G$, then $s(a), t(a) \in G^{(0)}$. In addition, if $a \in G^{(0)}$, then $s(a) = a$ and $t(a) = a$.
(5) If $a \in G$, then $(s(a), a), (a, t(a)) \in G^{(2)}$, $s(a) * a = a$, and $a * t(a) = a$.

Indeed, (1) is a consequence of (2.61), the equalities in (2) follow from (2.58), (2.11), and (2.59), while the statement in (3) is based on (2.64). The first part of (4) follows from (1) in Proposition 2.21, whereas the second part of (4) is seen by combining (2) in Proposition 2.21 with (2.16). Finally, (5) is a consequence of (2.58) and (2.66).

An equivalent characterization of groupoids, in the spirit of the original definition of Brandt [21] and which avoids direct references to semigroupoids, inverses, and cancellation properties, is as follows.

Proposition 2.26. *Let G be a nonempty set equipped with a partially defined binary operation "$*$" and, as before, denote by $G^{(2)}$ the collection of composable pairs in G. Then there exists an inversion $(\cdot)^{-1}$ on G such that $(G, *, (\cdot)^{-1})$ becomes a groupoid in the sense of Definition 2.17 if and only if the following five axioms are satisfied:*

(1) For every $a \in G$ there exist unique elements $a_1, a_2 \in G$ such that $(a_1, a), (a, a_2)$ belong to $G^{(2)}$ and $a_1 * a = a = a * a_2$.
(2) If $(a, b) \in G^{(2)}$ and $a * b = a$, or if $(b, a) \in G^{(2)}$ and $b * a = a$, then $(b, b) \in G^{(2)}$ and $b * b = b$.
(3) For every $a, b \in G$ there holds $(a, b) \in G^{(2)}$ if and only if there exists $c \in G$ such that $(a, c), (c, b) \in G^{(2)}$ and $a * c = a$, $c * b = b$.
(4) If $a, b, c \in G$ are such that $(a, b), (b, c) \in G^{(2)}$, then $(a*b, c), (a, b*c) \in G^{(2)}$ and $(a * b) * c = a * (b * c)$.
(5) If $a, \tilde{a}, \tilde{\tilde{a}} \in G$ are such that $(\tilde{a}, a), (a, \tilde{\tilde{a}}) \in G^{(2)}$ and $\tilde{a} * a = a = a * \tilde{\tilde{a}}$, then there exists $b \in G$ such that $(a, b), (b, a) \in G^{(2)}$ and $a * b = \tilde{a}$ and $b * a = \tilde{\tilde{a}}$.

Proof. If there exists an inversion $(\cdot)^{-1}$ on G such that $(G, *, (\cdot)^{-1})$ becomes a groupoid in the sense of Definition 2.17, then the five conditions in the statement of the proposition are readily verified based on the algebraic properties of groupoids discussed earlier. The crux of the matter is dealing with the implication in the converse direction. To this end, assume that properties (1)–(5) listed in the statement hold, and introduce $G^{(0)}$ as in (2.16). Next, for each $a \in G$ define $\sigma(a) := a_1$, $\tau(a) := a_2$, where $a_1, a_2 \in G$ are (uniquely) determined by $a \in G$ as in axiom (1). Translating axioms (1)–(3) in the language of the mappings $\sigma, \tau : G \to G$ then yields the following properties:

$$\sigma(a), \tau(a) \in G^{(0)}, \quad \forall\, a \in G, \tag{2.81}$$

$$\sigma(a) = \tau(a) = a, \quad \forall\, a \in G^{(0)}, \tag{2.82}$$

$$(\sigma(a), a), (a, \tau(a)) \in G^{(2)} \text{ and } \sigma(a) * a = a = a * \tau(a), \quad \forall\, a \in G, \tag{2.83}$$

$$(a, b) \in G^{(2)} \text{ if and only if } a, b \in G \text{ and } \sigma(b) = \tau(a). \tag{2.84}$$

Note that, in concert, (2.81) and (2.82) entail that the mappings σ, τ are involutive, i.e.,

$$\sigma(\sigma(a)) = \sigma(a) \;\text{ and }\; \tau(\tau(a)) = \tau(a) \;\text{ for every }\; a \in G. \tag{2.85}$$

In addition, based on axiom (1) it may also be verified that

$$\sigma(a * b) = \sigma(a) \;\text{ and }\; \tau(a * b) = \tau(b) \;\text{ for every }\; (a, b) \in G^{(2)}. \tag{2.86}$$

Going further, observe that axioms (1) and (5) tell us that

$$\begin{aligned} &\text{for every } a \in G \text{ there exists some } b \in G \text{ such that} \\ &(a, b), (b, a) \in G^{(2)} \text{and } \sigma(a) = a * b, \tau(a) = b * a. \end{aligned} \tag{2.87}$$

We claim that $b \in G$ in (2.87) is uniquely determined by the choice of $a \in G$. To prove this claim, assume that $a \in G$ has been fixed and that $b, c \in G$ are such that

$$\begin{aligned} &(a, b), (b, a) \in G^{(2)} \text{ and } \sigma(a) = a * b, \tau(a) = b * a, \\ &(a, c), (c, a) \in G^{(2)} \text{ and } \sigma(a) = a * c, \tau(a) = c * a. \end{aligned} \tag{2.88}$$

Then $\tau(b) = \tau(a * b) = \tau(\sigma(a)) = \sigma(a)$, by (2.86), (2.88) and (2.81), (2.82). In a similar manner, we also obtain that $\tau(c) = \sigma(a)$. Keeping these, as well as (2.88), in mind we may then write

$$\begin{aligned} c &= c * \tau(c) = c * \sigma(a) = c * (a * b) = (c * a) * b \\ &= \tau(a) * b = (b * a) * b = b * (a * b) = b * \sigma(a) = b * \tau(b) = b, \end{aligned} \tag{2.89}$$

where we have also made repeated use of the associativity axiom (4) and property (2.83). This completes the proof of the claim made just after (2.87). Granted this, it follows that it is unambiguous to define the inverse of $a \in G$ as $a^{-1} := b$ if $b \in G$ is related to $a \in G$ as in (2.87). In this notation, (2.87) then reads

$$(a, a^{-1}), (a^{-1}, a) \in G^{(2)} \;\text{ and }\; \sigma(a) = a * a^{-1}, \;\; \tau(a) = a^{-1} * a, \quad \forall\, a \in G. \tag{2.90}$$

There remains to show that $(G, *, (\cdot)^{-1})$ is a groupoid, and a convenient way to do so is to check properties (i) and (ii) listed in Remark 2.20. In fact, since the associativity axiom (i) coincides with the current axiom (4), it is only the cancellation axiom (ii) that concerns us. As a way to justify this, we make the following claim:

$$\text{for every } a \in G^{(0)} \text{ and } b \in G \text{ such that } (a,b) \in G^{(2)} \text{ one has } a * b = b. \tag{2.91}$$

To see that this is the case, note that if $a \in G^{(0)}$ and $b \in G$ are such that $(a,b) \in G^{(2)}$, then $\tau(a) = \sigma(b)$ by (2.84). In turn, based on this and (2.82), we conclude that $a = \sigma(b)$. Hence, $a * b = \sigma(b) * b = b$, by (2.83), completing the proof of (2.91). With this in hand, given $(a,b) \in G^{(2)}$, we may write

$$a^{-1} * a * b = \tau(a) * b = b, \tag{2.92}$$

where we have also made use of (2.90) and (2.81). This completes the verification of the first identity in part (ii) of Remark 2.20. The second identity in part (ii) of Remark 2.20 is checked in a similar fashion; hence the conclusion is that $(G, *, (\cdot)^{-1})$ is a groupoid. □

In the context of Remark 2.24, it may be verified directly that $-\vec{v}_a = \vec{v}_{a^{-1}}$ for every $a \in G$. As proved below, this is no accident since it turns out that any groupoid homomorphism intertwines inverses. Based on this, it is then possible to show that a groupoid homomorphism is an isomorphism if and only if it is a bijection. Concretely, we have the following proposition.

Proposition 2.27. *Suppose* $\big(G, *, (\cdot)^{-1}\big)$ *and* $\big(H, \circ, [\cdot]^{-1}\big)$ *are two groupoids, and assume that* $\phi \in \mathrm{Hom}\,(G, H)$. *Then the following assertions are valid.*

(1) $[\phi(a)]^{-1} = \phi\big(a^{-1}\big)$ *for every* $a \in G$.
(2) ϕ *is a groupoid isomorphism if and only if* ϕ *is a bijection.*

Proof. To prove (1), fix $a \in G$ arbitrary. Then, since by (2.58) we have $(a, a^{-1}) \in G^{(2)}$, it follows that $(\phi(a), \phi(a^{-1})) \in H^{(2)}$ and $\phi(a * a^{-1}) = \phi(a) \circ \phi(a^{-1})$. Note that by (2.10), this entails $([\phi(a)]^{-1}, \phi(a * a^{-1})) \in H^{(2)}$. Composing both sides of the last identity with $[\phi(a)]^{-1}$ we further obtain, on account of (2.59),

$$[\phi(a)]^{-1} \circ \phi(a * a^{-1}) = [\phi(a)]^{-1} \circ \phi(a) \circ \phi(a^{-1}) = \phi(a^{-1}). \tag{2.93}$$

On the other hand, since $a * a^{-1} \in G^{(0)}$, Proposition 2.11 gives that $\phi(a * a^{-1}) \in H^{(0)}$, which, in combination with part (3) of Proposition 2.21, further yields that $[\phi(a)]^{-1} \circ \phi(a * a^{-1}) = [\phi(a)]^{-1}$. Now the identity stated in (1) follows from this and (2.93).

It is immediate from the definition that if ϕ is an isomorphism of groupoids, then ϕ is bijective. Conversely, suppose that the homomorphism ϕ is bijective. To conclude that ϕ is an isomorphism, we need to show that ϕ^{-1} is a homomorphism. Let $(u, v) \in H^{(2)}$. Then there exist $a, b \in G$ such that $u = \phi(a)$ and $v = \phi(b)$. By (2.61), we have $[\phi(a)]^{-1} \circ \phi(a) = \phi(b) \circ [\phi(b)]^{-1}$. On the other hand, $(a^{-1}, a), (b, b^{-1}) \in G^{(2)}$, and since ϕ is a homomorphism, we have

$\phi(a^{-1} * a) = [\phi(a)]^{-1} \circ \phi(a)$ and $\phi(b * b^{-1}) = \phi(b) \circ [\phi(b)]^{-1}$. Consequently, $\phi(a^{-1} * a) = \phi(b * b^{-1})$. The latter, in combination with the fact that ϕ is injective, implies $a^{-1} * a = b * b^{-1}$. Returning with this to (2.61) it follows that $(a, b) \in G^{(2)}$. Furthermore, since ϕ is a homomorphism, $\phi(a * b) = \phi(a) \circ \phi(b) = u \circ v$. Applying ϕ^{-1} to this identity we obtain $\phi^{-1}(u \circ v) = a * b = \phi^{-1}(u) * \phi^{-1}(v)$. This completes the proof of the fact that ϕ^{-1} is a homomorphism and, with it, the proof of part (2). □

Remark 2.28. Suppose $\big(G, *, (\cdot)^{-1}\big)$ and $\big(H, \circ, [\cdot]^{-1}\big)$ are groupoids and assume $\phi : G \to H$ is a homomorphism. Then

$$\phi(G^{(0)}) = H^{(0)} \cap \operatorname{Im}\phi, \tag{2.94}$$

where $\operatorname{Im}\phi := \phi(G)$ denotes the image of the function ϕ. In particular, we have $\phi(G^{(0)}) \subseteq H^{(0)}$ and $\phi(G^{(0)}) = H^{(0)}$ if and only if $H^{(0)} \subseteq \operatorname{Im}\phi$.

To see why the equality in (2.94) holds, first note that, based on Proposition 2.11, we have $\phi(G^{(0)}) \subseteq H^{(0)}$, hence $\phi(G^{(0)}) \subseteq H^{(0)} \cap \operatorname{Im}\phi$. Conversely, if $u \in H^{(0)} \cap \operatorname{Im}\phi$, then there exists $a \in G$ such that $u = \phi(a)$ and $u \circ u = u$. Consequently,

$$u = u \circ [u]^{-1} = \phi(a) \circ [\phi(a)]^{-1} = \phi(a) \circ \phi(a^{-1}) = \phi(a * a^{-1}), \tag{2.95}$$

and since $a * a^{-1} \in G^{(0)}$, it follows that $u \in \phi(G^{(0)})$. Thus, $H^{(0)} \cap \operatorname{Im}\phi \subseteq \phi(G^{(0)})$, so that, ultimately, (2.94) holds.

Below we present several examples of groupoids. The aim here is twofold. On the one hand, such a list underscores the thesis that groupoids occur naturally and frequently in practical situations. On the other hand, this is intended to help the reader better understand the salient features of the abstract definition of groupoids. The first two examples illustrate the fact that the fundamental notions of group and set occur as particular, extreme cases of groupoids.

Example 2.29 (Groups). Any group can naturally be regarded as a groupoid.

Example 2.30 (Sets). Any set X carries a groupoid structure, namely, $(X, *, (\cdot)^{-1})$, where $X^{(2)} := \operatorname{diag}(X)$ and $x * x := x$, $x^{-1} := x$ for every $x \in X$.

Example 2.31 (Groupoid induced by an equivalence relation). Let X be an arbitrary, nonempty set, and let $G \subseteq X \times X$ be the graph of an equivalence relation on X. Then the triplet $(G, *, (\cdot)^{-1})$, where

$$\begin{aligned} &G^{(2)} := \big\{((x,y),(z,w)) \in G \times G : \ y = z\big\} \quad \text{and} \\ &(x,y) * (y,w) := (x,w), \quad \forall\, ((x,y),(y,w)) \in G^{(2)}, \end{aligned} \tag{2.96}$$

and

$$(x,y)^{-1} := (y,x), \qquad \forall\, (x,y) \in G, \tag{2.97}$$

is a groupoid. Note that in all cases, $G^{(0)} = \operatorname{diag}(X)$.

The special cases when $G := X \times X$ and $G := \operatorname{diag}(X)$ yield, respectively, the `pair groupoid` and the `diagonal groupoid` associated with the set X. The latter groupoid is isomorphic (in the sense of Definition 2.9) with the groupoid associated with the set X from Example 2.30.

It is worth pointing out that the groupoid induced by an equivalence relation on a set is isomorphic to a vector groupoid (as described in Example 2.18).

Example 2.32 (Brandt groupoids). Assume that X is an arbitrary, nonempty set and that $(H, \circ)$ is a group. Set $G := \{(x, a, y) : x, y \in X \text{ and } a \in H\}$, consider

$$G^{(2)} := \Big\{\big((x, a, z), (z, b, y)\big) : x, y, z \in X \text{ and } a, b \in H\Big\}, \tag{2.98}$$

and define

$$(x, a, z) * (z, b, y) := (x, a \circ b, y), \qquad \forall\, \big((x, a, z), (z, b, y)\big) \in G^{(2)}, \tag{2.99}$$

$$(x, a, y)^{-1} := (y, a^{-1}, x), \qquad \forall\, (x, a, y) \in G. \tag{2.100}$$

Then $(G, *, (\cdot)^{-1})$ is a groupoid. Any groupoid isomorphic (in the sense of Definition 2.9) to $(G, *, (\cdot)^{-1})$ is called a `Brandt groupoid`.

It is clear from the definitions given previously that any pair groupoid is a Brandt groupoid. Our next three examples give recipes for constructing new groupoids from older ones.

Example 2.33 (Direct sums of groupoids). If $(G, *, (\cdot)^{-1})$ and $(H, \circ, [\cdot]^{-1})$ are two given groupoids, then the semigroupoid $G \oplus H$ defined in Example 2.6 becomes a groupoid if we define $(a, b)^{-1} := ((a)^{-1}, [b]^{-1})$ for every $(a, b) \in G \times H$.

Parenthetically we note that, in the context of Example 2.32, G is isomorphic to $(X \times X) \oplus H$. The manner in which a given groupoid can be contracted via a subset of its unit space is illustrated below.

Example 2.34 (Contractions of a groupoid). Let $\big(G, *, (\cdot)^{-1}\big)$ be a groupoid, and let E be an arbitrary nonempty subset of $G^{(0)}$. Then the subset $G\big|_E$ of G, defined as

$$G\big|_E := \{a \in G : a^{-1} * a \in E,\ a * a^{-1} \in E\}, \tag{2.101}$$

can be given a natural groupoid structure by taking $\big(G\big|_E\big)^{(2)} := \big(G\big|_E \times G\big|_E\big) \cap G^{(2)}$ and by considering the restrictions to $G\big|_E$ of the groupoid operations on G. Call $\big(G\big|_E, *, (\cdot)^{-1}\big)$ the `contraction of` G `by` E.

Example 2.35 (Groupoid of composable pairs). Let $\big(G, *, (\cdot)^{-1}\big)$ be a groupoid, and recall the semigroupoid $(G^{(2)}, \circ)$ constructed as in Example 2.8. Equipped with the inversion $(a, b)^{-1} := (a * b, b^{-1})$ for each $(a, b) \in G^{(2)}$, we have that $(G^{(2)}, \circ, (\cdot)^{-1})$ becomes a groupoid.

We continue our discussion by presenting the definition of the fundamental groupoid associated with a topological space.

Example 2.36 (Fundamental groupoid). Assume that (X, τ) is a topological space. Define the fundamental groupoid $(\Pi(X), *, (\cdot)^{-1})$ by taking $\Pi(X)$ to be the set of equivalence classes (homotopy classes of continuous paths in X)

$$\Pi(X) := \{\gamma : [0,1] \to X \, : \, \gamma \text{ continuous}\} \big/ \sim, \tag{2.102}$$

where for two continuous paths γ_1, γ_2 in X we write $\gamma_1 \sim \gamma_2$ if there exists a homotopy between γ_1 and γ_2 fixing the endpoints (cf., e.g., [24, Sect. 6.2, p. 207]). Denote by $[\gamma]$ the equivalence class of γ, and define

$$(\Pi(X))^{(2)} := \big\{([\gamma_1], [\gamma_2]) \in \Pi(X) \times \Pi(X) \, : \, \gamma_1(1) = \gamma_2(0)\big\}, \tag{2.103}$$

and for each $([\gamma_1], [\gamma_2]) \in (\Pi(X))^{(2)}$ set $[\gamma_1] * [\gamma_2] := [\gamma_1 * \gamma_2]$, where $\gamma_1 * \gamma_2$ is the concatenation of γ_1 with γ_2 as in (2.3). Finally, for $[\gamma] \in \Pi(X)$ define

$$[\gamma]^{-1} := [\gamma^{-1}] \in \Pi(X), \quad \text{ where } \gamma^{-1}(t) := \gamma(1-t), \quad \forall\, t \in [0,1]. \tag{2.104}$$

Example 2.37 (Bundle of groups). Let U be an arbitrary set, and let $\{\Gamma_u\}_{u \in U}$ be a family of groups indexed by U. Consider $G := \{(u,a) : u \in U, \, a \in \Gamma_u\}$, called a bundle of groups over the set U. Furthermore, define

$$G^{(2)} := \Big\{\big((u,a),(v,b)\big) \in G \times G \, : \, u = v\Big\}, \tag{2.105}$$

and set $(u,a) * (v,b) := (u, a \circ b)$ for every $\big((u,a),(v,b)\big) \in G^{(2)}$ and $(u,a)^{-1} := (u, a^{-1})$ for every $(u,a) \in G$, where $\circ$ denotes the composition in the group Γ_u and a^{-1} is the inverse of a in Γ_u. Then $(G, *, (\cdot)^{-1})$ is a groupoid.

Example 2.38 (Transformation groupoids induced by groups). Let Γ be a group acting on a set X from the right, and, given $x \in X$ and $\alpha \in \Gamma$, denote by $x \cdot \alpha \in X$ the transformation of x by α. Set $G := X \times \Gamma$, and define

$$G^{(2)} := \Big\{\big((x,\alpha),(y,\beta)\big) \in G \times G \, : \, x \cdot \alpha = y\Big\}. \tag{2.106}$$

For every $\big((x,\alpha),(y,\beta)\big) \in G^{(2)}$ set $(x,\alpha) * (y,\beta) := (x, \alpha\beta)$, and for every $(x,\alpha) \in G$ set $(x,\alpha)^{-1} := (x \cdot \alpha, \alpha^{-1})$. Then $(G, *, (\cdot)^{-1})$ is a groupoid (the transformation groupoid determined by the action of Γ on X). Note that in such a setting, $G^{(0)} = X \times \{e\}$, where e is the neutral element in Γ.

There is an abundance of transformation groupoids induced by actions of groups on sets, generated according to the abstract scheme described in Example 2.38. Here are a couple of examples.

Example 2.39. (i) Given an arbitrary, nonempty set X, define G as the collection of triplets (ρ, ϕ, ρ'), where $\phi : X \to X$ is a bijective function and ρ, ρ' are quasidistances on X with the property that

$$\rho' = \rho \circ \phi, \quad \text{i.e.,} \quad \rho'(x, y) := \rho(\phi(x), \phi(y)) \quad \forall\, x, y \in X. \tag{2.107}$$

Two such triplets $(\rho_1, \phi_1, \rho_1')$ and $(\rho_2, \phi_2, \rho_2')$ are said to be composable if $\rho_1' = \rho_2$, in which case we define $(\rho_1, \phi_1, \rho_1') * (\rho_2, \phi_2, \rho_2') := (\rho_1, \phi_1 \circ \phi_2, \rho_2')$. Finally, for every $(\rho, \phi, \rho') \in G$ set $(\rho, \phi, \rho')^{-1} := (\rho', \phi^{-1}, \rho)$. Then $(G, *, (\cdot)^{-1})$ is a groupoid.

(ii) Assume that M is a smooth differentiable manifold, and define G as the collection of triplets (g, ϕ, g'), where g, g' are smooth Riemannian metrics on M and ϕ is a diffeomorphism of M that takes g to g' (i.e., g' is the push-forward of g into g'). Then, with $*, (\cdot)^{-1}$ defined analogously to part (i) of this example, $(G, *, (\cdot)^{-1})$ is a groupoid.

An example of a groupoid that does not fall under the scope of any of the previous constructions is the so-called Deaconu–Renault groupoid described next.

Example 2.40 (Deaconu–Renault groupoids). Let X be a compact, Hausdorff topological space, and assume that $\sigma : X \to X$ is a covering map. Define

$$G := \{(x, n, y) \in X \times \mathbb{Z} \times X : \exists\, k, \ell \in \mathbb{Z}_+ \text{ so that } n = k - \ell,\ \sigma^k x = \sigma^\ell y\}. \tag{2.108}$$

Then, if $G^{(2)} := \{((x, n, y), (w, m, z)) \in G \times G : y = w\}$ and

$$\begin{aligned} &(x, n, y) * (y, m, z) := (x, n + m, z) \quad \forall\, ((x, n, y), (y, m, z)) \in G^{(2)} \\ &\text{and} \quad (x, n, y)^{-1} := (y, -n, x), \qquad \forall\, (x, n, y) \in G, \end{aligned} \tag{2.109}$$

it follows that $(G, *, (\cdot)^{-1})$ is a groupoid.

Example 2.41 (Isomorphism groupoid of a fibered set). Let X, U be sets, and suppose $\pi : X \to U$ is a surjective function. For each $u \in U$ call the set $X_u := \pi^{-1}(\{u\})$ the fiber over u. Furthermore, define

$$\operatorname{Iso}(X, \pi, U) := \{(u, \phi, v) : u, v \in U \text{ and } \phi : X_u \to X_v \text{ bijective}\}, \tag{2.110}$$

and if $(u, \phi, v), (u', \phi', v') \in \operatorname{Iso}(X, \pi, U)$, then say that they are compatible provided $v = u'$, in which case set $(u, \phi, v) \bullet (u', \phi', v') := (u, \phi' \circ \phi, v')$. Also for each $(u, \phi, v) \in \operatorname{Iso}(X, \pi, U)$ define $(u, \phi, v)^{-1} := (v, \phi^{-1}, u)$. Then $(\operatorname{Iso}(X, \pi, U), \bullet, (\cdot)^{-1})$ is a groupoid called the isomorphism groupoid of the fibered set X.

A more general scheme for constructing groupoids that contains as particular cases both the bundle of groups from Example 2.37 and, up to isomorphisms, all the Brandt groupoids from Example 2.32 (see the comments in Remark 2.43) is described below. Indeed, as discussed later in Theorem 2.54, this constitutes the “blueprint” according to which all groupoids can be generated.

Example 2.42 (Group bundles over equivalence relations). Let X be a set, “ $\sim$” an equivalence relation on X, and $\Gamma := \{\Gamma_\xi\}_{\xi \in X/_\sim}$ an arbitrary family of groups indexed by the equivalence classes induced by $\sim$ on X. Consider the set

$$H := \big\{(x, g, y) : \exists\, \xi \in X/_\sim \text{ such that } x, y \in \xi \text{ and } g \in \Gamma_\xi\big\}, \tag{2.111}$$

and introduce

$$H^{(2)} := \big\{\big((x, g, y), (z, h, w)\big) \in H \times H : \ y = z\big\}. \tag{2.112}$$

Also, define

$$(x, g, y) * (z, h, w) := (x, g \cdot h, w), \quad \forall\, \big((x, g, y), (z, h, w)\big) \in H^{(2)}, \tag{2.113}$$

$$(x, g, y)^{-1} := (y, g^{-1}, x), \quad \forall\, (x, g, y) \in H, \tag{2.114}$$

where, if ξ denotes the equivalence class of x, y, then $g \cdot h$ stands for the product of $g, h \in \Gamma_\xi$ and $g^{-1} \in \Gamma_\xi$ denotes the inverse of g in Γ_ξ. Then $(H, *, (\cdot)^{-1})$ is a groupoid to be denoted by $\big[X, \sim, \{\Gamma_\xi\}_{\xi \in X/_\sim}\big]$.

Remark 2.43. In the context of Example 2.42, when the equivalence relation $\sim$ is such that its graph is $X \times X$, in which case there exists a unique class of equivalence and thus the family of groups reduces to one group Γ, the groupoid $\big[X, \sim, \{\Gamma_\xi\}_\xi\big]$ is the Brandt groupoid $X \times \Gamma \times X$. On the other hand, if the equivalence relation $\sim$ is such that its graph is $\operatorname{diag}(X)$, then $\big[X, \sim, \{\Gamma_\xi\}_\xi\big]$ becomes the bundle of the family of groups $\{\Gamma_\xi\}_\xi$ (which can be thought of as being indexed by the elements of X).

In summary, the notion of groupoid generalizes that of set, group, equivalence relation, transformation group, etc. In this vein, let us note that, remarkably, a groupoid contains many (generally speaking) different groups, in a natural fashion. Specifically, the following result holds.

Proposition 2.44. *Let $\big(G, *, (\cdot)^{-1}\big)$ be a groupoid, and for each $a \in G^{(0)}$ define*

$$S_a := G\big|_{\{a\}} = \{b \in G : \ b^{-1} * b = a = b * b^{-1}\}. \tag{2.115}$$

Then, with the groupoid operations inherited from G, $(S_a, *, (\cdot)^{-1})$ becomes a group whose neutral element is $a \in S_a$.

Proof. This is clear from definitions, (2.61), and Proposition 2.21. □

We continue by discussing the concept of subgroupoid.

Definition 2.45. Let $\big(G, *, (\cdot)^{-1}\big)$ be a groupoid. Call H a `subgroupoid` of G if the following conditions are satisfied:

(i) $G^{(0)} \subseteq H \subseteq G$.
(ii) For every $(a, b) \in (H \times H) \cap G^{(2)}$ there holds $a * b \in H$.
(iii) For every $a \in H$ there holds $a^{-1} \in H$.

Remark 2.46. (1) Note that if H is a subgroupoid of groupoid G, then $\big(H, *, (\cdot)^{-1}\big)$, where $H^{(2)} := (H \times H) \cap G^{(2)}$, becomes itself a groupoid, and $H^{(0)} = G^{(0)}$.
(2) Let $\big(G, *, (\cdot)^{-1}\big)$ be a groupoid, and let E be an arbitrary nonempty subset of $G^{(0)}$. Then the groupoid $G\big|_E$, the contraction of G by E (cf. Example 2.34), is a subgroupoid of G if and only if $E = G^{(0)}$ (in which case $G\big|_E$ actually becomes G itself).

Our next result catalogs all subgroupoids of a given pair groupoid.

Proposition 2.47. *Let X be an arbitrary set, and assume that $G \subseteq X \times X$. Then*

$$\begin{array}{l} G \text{ is a subgroupoid of} \\ \text{the pair groupoid } X \times X \end{array} \iff \begin{array}{l} G \text{ is the graph of} \\ \text{an equivalence relation on } X. \end{array} \tag{2.116}$$

Proof. The right-to-left implication follows from Example 2.31 and Definition 2.45. Conversely, suppose now that G is a subgroupoid of the pair groupoid $X \times X$. Since $(X \times X)^{(0)} = \operatorname{diag}(X)$, it follows that $\operatorname{diag}(X) \subseteq G \subseteq X \times X$. Also, if $(a, b) \in G$, then $(b, a) = (a, b)^{-1} \in G$ by (iii) in Definition 2.45. Finally, if $(a, b), (b, c) \in G$ then $(a, c) = (a, b) * (b, c) \in G$ by (ii) in Definition 2.45. Hence, G is the graph of an equivalence relation on X. □

We continue by identifying a distinguished subclass of homomorphisms between two given groupoids whose relevance will become more apparent in Proposition 2.50 below.

Definition 2.48. Let $\big(G, *, (\cdot)^{-1}\big)$ and $\big(H, \circ, [\cdot]^{-1}\big)$ be two groupoids, and assume that $\phi \in \operatorname{Hom}(G, H)$. Call ϕ a `tight homomorphism` provided the function $\phi : G^{(0)} \to H^{(0)}$ (which, by Remark 2.28, is well defined) is a bijection.

Remark 2.49. (i) If G_1, G_2, G_3 are groupoids, and if the functions $\phi_1 : G_1 \to G_2$ and $\phi_2 : G_2 \to G_3$ are tight homomorphisms, then $\phi_2 \circ \phi_1 : G_1 \to G_3$ is a tight homomorphism as well.
(ii) Any groupoid isomorphism is a tight homomorphism.

Proposition 2.50. *Let $\big(G, *, (\cdot)^{-1}\big)$ and $\big(H, \circ, [\cdot]^{-1}\big)$ be two groupoids, and assume that $\phi \in \operatorname{Hom}(G, H)$. Then the following claims are true.*

(i) A necessary condition for $\operatorname{Im}\phi$ to be a subgroupoid of H is that the function $\phi : G^{(0)} \to H^{(0)}$ must be surjective.
(ii) If ϕ is a tight homomorphism, then ϕ maps any subgroupoid of G into a subgroupoid of H. In particular, $\operatorname{Im}\phi$ is a subgroupoid of H and $(\operatorname{Im}\phi)^{(0)} = H^{(0)}$.

(iii) The preimage under the homomorphism ϕ of any subgroupoid of H is a subgroupoid of G.

Proof. If $\operatorname{Im}\phi$ is a subgroupoid of H, then, by Definition 2.45, $H^{(0)} \subseteq \operatorname{Im}\phi$, hence further, by Remark 2.28, $\phi(G^{(0)}) = H^{(0)}$, i.e., the function $\phi : G^{(0)} \to H^{(0)}$ is surjective. This proves (i).

To prove claim (ii), assume that $\phi : G \to H$ is a tight homomorphism and fix an arbitrary subgroupoid K of G. Then $G^{(0)} \subseteq K$ and $\phi(G^{(0)}) = H^{(0)}$, which in concert imply that $H^{(0)} \subseteq \phi(K) \subseteq H$. This proves that the analog of (i) (for the present situation) in Definition 2.45 holds. Moving on, if $(u, v) \in (\phi(K) \times \phi(K)) \cap H^{(2)}$, then there exist $a, b \in K$ such that $(\phi(a), \phi(b)) \in H^{(2)}$ and $u = \phi(a)$, $v = \phi(b)$. Making use of (2.61) we can write

$$\begin{aligned}\big(\phi(a), \phi(b)\big) \in H^{(2)} &\Leftrightarrow [\phi(a)]^{-1} \circ \phi(a) = \phi(b) \circ [\phi(b)]^{-1} \\ &\Leftrightarrow \phi(a^{-1}) \circ \phi(a) = \phi(b) \circ \phi(b^{-1}) \\ &\Leftrightarrow \phi(a^{-1} * a) = \phi(b * b^{-1}) \Leftrightarrow a^{-1} * a = b * b^{-1}\end{aligned} \tag{2.117}$$

since $a^{-1} * a, b * b^{-1} \in G^{(0)}$ and $\phi : G^{(0)} \to H^{(0)}$ is injective. Consequently, $(a, b) \in G^{(2)}$ [by (2.61)]. Given that ϕ is a homomorphism, this entails $(\phi(a), \phi(b)) \in H^{(2)}$ and $u \circ v = \phi(a) \circ \phi(b) = \phi(a * b) \in \phi(K)$ since $a * b \in K$, given that K is a subgroupoid of G. This shows that $\phi(K)$ satisfies the natural analog, for the present situation, of condition (ii) in Definition 2.45. Finally, as far as condition (iii) in Definition 2.45 is concerned, observe that, whenever $u \in \phi(K)$, there exists $a \in K$ such that $u = \phi(a)$; hence, by Proposition 2.27, $u^{-1} = [\phi(a)]^{-1} = \phi(a^{-1}) \in \phi(K)$ since $a^{-1} \in K$ given that K is a subgroupoid of G. Thus, $\phi(K)$ is a subgroupoid of H. The very last claim in (ii) follows from what we have proved so far and Remark 2.46. This completes the proof of claim (ii).

There remains to justify claim (iii). To see this, assume that K is a subgroupoid of H and denote by $\phi^{-1}(K)$ its preimage under the mapping ϕ. Since $\phi(G^{(0)}) \subseteq H^{(0)} \subseteq K$, it follows that $G^{(0)} \subseteq \phi^{-1}(K)$, hence $G^{(0)} \subseteq \phi^{-1}(K) \subseteq G$. Next, if $a, b \in \phi^{-1}(K)$ are such that $(a, b) \in G^{(2)}$, then, since ϕ is a homomorphism, we have that, on the one hand, $(\phi(a), \phi(b)) \in H^{(2)}$ and $\phi(a * b) = \phi(a) * \phi(b)$ and, on the other hand, the fact that $\phi(a), \phi(b) \in K$, $(\phi(a), \phi(b)) \in H^{(2)}$, and K is a subgroupoid of H entails $\phi(a) * \phi(b) \in K$. All things considered, we arrive at the conclusion that $\phi(a * b) \in K$ or, equivalently, $a * b \in \phi^{-1}(K)$. Finally, if $a \in \phi^{-1}(K)$, then $\phi(a) \in K$, which, given that K is a subgroupoid of H, implies $[\phi(a)]^{-1} \in K$, or $\phi(a^{-1}) \in K$ by assertion (1) in Proposition 2.27. Hence, $a^{-1} \in \phi^{-1}(K)$. Collectively, these arguments prove that $\phi^{-1}(K)$ is a subgroupoid of G. This concludes the proof of *(iii)* and, with it, the proof of the proposition. □

A basic example of a tight groupoid homomorphism is offered by the next proposition.

Proposition 2.51. *Let* $(G, *, (\cdot)^{-1})$ *be a groupoid. Then the function*

$$\eta_G : G \longrightarrow G^{(0)} \times G^{(0)}, \quad \eta_G(a) := \big(a * a^{-1}, a^{-1} * a\big), \quad \forall\, a \in G, \tag{2.118}$$

is a tight homomorphism if $G^{(0)} \times G^{(0)}$ *is regarded as the pair groupoid on* $G^{(0)}$ *(cf. Example 2.31). Furthermore, if* H *is another groupoid and if* $\phi \in \mathrm{Hom}\,(G, H)$*, then the following intertwining identity holds:*

$$\eta_H \circ \phi = \Phi \circ \eta_G, \quad \textit{where} \quad \Phi := (\phi, \phi) : G^{(0)} \times G^{(0)} \longrightarrow H^{(0)} \times H^{(0)}. \tag{2.119}$$

Proof. That η_G is a homomorphism can be seen from an inspection of the discussion in Remark 2.24. A direct argument is as follows. By (1) in Proposition 2.21, the mapping (2.118) is well defined. Assume that $(a, b) \in G^{(2)}$, so that $a^{-1}*a = b*b^{-1}$ by (2.61). This and the definition of η_G imply $(\eta_G(a), \eta_G(b)) \in (G^{(0)} \times G^{(0)})^{(2)}$ and

$$\begin{aligned} \eta_G(a * b) &= \big(a * b * b^{-1} * a^{-1}\,,\, b^{-1} * a^{-1} * a * b\big) = \big(a * a^{-1}, b^{-1} * b\big) \\ &= \eta_G(a) * \eta_G(b). \end{aligned} \tag{2.120}$$

Thus, η_G is a homomorphism. As regards the claim that this homomorphism is tight, observe that $\eta_G(a) = (a, a)$ for every $a \in G^{(0)}$. Since, as already noted in Example 2.31, $(G^{(0)} \times G^{(0)})^{(0)} = \mathrm{diag}\,(G^{(0)})$, the desired conclusion follows.

Finally, the function Φ in (2.119) is well defined by Proposition 2.11, and the intertwining identity in (2.119) is a consequence of (2.118) and assertion (1) in Proposition 2.27. □

We now discuss an important example of a subgroupoid of a given groupoid. To present it, it is useful to recall the groups defined in (2.115).

Proposition 2.52. *Given a groupoid* $\big(G, *, (\cdot)^{-1}\big)$*, it follows that*

$$\mathrm{St}\,(G) := \bigcup_{a \in G^{(0)}} S_a = \{b \in G : b^{-1} * b = b * b^{-1}\} \tag{2.121}$$

is a subgroupoid of G*, called the* `stabilizer` *subgroupoid of* G*. In fact,*

$$\begin{aligned} &S_a = \eta_G^{-1}\big(\{(a, a)\}\big) \;\; \textit{for every} \;\; a \in G^{(0)} \quad \textit{and} \\ &\mathrm{St}\,(G) = \eta_G^{-1}\big(\mathrm{diag}\,(G^{(0)})\big), \quad \eta_G\big(\mathrm{St}\,(G)\big) = \mathrm{diag}\,(G^{(0)}), \end{aligned} \tag{2.122}$$

where η_G *is the mapping defined in* (2.118).

Proof. The identities in (2.122) are easily seen from definitions. From Proposition 2.51 we know that the mapping (2.118) is a homomorphism. Hence, since the diagonal groupoid associated with $G^{(0)}$ (see the last part in Example 2.31) is a subgroupoid of the pair groupoid $G^{(0)} \times G^{(0)}$, then, based on claim (*iii*) in Proposition 2.50 and the first identity in the second line of (2.122), we obtain that $\mathrm{St}\,(G)$ is a subgroupoid of G. □

The relevance of Example 2.41 is underscored by the next theorem.

Theorem 2.53 (Cayley's theorem). *Every groupoid is isomorphic to a subgroupoid of* $\mathrm{Iso}\,(X, \pi, U)$ *for some choice of* X, π, U.

Proof. The proof below elaborates on the brief outline given in [86]. Suppose $(G, *, (\cdot)^{-1})$ is an arbitrary groupoid, and let $\pi : G \to G^{(0)}$ be the mapping defined by $\pi(a) := a * a^{-1}$ for $a \in G$. By (1) in Proposition 2.21, the mapping π is well defined and surjective. We will prove that G is isomorphic to a subgroupoid of $\mathrm{Iso}\,(G, \pi, G^{(0)})$. To this end, consider the function

$$\Phi : G \longrightarrow \mathrm{Iso}\,(G, \pi, G^{(0)}), \quad \Phi(a) := \big(a * a^{-1}, \phi_a, a^{-1} * a\big), \quad \forall\, a \in G, \tag{2.123}$$

where, for each $a \in G$,

$$\begin{aligned} &\phi_a : \pi^{-1}(\{a * a^{-1}\}) \longrightarrow \pi^{-1}(\{a^{-1} * a\}), \\ &\phi_a(b) := a^{-1} * b, \quad \forall\, b \in \pi^{-1}(\{a * a^{-1}\}). \end{aligned} \tag{2.124}$$

First note that for each $a \in G$ the mapping ϕ_a is well defined. Indeed, for every $b \in \pi^{-1}(\{a * a^{-1}\}) = \{b \in G : b * b^{-1} = a * a^{-1}\}$ we have $(a^{-1}, b) \in G^{(2)}$ and $a^{-1} * b \in \pi^{-1}(\{a^{-1} * a\}) = \{c \in G : c * c^{-1} = a^{-1} * a\}$. Second, we claim that ϕ_a is a bijection for each $a \in G$. That ϕ_a is surjective is a consequence of the identity $\phi_a(a * c) = c$ for every $c \in \pi^{-1}(\{a^{-1} * a\})$. Also, if $b, b' \in \pi^{-1}(\{a * a^{-1}\})$ are such that $\phi_a(b) = \phi_a(b')$, that is, $a^{-1} * b = a^{-1} * b'$, then applying $a*$ to the left-hand side of the last equality yields $b = b'$, hence ϕ_a is injective. By combining what we proved so far, we can conclude that Φ as in (2.123) is well defined.

Next we show that Φ is a homomorphism. Let $(a, b) \in G^{(2)}$. Then by (2.61) we have $a^{-1} * a = b * b^{-1}$, so that $(\Phi(a), \Phi(b)) \in (\mathrm{Iso}\,(G, \pi, G^{(0)}))^{(2)}$ and

$$\Phi(a) \bullet \Phi(b) = \big(a * a^{-1}, \phi_b \circ \phi_a, b^{-1} * b\big) = \big(a * a^{-1}, \phi_{a*b}, b^{-1} * b\big) = \Phi(a * b), \tag{2.125}$$

proving that Φ is a homomorphism. In fact, this homomorphism satisfies

$$\Phi(G^{(0)}) = \big\{(a, \mathrm{id}_{\pi^{-1}(\{a\})}, a) : a \in G^{(0)}\big\} = \big(\mathrm{Iso}\,(G, \pi, G^{(0)})\big)^{(0)}, \tag{2.126}$$

where $\mathrm{id}_{\pi^{-1}(\{a\})}$ is the identity function on $\pi^{-1}(\{a\})$. In addition, Φ is injective since $\Phi(a) = \Phi(b)$ for some $a, b \in G$ implies $\phi_a = \phi_b$, which further yields $a = b$. Hence, we can apply claim (ii) in Proposition 2.50 in concert with (2.126) to conclude that $\mathrm{Im}\,\Phi$ is a subgroupoid of $\mathrm{Iso}\,(G, \pi, G^{(0)})$. The proof of the theorem is now complete. □

A drawback of Cayley's theorem (formulated in Theorem 2.53) is that there is no intrinsic description of all the subgroupoids of Iso $(G, \pi, G^{(0)})$, where G is some given groupoid. A more satisfactory answer to the question of providing a transparent scheme according to which all groupoids can be constructed is given in Theorem 2.54 below.

Theorem 2.54. *Every groupoid G is isomorphic to a group bundle over an equivalence relation groupoid, constructed as in Example 2.42 with $X := G^{(0)}$.*

Proof. Fix a groupoid $(G, *, (\cdot)^{-1})$, and recall the function in (2.118). From (1) in Theorem 2.58 we know that $\operatorname{Im} \eta_G$ is the graph of an equivalence relation on $G^{(0)}$, which will be denoted by "$\sim$." Denote by $[a]$ the equivalence class of $a \in G^{(0)}$ relative to the equivalence relation $\sim$. For each equivalence class $\xi \in G^{(0)}/_\sim$ pick an element $e_\xi \in \xi$. That such a selection is possible is ensured by the axiom of choice. Furthermore, if

$$\Gamma_\xi := \eta_G^{-1}\big(\{(e_\xi, e_\xi)\}\big), \quad \forall\, \xi \in G^{(0)}/_\sim, \tag{2.127}$$

then for each $\xi \in G^{(0)}/_\sim$ we have that $\Gamma_\xi = S_\xi$, where S_ξ is as in (2.115). Hence, by Proposition 2.44, it follows that Γ_ξ is a group for each $\xi \in G^{(0)}/_\sim$. Also, define

$$H := \big\{(a, g, b) : \exists\, \xi \in G^{(0)}/_\sim \text{ such that } x, y \in \xi \text{ and } g \in \Gamma_\xi\big\}, \tag{2.128}$$

regarded as the groupoid $\big[G^{(0)}, \sim, \{\Gamma_\xi\}_{\xi \in G^{(0)}/_\sim}\big]$ in the sense of Example 2.42. Our goal is to prove that the groupoid G is isomorphic to the groupoid H.

To this end, note that the fact that $e_{[a]} \in [a]$ for each $a \in G^{(0)}$ entails $e_{[a]} \sim a$ for each $a \in G^{(0)}$, hence $(a, e_{[a]}) \in \operatorname{Im} \eta_G$ for each $a \in G^{(0)}$, which in turn implies

$$\eta_G^{-1}\big(\{(a, e_\xi)\}\big) \neq \emptyset, \qquad \forall\, \xi \in G^{(0)}/_\sim, \ \ \forall\, a \in \xi. \tag{2.129}$$

Based on (2.129) and the axiom of choice, for every $a \in G^{(0)}$ select $\tau_a \in \eta_G^{-1}\big(\{(a, e_{[a]})\}\big)$. This selection ensures that $\eta_G(\tau_a) = (a, e_{[a]})$ for every $a \in G^{(0)}$ or, equivalently,

$$\tau_a * \tau_a^{-1} = a \quad \text{and} \quad \tau_a^{-1} * \tau_a = e_{[a]} \quad \text{for all } a \in G^{(0)}. \tag{2.130}$$

Now define

$$\Phi(a) := \Big(a * a^{-1}, (\tau_{a*a^{-1}})^{-1} * a * \tau_{a^{-1}*a},\ a^{-1} * a\Big), \quad \text{for every } a \in G. \tag{2.131}$$

Let us show that this definition of $\Phi(a)$ is meaningful for each $a \in G$. Indeed, from (2.130) we have $(a, \tau_a) \in G^{(2)}$ for every $a \in G$, so in particular, $(a * a^{-1}, \tau_{a*a^{-1}}) \in G^{(2)}$ for every $a \in G$. The latter implies $(a^{-1}, \tau_{a*a^{-1}}) \in G^{(2)}$ for every $a \in G$, hence

$$\big((\tau_{a*a^{-1}})^{-1}, a\big) \in G^{(2)} \quad \forall\, a \in G. \tag{2.132}$$

Similarly, from (2.130) we have $\big((\tau_{a^{-1}*a})^{-1}, a^{-1} * a\big) \in G^{(2)}$ for every $a \in G$, which yields

$$\big(a, \tau_{a*a^{-1}}\big) \in G^{(2)}, \quad \forall\, a \in G. \tag{2.133}$$

Combining (2.132) and (2.133), it follows that (2.131) is meaningful.

The next order of business is to check that

$$(\tau_{a*a^{-1}})^{-1} * a * \tau_{a^{-1}*a} \in \Gamma_{[a^{-1}*a]} = \eta_G^{-1}\big(\{(e_{[a^{-1}*a]}, e_{[a^{-1}*a]})\}\big), \quad \forall\, a \in G, \tag{2.134}$$

which is true if

$$\begin{aligned}(e_{[a^{-1}*a]}, e_{[a^{-1}*a]}) &= \eta_G\big((\tau_{a*a^{-1}})^{-1} * a * \tau_{a^{-1}*a}\big)\\ &= \big((\tau_{a*a^{-1}})^{-1} * \tau_{a*a^{-1}}, (\tau_{a^{-1}*a})^{-1} * \tau_{a^{-1}*a}\big), \quad \forall\, a \in G.\end{aligned} \tag{2.135}$$

The fact that (2.135) holds is a consequence of the second condition in (2.130) and the identity $[a * a^{-1}] = [a^{-1} * a]$ for all $a \in G$, given that $\eta_G(a) \in \operatorname{Im} \eta_G$ for all $a \in G$. This completes the proof of (2.134). In summary, we have proved that

$$\Phi : G \longrightarrow H \quad \text{is a well-defined mapping.} \tag{2.136}$$

In the next stage, the goal is to show that Φ is a homomorphism from the groupoid $(G, *, (\cdot)^{-1})$ into the groupoid $H = \big[G^{(0)}, \sim, \{\Gamma_\xi\}_{\xi \in G^{(0)}/\sim}\big]$. To this end, fix $(a, b) \in G^{(2)}$ arbitrary. Then $a^{-1} * a = b * b^{-1}$, from which we conclude that $(\Phi(a), \Phi(b)) \in H^{(2)}$. In addition,

$$\Phi(a) * \Phi(b) = \big(a * a^{-1}, (\tau_{a*a^{-1}})^{-1} * a * b * \tau_{b^{-1}*b}, b^{-1} * b\big) = \Phi(a * b), \tag{2.137}$$

proving that $\Phi \in \operatorname{Hom}(G, H)$. To complete the proof of the theorem, we are therefore left with showing that the function Φ, defined as in (2.136) and (2.131), is a bijection [here, part (2) of Proposition 2.27 is used]. The injectivity of the function Φ readily follows by observing that if $\Phi(a) = \Phi(b)$ for some $a, b \in G$, then $a * a^{-1} = b * b^{-1}$, $a^{-1} * a = b^{-1} * b$ and $(\tau_{a*a^{-1}})^{-1} * a * \tau_{a^{-1}*a} = (\tau_{b*b^{-1}})^{-1} * b * \tau_{b^{-1}*b}$, so that

$$\begin{aligned} a &= \tau_{a*a^{-1}} * (\tau_{a*a^{-1}})^{-1} * a * \tau_{a^{-1}*a} * (\tau_{a^{-1}*a})^{-1}\\ &= \tau_{b*b^{-1}} * (\tau_{b*b^{-1}})^{-1} * b * \tau_{b^{-1}*b} * (\tau_{b^{-1}*b})^{-1} = b,\end{aligned} \tag{2.138}$$

as desired. To prove that Φ is surjective, pick an arbitrary triplet $(a, g, b) \in H$. Then $a, b \in G^{(0)}$, $[a] = [b]$ and $g \in \Gamma_{[a]}$. Hence, $g * g^{-1} = e_{[a]} = e_{[b]} = g^{-1} * g$ and, further,

$$(e_{[a]}, g),\ (g, e_{[b]}) \in G^{(2)}. \tag{2.139}$$

In addition, the second condition in (2.130) implies

$$(\tau_a, (e_{[a]})^{-1}),\ ((e_{[b]})^{-1}, \tau_b^{-1}) \in G^{(2)}. \tag{2.140}$$

In concert, conditions (2.139) and (2.140) ensure that $\tau_a * g * \tau_b^{-1}$ is meaningfully defined in G. Moreover, we have $\Phi(\tau_a * g * \tau_b^{-1}) = (a, g, b)$, proving that Φ is surjective. This completes the proof of the bijectivity of Φ and, with it, the proof of the theorem. □

Proposition 2.55. *Let $(G, *, (\cdot)^{-1})$ be a groupoid. Then the following statements are equivalent:*

(1) The function η_G defined in (2.118) is injective.
(2) If $a \in G$ is such that $a * a^{-1} = a^{-1} * a$, then $a \in G^{(0)}$.
(3) $\mathrm{St}\,(G) = G^{(0)}$.

Proof. Suppose the function η_G is injective, and let $a \in G$ be such that $a * a^{-1} = a^{-1} * a$. Then $\eta_G(a) = \eta_G(a^{-1})$ which implies that $a = a^{-1}$. The latter yields $(a, a) \in G^{(2)}$. Furthermore,

$$\begin{aligned}\eta_G(a * a) &= (a * a * a^{-1} * a^{-1}, a^{-1} * a^{-1} * a * a)\\ &= (a * a^{-1}, a^{-1} * a) = \eta_G(a),\end{aligned} \tag{2.141}$$

hence $a * a = a$. This proves that $a \in G^{(0)}$, so (1) $\Longrightarrow$ (2).

Suppose now that (2) is true, and let $a, b \in G$ be such that $\eta_G(a) = \eta_G(b)$. Thus, $a*a^{-1} = b*b^{-1}$ and $a^{-1}*a = b^{-1}*b$, which implies $(a, b^{-1}), (b^{-1}, a) \in G^{(2)}$. Let $c := a*b^{-1}$. Then $c^{-1}*c = b*b^{-1} = a*a^{-1} = c*c^{-1}$. Given that (2) holds, this forces $c \in G^{(0)}$. Knowing that $a * b^{-1} \in G^{(0)}$, we may use (4) in Proposition 2.21 to obtain $a = a * b^{-1} * b = c * b = b$. Hence, η is injective, proving (2) $\Longrightarrow$ (1). Finally, the fact that (2) $\Longleftrightarrow$ (3) is an immediate consequence of (2.121). □

Definition 2.56. A groupoid G is called `principal` if the function η_G defined in (2.118) is injective, and it is called `transitive` if the function η_G defined in (2.118) is surjective.

For instance, if X is an arbitrary set, then

$$\text{the pair groupoid } X \times X \text{ is both principal and transitive} \tag{2.142}$$

since $\eta_{X\times X}\big((x, y)\big) = \big((x, x), (y, y)\big)$ for every $x, y \in X$. Let us also note here that if G is an arbitrary, given groupoid, then $G^{(2)}$, regarded as a groupoid in the sense explained in Example 2.35, is always principal. In addition, $G^{(2)}$ is transitive if and only if G is a group. Furthermore, from (2.119) and (ii) in Remark 2.49 we also have the following useful result:

Remark 2.57. Each of the qualities of being principal and transitive is invariant under groupoid isomorphisms.

The aim of the next theorem is to elaborate on the following issue: how typical are the following types of groupoids, listed in increasing order of generality: (1) pair groupoids, (2) groupoids induced by an equivalence relation, and (3) Brandt groupoids?

Theorem 2.58. *Let* $(G, *, (\cdot)^{-1})$ *be a groupoid, and recall the function* η_G *from* (2.118)*. Then the following statements hold:*

(1) $\operatorname{Im}\eta_G$*, the image of the function* η_G*, is the graph of an equivalence relation on* $G^{(0)}$*. Hence, in the sense of Example 2.31,* $\operatorname{Im}\eta_G$ *can be regarded as a groupoid. Denote this groupoid by* $\operatorname{Or}(G)$*, and call it the* `orbit groupoid` *associated with* G*.*
(2) $\operatorname{Or}(G)$ *is a principal groupoid with unit space* $(\operatorname{Or}(G))^{(0)} = \operatorname{diag}(G^{(0)})$*, and the function* $\eta_G : G \to \operatorname{Or}(G)$ *is a surjective homomorphism.*
(3) G *is a principal groupoid if and only if* G *is isomorphic to* $\operatorname{Or}(G)$*, if and only if* G *is isomorphic to a groupoid induced by an equivalence relation (as defined in Example 2.31).*
(4) The following conditions are equivalent:

$$G \text{ is isomorphic to a pair groupoid,} \tag{2.143}$$

$$\text{the function } \eta_G \text{ defined in (2.118) is a bijection,} \tag{2.144}$$

$$G \text{ is simultaneously principal and transitive,} \tag{2.145}$$

$$G \text{ is principal and } \operatorname{Or}(G) \text{ is the canonical pair groupoid on } G^{(0)}, \tag{2.146}$$

$$G \text{ is transitive and } \operatorname{St}(G) = G^{(0)}. \tag{2.147}$$

(5) G *is a Brandt groupoid if and only if* G *is transitive.*

Proof. By Proposition 2.51, the mapping in (2.118) is a tight homomorphism. As such, we can apply (ii) in Proposition 2.50 to conclude that $\operatorname{Im}\eta_G$ is a subgroupoid of the pair groupoid $G^{(0)} \times G^{(0)}$, and $(\operatorname{Im}\eta_G)^{(0)} = (G^{(0)} \times G^{(0)})^{(0)} = \operatorname{diag}(G^{(0)})$. Hence, by Proposition 2.47, $\operatorname{Or}(G)$ is the graph of an equivalence relation on $G^{(0)}$ and, as noted in Remark 2.46, $\operatorname{Or}(G)$ may naturally be viewed as a groupoid $(\operatorname{Or}(G), \circ, (\cdot)^{-1})$. This completes the proof of (1).

Turning our attention to (2), based on what we have proved so far, we are only left with checking that the groupoid $(\operatorname{Or}(G), \circ, (\cdot)^{-1})$ is principal. Based on Proposition 2.55, it suffices to show that if $\alpha \in \operatorname{Or}(G)$ satisfies $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha$, then $\alpha \in (\operatorname{Or}(G))^{(0)}$. Fix such an α, and pick $a \in G$ such that $\eta_G(a) = \alpha$. Then

$$(a * a^{-1}, a * a^{-1}) = \alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = (a^{-1} * a, a^{-1} * a), \tag{2.148}$$

which implies $a * a^{-1} = a^{-1} * a$. In turn, this shows that $\alpha = \eta_G(a) \in \operatorname{diag}(G^{(0)})$. The fact that $\alpha \in (\operatorname{Or}(G))^{(0)}$ now follows since $\operatorname{diag}(G^{(0)}) = (\operatorname{Or}(G))^{(0)}$, as already seen in (1). This completes the proof of (2).

Next, the first equivalence in claim in (3) follows from what we proved so far and Definition 2.56. In turn, the second equivalence in claim (3) is implied by the first equivalence in (3), by (1), the fact that any groupoid induced by an equivalence relation (as in Example 2.31) is principal, and Proposition 2.51.

Let us now deal with (4). The implication (2.143) $\Longrightarrow$ (2.144) is a consequence of Remark 2.57 and (2.142). Next, the fact that (2.144) $\Longrightarrow$ (2.143) is trivial. The equivalence of (2.144) with each of (2.145)–(2.147) is immediate from Definition 2.56, the definition of $\mathrm{Or}\,(G)$, and Proposition 2.55. This completes the proof of (4).

We are left with proving the statement in (5). If G is a Brandt groupoid, then, in view of Remark 2.57, in order to show that η_G is surjective, it suffices to assume that G is of the form described in Example 2.32. Then, retaining the notation from Example 2.32, it is easy to check that $G^{(0)} = \{(x, e_H, x) : x \in X\}$, with e_H denoting the unit element in H, and if $\big((x, e_H, x), (y, e_H, y)\big) \in G^{(0)} \times G^{(0)}$ is arbitrary, then $\eta_G\big((x, e_H, y)\big) = \big((x, e_H, x)\,,\,(y, e_H, y)\big)$. This proves that η_G is surjective. Conversely, if η_G is surjective, then $\mathrm{Im}\,\eta_G = G^{(0)} \times G^{(0)}$. Hence, there exists a unique class of equivalence for the equivalence relation whose graph is $\mathrm{Im}\,\eta_G$, which implies that the family of groups $\{\Gamma_\xi\}_{\xi \in G^{(0)}/\sim}$ from Theorem 2.54 reduces to just one group Γ. Consequently, by Theorem 2.54, G is isomorphic to $G^{(0)} \times \Gamma \times G^{(0)}$, proving that G is a Brandt groupoid. □

Remark 2.59. Proposition 2.26, together with statement (5) of Theorem 2.58 and the discussion in Example 2.32, permits us to characterize Brandt groupoids as follows. Assume that G is a nonempty set equipped with a partially defined binary operation "$*$" and denote by $G^{(2)}$ the collection of composable pairs in G. Then there exists an inversion $(\cdot)^{-1}$ on G such that $(G, *, (\cdot)^{-1})$ becomes a Brandt groupoid if and only if the following six axioms are satisfied:

(1) For every $a \in G$ there exist unique elements $a_1, a_2 \in G$ with the property that $(a_1, a), (a, a_2) \in G^{(2)}$ and $a_1 * a = a = a * a_2$.
(2) If $(a, b) \in G^{(2)}$ and $a * b = a$, or if $(b, a) \in G^{(2)}$ and $b * a = a$, then $(b, b) \in G^{(2)}$ and $b * b = b$.
(3) For every $a, b \in G$ there holds $(a, b) \in G^{(2)}$ if and only if there exists $c \in G$ such that $(a, c), (c, b) \in G^{(2)}$ and $a * c = a$, $c * b = b$.
(4) If $a, b, c \in G$ are such that $(a, b), (b, c) \in G^{(2)}$, then $(a * b, c), (a, b * c) \in G^{(2)}$ and $(a * b) * c = a * (b * c)$.
(5) If $a, \tilde{a}, \tilde{\tilde{a}} \in G$ are such that $(\tilde{a}, a), (a, \tilde{\tilde{a}}) \in G^{(2)}$ and $\tilde{a} * a = a = a * \tilde{\tilde{a}}$, then there exists $b \in G$ such that $(a, b), (b, a) \in G^{(2)}$ and $a * b = \tilde{a}$ and $b * a = \tilde{\tilde{a}}$.
(6) If $a', a'' \in G$ are such that $(a', a'), (a'', a'') \in G^{(2)}$ and $a' * a' = a'$, $a'' * a'' = a''$, then there exists $a \in G$ such that $(a', a), (a, a'') \in G^{(2)}$ and $a' * a = a = a * a''$.

In closing, we wish to point out that it is a common occurrence in the literature (cf., e.g., [25]) to take the axiomatic setup from Remark 2.59 as the very definition of a Brandt groupoid. As the discussion in Remark 2.59 shows, this point of view leads to the same class of Brandt groupoids as the one considered here.

2.2 Topological Considerations

This section is reserved for a discussion of results and concepts of topological flavor that are pertinent to the present work. We begin by recalling the notion of topological groupoid.

Definition 2.60. A `topological groupoid` is a groupoid $(G, *, (\cdot)^{-1})$ endowed with a topology τ on G with the property that the groupoid operations $(\cdot)^{-1} : G \to G$ and $* : G^{(2)} \to G$ are continuous functions (in the latter case, considering the topology on $G^{(2)}$ inherited from $(G \times G, \tau \times \tau)$).

Topological groupoids generalize the familiar notions of topological groups and topological space. Concretely, any topological group is a topological groupoid. At the other extreme, if (X, τ) is a topological space, then X, considered as a set groupoid in the sense of Example 2.30, can naturally be regarded as a topological groupoid. In addition, the pair groupoid $X \times X$ also becomes a topological groupoid when equipped with the product topology $\tau \times \tau$.

Most of the examples of groupoids presented in Sect. 3 naturally become topological groupoids if suitable topologies are given in the background. To illustrate this point note that if Γ is a topological group acting continuously on a topological space X, then the groupoid $X \times \Gamma$ (defined in Example 2.38) becomes a topological groupoid when equipped with the product topology (cf., e.g., [87, p. 86]). Furthermore, $X \times \Gamma$ is Hausdorff if Γ and X are so and is locally compact if Γ and X are so.

To describe another instance of how one of the groupoids from Sect. 3 can be naturally turned into a topological groupoid, we first recall some definitions. Let (X, τ) be a topological space. A family $\mathcal{B}$ of open sets in X is called a `base` for τ if every open set in X can be written as a union of members of $\mathcal{B}$ (or, equivalently, whenever U is open in X and $x \in U$, then there is an open set $V \in \mathcal{B}$ such that $x \in V \subseteq U$; cf. [71, pp. 46–47] and [87, p. 78]). Given a topological space (X, τ), call a family $\mathcal{B}_o$ of open sets in X a `subbase` for τ if the collection $\mathcal{B}$ of finite intersections of elements of $\mathcal{B}_o$ is a base for τ. Going further, the `compact-open topology` is the topology on the space $C^0(X, Y)$ of all continuous functions from a topological X into a topological space Y, characterized by the fact that a subbase for this topology is given by the family of sets of the form $W_{K,U} := \{f \in C^0(X, Y) : f(K) \subseteq U\}$, where K is compact in X and U is open in Y. Then the groupoid in part (ii) of Example 2.39 becomes a topological groupoid if one considers the corresponding compact-open topologies both on the space of all diffeomorphisms of the manifold M and the space of metrics on M.

Remark 2.61. (i) If $(G, *, (\cdot)^{-1})$ is a topological groupoid, then the inversion mapping $G \ni a \mapsto a^{-1} \in G$ is a homeomorphism, while the source and target functions s, t defined in (2.80) are continuous.

(ii) Given a Hausdorff topological groupoid G, it follows from (i) and the descriptions $G^{(0)} = \{a \in G : t(a) = s(a)\}$ and $G^{(2)} = \{(a, b) \in G \times G :$

$t(a) = s(b)\}$ that the unit space $G^{(0)}$ is a closed subset of G and the collection of composable pairs $G^{(2)}$ is a closed subset of $G \times G$ (where the latter set is equipped with the natural product topology; cf., e.g., [87, p. 86]).

We next describe the topology induced on a given groupoid G by an arbitrary nonnegative (and possibly infinite) function ψ defined on G.

Definition 2.62. Given a groupoid $(G, *, (\cdot)^{-1})$ and a function $\psi : G \to [0, +\infty]$, define τ_ψ^{R}, the `right-topology` induced by ψ on G, by taking

$$\begin{aligned} O \subseteq G \text{ is open in } \tau_\psi^{\mathrm{R}} &\overset{\text{def}}{\Longleftrightarrow} \forall\, a \in O \;\; \exists\, r > 0 \text{ with the property that} \\ B_\psi^{\mathrm{R}}(a, r) &:= \big\{ b \in G : (a, b^{-1}) \in G^{(2)} \text{ and } \psi(a * b^{-1}) < r \big\} \subseteq O. \end{aligned} \tag{2.149}$$

The `left-topology` induced by ψ on G is denoted by τ_ψ^{L} and is defined analogously, this time taking

$$\begin{aligned} O \subseteq G \text{ is open in } \tau_\psi^{\mathrm{L}} &\overset{\text{def}}{\Longleftrightarrow} \forall\, a \in O \;\; \exists\, r > 0 \text{ with the property that} \\ B_\psi^{\mathrm{L}}(a, r) &:= \big\{ b \in G : (b^{-1}, a) \in G^{(2)} \text{ and } \psi(b^{-1} * a) < r \big\} \subseteq O. \end{aligned} \tag{2.150}$$

The reader should have no difficulties in verifying that τ_ψ^{R} and τ_ψ^{L}, introduced in (2.149) and (2.150), are indeed topologies on the set G.

Recall the convention made in (2.14), which is used in the formulation of the conclusion in the following lemma.

Lemma 2.63. *Assume that* $(G, *, (\cdot)^{-1})$ *is a groupoid and that* $\psi : G \to [0, +\infty]$ *is an arbitrary function. Then*

$$B_\psi^{\mathrm{R}}(a, r) * b = B_\psi^{\mathrm{R}}(a * b, r), \qquad \forall\, (a, b) \in G^{(2)}, \;\; \forall\, r \in (0, +\infty), \tag{2.151}$$

$$b * B_\psi^{\mathrm{L}}(a, r) = B_\psi^{\mathrm{L}}(b * a, r), \qquad \forall\, (b, a) \in G^{(2)}, \;\; \forall\, r \in (0, +\infty). \tag{2.152}$$

Proof. Pick $(a, b) \in G^{(2)}$ along with $r \in (0, +\infty)$. Now, if $c \in B_\psi^{\mathrm{R}}(a, r) * b$, then there exists some $u \in B_\psi^{\mathrm{R}}(a, r)$ such that $(u, b) \in G^{(2)}$ and $u * b = c$, hence also $(a, u^{-1}) \in G^{(2)}$, $\psi(a * u^{-1}) < r$, and $(b^{-1}, u^{-1}) \in G^{(2)}$, $b^{-1} * u^{-1} = c^{-1}$. Thus, $(a * b, c^{-1}) \in G^{(2)}$ and $\psi(a * b * c^{-1}) = \psi(a * u^{-1}) < r$, which shows that $c \in B_\psi^{\mathrm{R}}(a * b, r)$. This proves the left-to-right inclusion in (2.151). To justify the opposite inclusion, suppose that $c \in B_\psi^{\mathrm{R}}(a * b, r)$, and note that this entails $(a * b, c^{-1}) \in G^{(2)}$ and $\psi\big((a * b) * c^{-1}\big) < r$. Thus $(b, c^{-1}) \in G^{(2)}$ and, further, $(c, b^{-1}) \in G^{(2)}$. If we now set $u := c * b^{-1} \in G$, then it follows that $u^{-1} = b * c^{-1}$, hence $(a, u^{-1}) \in G^{(2)}$ since $(a, b) \in G^{(2)}$. Moreover, $\psi(a * u^{-1}) = \psi\big((a * b) * c^{-1}\big) < r$, which shows that $u \in B_\psi^{\mathrm{R}}(a, r)$. In addition, $(u, b) \in G^{(2)}$ and $u * b = c$, which ultimately implies that $c \in B_\psi^{\mathrm{R}}(a, r) * b$. This gives the right-to-left inclusion and completes the proof of (2.151). Finally, (2.152) is established in a similar manner. □

Note that, in the context of Definition 2.62,

$$G^{(0)} \subseteq \psi^{-1}(\{0\}) \implies a \in B^{\mathrm{L}}_{\psi}(a,r) \cap B^{\mathrm{R}}_{\psi}(a,r), \qquad \forall\, a \in G, \quad \forall\, r > 0. \tag{2.153}$$

Our next lemma describes a setting in which the center of a left-ball or right-ball actually belongs to the interior of the ball in question.

Lemma 2.64. *Let* $(G, *, (\cdot)^{-1})$ *be a groupoid, and assume that* $\psi : G \to [0, +\infty]$ *is a function for which there exists a finite constant* $C_1 \geq 1$ *such that*

$$G^{(0)} \subseteq \psi^{-1}(\{0\}), \ \text{ and } \ \psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\} \quad \forall\, (a,b) \in G^{(2)}. \tag{2.154}$$

Then

$$a \in \big(B^{\mathrm{R}}_{\psi}(a,r)\big)^{\circ}, \qquad \forall\, a \in G, \ \ \forall\, r > 0, \tag{2.155}$$

where the interior $(\dots)^{\circ}$ *is taken with respect to* τ^{R}_{ψ}*, the right-topology induced by* ψ *on* G*, as in Definition 2.62.*

Moreover, a similar result holds for the left-balls associated with ψ *given that this time the interior is taken with respect to* τ^{L}_{ψ}*, the left-topology induced by* ψ *on* G *in the sense of Definition 2.62.*

Proof. First, it is clear from the quasisubadditivity condition on ψ that the quantity (3.207) is, in the case when G does not reduce to $G^{(0)}$, a well-defined number that belongs to the interval $[1, C_1]$ and satisfies

$$\psi(a * b) \leq A\big(\psi(a) + \psi(b)\big), \qquad \forall\, (a,b) \in G^{(2)}. \tag{2.156}$$

In fact, inequality (2.156) also holds when G reduces to $G^{(0)}$ (in which scenario $A := 1$) since, in this case, $\psi \equiv 0$ given that $G^{(0)} \subseteq \psi^{-1}(\{0\})$. To proceed, fix $a \in G$ and some finite $r > 0$. Consider

$$O := \big\{x \in B^{\mathrm{R}}_{\psi}(a,r) : \exists\, r_x > 0 \ \text{such that} \ B^{\mathrm{R}}_{\psi}(x, r_x) \subseteq B^{\mathrm{R}}_{\psi}(a,r)\big\}, \tag{2.157}$$

and note that, thanks to (2.153),

$$a \in O \subseteq B^{\mathrm{R}}_{\psi}(a,r). \tag{2.158}$$

We claim that $O \in \tau^{\mathrm{R}}_{\psi}$. To justify this claim, pick an arbitrary $x_o \in O$. Then there exists $r_o > 0$ such that

$$B^{\mathrm{R}}_{\psi}(x_o, r_o) \subseteq B^{\mathrm{R}}_{\psi}(a,r). \tag{2.159}$$

Select a number $\delta \in (0, r_o/A)$ and set $\varepsilon := r_o/A - \delta > 0$. At this stage, we make the observation that

$$B^{\mathrm{R}}_{\psi}(x_o, \delta) \subseteq O. \tag{2.160}$$

Indeed, assume that $y \in B^{\mathrm{R}}_{\psi}(x_o, \delta)$ and $z \in B^{\mathrm{R}}_{\psi}(y, \varepsilon)$. Then $(x_o, y^{-1}), (y, z^{-1}) \in G^{(2)}$, hence also $(x_o, z^{-1}) \in G^{(2)}$. Based on these considerations and (2.156), we may estimate

$$\psi(x_o * z^{-1}) \leq A\big(\psi(x_o * y^{-1}) + \psi(y * z^{-1})\big) < A(\delta + \varepsilon) = r_o, \tag{2.161}$$

which further entails $\psi(x_o * z^{-1}) < r_o$. As such, $z \in B^{\mathrm{R}}_{\psi}(x_o, r_o)$, hence, ultimately, $B^{\mathrm{R}}_{\psi}(y, \varepsilon) \subseteq B^{\mathrm{R}}_{\psi}(x_o, r_o)$. Thus, $B^{\mathrm{R}}_{\psi}(y, \varepsilon) \subseteq B^{\mathrm{R}}_{\psi}(a, r)$ by (2.159), which proves that $y \in O$. Given that y has been arbitrarily chosen in $B^{\mathrm{R}}_{\psi}(x_o, \delta)$, we deduce that $B^{\mathrm{R}}_{\psi}(x_o, \delta) \subseteq O$, completing the proof of (2.160). In turn, the latter condition implies that $O \in \tau^{\mathrm{R}}_{\psi}$. In light of (2.158), we may therefore conclude that (2.155) holds. The version of this result for left-balls is proved in an analogous fashion. □

Given an arbitrary topological space (X, τ), denote by $\mathscr{N}(x;\tau)$ the family of neighborhoods of the point $x \in X$, relative to the topology τ. Also, for any $A \subseteq X$ let $\mathrm{Int}(A;\tau)$ stand for the interior of the set A, relative to the topology τ. Finally, recall that, given a topological space (X, τ) and a point $x \in X$, a collection $\mathcal{F}$ of neighborhoods of x is called a `fundamental system of neighborhoods of` x if for every $W \in \mathscr{N}(x;\tau)$ there exists a $V \in \mathcal{F}$ such that $V \subseteq U$.

Proposition 2.65. *Let X be an arbitrary set, and consider the assignment*

$$X \ni x \;\mapsto\; \mathcal{F}_x \subseteq 2^X, \tag{2.162}$$

satisfying the following properties:

(i) For each $x \in X$ there holds $\mathcal{F}_x \neq \emptyset$.
(ii) For each $x \in X$ and for each $B_1, B_2 \in \mathcal{F}_x$ there exists some $B \in \mathcal{F}_x$ such that $B \subseteq B_1 \cap B_2$.

Define

$$\tau := \big\{\mathcal{O} \subseteq X : \forall\, x \in \mathcal{O} \quad \exists\, B \in \mathcal{F}_x \;\; \textit{such that} \;\; B \subseteq \mathcal{O}\big\}. \tag{2.163}$$

Then the following statements hold:

(1) τ is a topology on X.
(2) $\mathrm{Int}(A;\tau) \subseteq \big\{x \in A : \exists\, B \in \mathcal{F}_x \;\; \textit{such that} \;\; B \subseteq A\big\}$ for each $A \subseteq X$.
(3) For each $x \in X$ and each $V \in \mathscr{N}(x;\tau)$ there exists $B \in \mathcal{F}_x$ such that $B \subseteq V$.
(4) τ is the largest topology on X for which (3) holds.

Moreover,

$$\mathcal{F}_x \, is\, a\; fundamental\; system\, of\; neighborhoods\, (in\;\tau)\, of\;\; x\;\; for\, each\, x \in X \tag{2.164}$$

if and only if the following two (additional) properties hold:

(iii) For each $x \in X$ and each $B \in \mathcal{F}_x$ one has $x \in B$.

(iv) For each $x \in X$ *and each* $B \in \mathcal{F}_x$ *there exists* $\widetilde{B} \subseteq B$ *such that* $x \in \widetilde{B}$*, and for each* $y \in \widetilde{B}$ *one can find* $B_y \in \mathcal{F}_y$ *with* $B_y \subseteq \widetilde{B}$*.*

Finally, if in addition to (i) *and* (ii)*, properties* (iii) *and* (iv) *are also satisfied by the assignment from* (2.162)*, then one has equality in (2), i.e.,*

$$\mathrm{Int}(A;\tau) = \big\{x \in A : \exists\, B \in \mathcal{F}_x \ \ \textit{such that} \ \ B \subseteq A\big\}, \ \ \textit{for each} \ \ A \subseteq X, \tag{2.165}$$

and τ *is the largest topology on* X *for which* $\mathcal{F}_x$ *is a fundamental system of neighborhoods of* x *in* τ*, for each* $x \in X$*.*

Proof. Assume first that *(i)* and *(ii)* hold and consider τ as in (2.163). Tautologically, $\emptyset \in \tau$. Property *(i)* entails $X \in \tau$, while property *(ii)* ensures that τ is stable under intersection. Finally, since by design τ is stable under arbitrary unions, it follows that τ is a topology on X. The claims made in parts (2) and (3) follow simply by unraveling definitions. Next, consider a topology τ_1 on X with the property that for each element $x \in X$ and each $V \in \mathcal{N}(x;\tau_1)$ there exists $B \in \mathcal{F}_x$ such that $B \subseteq V$. Let $O \in \tau_1$ be arbitrary. Then $O \in \mathcal{N}(x;\tau_1)$ for each $x \in O$. Hence, for each $x \in O$ there exists $B \in \mathcal{F}_x$ such that $B \subseteq O$. This forces $O \in \tau$ and, ultimately, $\tau_1 \subseteq \tau$. This proves the claim in (4).

Let us prove the fact that (2.164) implies *(iii)* and *(iv)*. To this end, assume that for each $x \in X$ the collection $\mathcal{F}_x$ is a fundamental system of neighborhoods in τ of x. In particular, for each $x \in X$ there holds $\mathcal{F}_x \subseteq \mathcal{N}(x;\tau)$. Consequently, if $x \in X$, then $x \in \bigcap_{B \in \mathcal{F}_x} B$, proving *(iii)*.

Next, let $x \in X$ and $B \in \mathcal{F}_x$. Since by hypothesis $\mathcal{F}_x \subseteq \mathcal{N}(x;\tau)$ it follows that there exists $\widetilde{B} \in \tau$ such that $x \in \widetilde{B} \subseteq B$. Moreover, that for each $y \in \widetilde{B}$ one can find $B_y \in \mathcal{F}_y$ with $B_y \subseteq \widetilde{B}$ follows from the fact that $\widetilde{B} \in \tau$ (cf. (2.163)). This establishes *(iv)* and completes the proof of the implication (2.164) $\Rightarrow$ *(iii)* and *(iv)*. Conversely, consider the implication *(i)–(iv)* $\Rightarrow$ (2.164). First notice that assumptions *(iii)* and *(iv)* guarantee that for each $x \in X$ and each $B \in \mathcal{F}_x$ one has $B \in \mathcal{N}(x;\tau)$. Combining this with (3), the claim from (2.164) immediately follows.

Moving on, assume that *(i)–(iv)* hold. With the goal of proving (2.165), fix $A \subseteq X$ and $x \in A$ such that there exists $B \in \mathcal{F}_x$ with the property that $B \subseteq A$. Appealing to *(iv)* it follows that there exists $\widetilde{B}$ such that $x \in \widetilde{B}$, $\widetilde{B} \subseteq B$, and for each $y \in \widetilde{B}$ there exists $B_y \in \mathcal{F}_y$ with the property that $B_y \subseteq \widetilde{B} \subseteq B \subseteq A$. Consequently, $\widetilde{B} \in \tau$ and

$$\widetilde{B} \subseteq \Big\{x \in A : \exists B \in \mathcal{F}_x \ \ \text{such that} \ \ B \subseteq A\Big\}. \tag{2.166}$$

This shows that the set $\big\{x \in A : \exists B \in \mathcal{F}_x \ \ \text{such that} \ \ B \subseteq A\big\}$ is open in τ. Since this set is also, by design, a subset of A, it follows that the right-to-left inclusion in (2.165) holds. This, together with (2), completes the proof of (2.165). Finally, the last claim made in the statement of the proposition is a direct consequence of (4). □

The following proposition ensures that Definition 2.62, introducing the topologies $\tau_\psi^{\rm L}$ and $\tau_\psi^{\rm R}$, is in agreement with the descriptions of $\tau_\psi^{\rm L}$ and $\tau_\psi^{\rm R}$ utilized in the statement of Theorem 3.26.

Proposition 2.66. *Let* $(G, *, (\cdot)^{-1})$ *be a groupoid, and assume that* $\psi : G \to [0, +\infty]$ *is a function satisfying the conditions in* (2.154) *(for some finite constant* $C_1 \geq 1$*). Then* $\tau_\psi^{\rm L}$ *and* $\tau_\psi^{\rm R}$ *are the largest topologies on* G *with the property that for any point* $a \in G$ *a fundamental system of neighborhoods is given by* $\{B_\psi^{\rm L}(a, r)\}_{r>0}$ *and by* $\{B_\psi^{\rm R}(a, r)\}_{r>0}$*, respectively.*

Moreover, for every $A \subseteq G$ *one has*

$$\mathrm{Int}(A; \tau_\psi^{\rm L}) = \big\{a \in A : \exists\, r \in (0, +\infty) \text{ such that } B_\psi^{\rm L}(a, r) \subseteq A\big\}, \tag{2.167}$$

$$\mathrm{Int}(A; \tau_\psi^{\rm R}) = \big\{a \in A : \exists\, r \in (0, +\infty) \text{ such that } B_\psi^{\rm R}(a, r) \subseteq A\big\}. \tag{2.168}$$

Proof. The claim in the first part of the statement is a direct consequence of the fact that the families $\mathcal{F}_x^{\rm L} := \{B_\psi^{\rm L}(x, r)\}_{r>0}$ and $\mathcal{F}_x^{\rm R} := \{B_\psi^{\rm R}(x, r)\}_{r>0}$, $x \in G$, satisfy properties *(i)–(iv)* from Proposition 2.65. Indeed, properties *(i)–(iii)* are easily checked from definitions, while property *(iv)* is a consequence of Lemma 2.64 (taking $\widetilde{B}$ to be the interior of B). Finally, the claim in the second part of the statement is seen from (2.165). □

It should be noted that while, as established in Proposition 2.66, for every $a \in G$ the family $\{B_\psi^{\rm R}(a, r)\}_{r>0}$ constitutes a fundamental system of neighborhoods for a in $\tau_\psi^{\rm R}$, the right-topology induced by ψ on G, the balls $B_\psi^{\rm R}(a, r)$, $r > 0$, are not, generally speaking, open in the topology $\tau_\psi^{\rm R}$ [likewise, the balls $B_\psi^{\rm L}(a, r)$, $r > 0$, are not necessarily open in the topology $\tau_\psi^{\rm L}$].

We continue by recording a useful result pertaining to the continuity properties of the multiplication and inverse operations in a groupoid.

Proposition 2.67. *Suppose* $(G, *, (\cdot)^{-1})$ *is a groupoid and* $\psi : G \to [0, +\infty)$ *is a given function for which there exists a finite constant* $C_0 \geq 0$ *such that the following properties hold:*

$$\psi(a^{-1}) \leq C_0 \psi(a) \;\; \text{for all } a \in G, \;\; \text{and} \;\; G^{(0)} \subseteq \psi^{-1}(\{0\}). \tag{2.169}$$

Let $\tau_\psi^{\rm R}$ *and* $\tau_\psi^{\rm L}$ *denote, respectively, the right-topology and left-topology induced by* ψ *on* G*, according to Definition* 2.62*. Then the mappings*

$$(\cdot)^{-1} : (G, \tau_\psi^{\rm R}) \longrightarrow (G, \tau_\psi^{\rm L}), \qquad (\cdot)^{-1} : (G, \tau_\psi^{\rm L}) \longrightarrow (G, \tau_\psi^{\rm R}) \tag{2.170}$$

are homeomorphisms (and, in fact, are inverse to one another). Furthermore, if $a \in G$ *is arbitrarily fixed and*

$$\Lambda^{\rm L}(a) := \{b \in G : (a, b) \in G^{(2)}\}, \quad \Lambda^{\rm R}(a) := \{b \in G : (b, a) \in G^{(2)}\}, \tag{2.171}$$

then the mappings

$$a * (\cdot) : \big(\Lambda^{\rm L}(a), \tau^{\rm L}_{\psi}\big|_{\Lambda^{\rm L}(a)}\big) \longrightarrow \big(G, \tau^{\rm L}_{\psi}\big), \tag{2.172}$$

$$(\cdot) * a : \big(\Lambda^{\rm R}(a), \tau^{\rm R}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big) \longrightarrow \big(G, \tau^{\rm R}_{\psi}\big) \tag{2.173}$$

are continuous (where, generally speaking, $\tau^{\rm L}_{\psi}\big|_E$ *and* $\tau^{\rm R}_{\psi}\big|_E$ *stand, respectively, for the topologies induced by* $\tau^{\rm L}_{\psi}$ *and* $\tau^{\rm R}_{\psi}$ *on the subset* E *of* G*). Finally, for each fixed* $a \in G$ *the mappings*

$$(\cdot)^{-1} : \big(\Lambda^{\rm L}(a), \tau^{\rm L}_{\psi}\big|_{\Lambda^{\rm L}(a)}\big) \to \big(\Lambda^{\rm R}(a^{-1}), \tau^{\rm R}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big), \tag{2.174}$$

$$(\cdot)^{-1} : \big(\Lambda^{\rm L}(a), \tau^{\rm R}_{\psi}\big|_{\Lambda^{\rm L}(a)}\big) \to \big(\Lambda^{\rm R}(a^{-1}), \tau^{\rm L}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big), \tag{2.175}$$

$$(\cdot)^{-1} : \big(\Lambda^{\rm R}(a), \tau^{\rm R}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big) \longrightarrow \big(\Lambda^{\rm L}(a^{-1}), \tau^{\rm L}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big), \tag{2.176}$$

$$(\cdot)^{-1} : \big(\Lambda^{\rm R}(a), \tau^{\rm L}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big) \longrightarrow \big(\Lambda^{\rm L}(a^{-1}), \tau^{\rm R}_{\psi}\big|_{\Lambda^{\rm R}(a)}\big) \tag{2.177}$$

are homeomorphisms.

Proof. That the mappings (2.170) are homeomorphisms is clear from (2.64) (which shows that $(\cdot)^{-1} : G \to G$ is a bijection, indeed, its own inverse) and the fact that for each $a \in G$ and $r > 0$ we have (again, based on (2.64))

$$\begin{aligned}
\big(B^{\rm L}_{\psi}(a^{-1}, r)\big)^{-1} &= \{b^{-1} \in G : (b^{-1}, a^{-1}) \in G^{(2)},\ \psi(b^{-1} * a^{-1}) < r\} \\
&= \{b \in G : (b, a^{-1}) \in G^{(2)},\ \psi((b * a^{-1}) < r\} \\
&\subseteq \{b \in G : (a, b^{-1}) \in G^{(2)},\ \psi(a * b^{-1}) < C_0 r\} \\
&= B^{\rm R}_{\psi}(a, C_0 r)
\end{aligned} \tag{2.178}$$

and, similarly, $\big(B^{\rm R}_{\psi}(a^{-1}, r)\big)^{-1} \subseteq B^{\rm L}_{\psi}(a, C_0 r)$, which shows that the two mappings in (2.170) are continuous.

Next, fix $a \in G$, and consider the issue of the continuity of the mapping (2.172) at some point $b_o \in \Lambda^{\rm L}(a)$. Concretely, given an arbitrary $r > 0$, we wish to find $\varepsilon > 0$ with the property that

$$b \in G,\ (a, b) \in G^{(2)},\ b \in B^{\rm L}_{\psi}(b_o, \varepsilon) \Longrightarrow a * b \in B^{\rm L}_{\psi}(a * b_o, r). \tag{2.179}$$

However, since $\psi(b_o^{-1} * a^{-1} * a * b) = \psi(b_o^{-1} * b) < \varepsilon$, it follows that (2.179) holds for the choice $\varepsilon := r$. This proves that the mapping (2.172) is continuous at any $b_o \in \Lambda^{\rm L}(a)$. A similar argument shows that the mapping (2.173) is also continuous at any point in $\Lambda^{\rm R}(a)$, and this completes the proof of the continuity of the mappings (2.172)–(2.173).

Finally, given $a \in G$, the mappings in (2.174) and (2.176) are clearly well defined and inverse to one another, as are the mappings (2.175) and (2.177). Since for each

$a \in G$ we have $\big(\Lambda^{\rm L}(a)\big)^{-1} = \Lambda^{\rm R}(a^{-1})$ and $\big(\Lambda^{\rm R}(a)\big)^{-1} = \Lambda^{\rm L}(a^{-1})$, using the fact that the map in (2.170) is a homeomorphism it follows that the mappings in (2.174)–(2.177) are continuous and, thus, homeomorphisms. This completes the proof of the proposition. □

At this stage, we momentarily digress for the purpose of introducing the notion of partially defined pseudodistance on a given arbitrary set, as well as other related concepts.

Definition 2.68. Fix an arbitrary, nonempty set X. For a given set $\mathscr{R} \subseteq X \times X$ and a given function $d : \mathscr{R} \to [0, +\infty]$ consider the following conditions:

(1) `Nondegeneracy:` there holds $\operatorname{diag}(X) \subseteq \mathscr{R}$ and, for any (a,b) belonging to $\mathscr{R}$, one has $d(a,b) = 0$ if and only if $a = b$ (i.e., $d^{-1}(\{0\}) = \operatorname{diag}(X)$).
(2) `Symmetry:` the set $\mathscr{R}$ is symmetric (i.e., $(a,b) \in \mathscr{R}$ implies $(b,a) \in \mathscr{R}$) and the function d is symmetric, i.e., one has $d(a,b) = d(b,a)$ for every $(a,b) \in \mathscr{R}$.
(3) `Triangle inequality:` whenever $(a,c),(c,b) \in \mathscr{R}$, it follows that $(a,b) \in \mathscr{R}$ and $d(a,b) \le d(a,c) + d(c,b)$.

If the function d is such that conditions (1)–(3) above are satisfied, then d is called a `partially defined distance` on X with domain $\operatorname{Dom}(d) := \mathscr{R}$.

Furthermore, call $d : \mathscr{R} \to [0, +\infty]$ a `partially defined pseudo-distance` (with domain $\operatorname{Dom}(d) := \mathscr{R}$) if conditions (2) and (3) above hold as stated and, in place of the nondegeneracy condition (1), the following, weaker, version is satisfied:

(1′) `Pseudo-nondegeneracy:` there holds $\operatorname{diag}(X) \subseteq \mathscr{R}$ and $\operatorname{diag}(X) \subseteq d^{-1}(\{0\})$ (i.e., $(a,a) \in \mathscr{R}$ and $d(a,a) = 0$ for every $a \in X$).

It is useful to note that the domain of a partially defined pseudodistance on a set X is always the graph of an equivalence relation on X. Of course, any partially defined distance is a partial pseudodistance, and a partial distance on a set X is a genuine distance on X if and only if it is finite and its domain is $X \times X$. A partially defined pseudodistance on a set X whose domain is $X \times X$ will be referred to simply as a `pseudodistance` on X.

The manner in which a partially defined pseudodistance on a set X induces a topology on X is described next.

Definition 2.69. Let X be an arbitrary nonempty set, and assume that d is a partially defined pseudodistance on X, with domain $\operatorname{Dom}(d)$. Then τ_d, the topology induced by d on G, is defined by demanding that

$$O \subseteq X \text{ is open in } \tau_d \overset{\text{def}}{\Longleftrightarrow} O \subseteq X \text{ and } \forall\, x \in O\ \exists\, r > 0 \text{ with the property that}$$
$$\mathscr{B}_d(x,r) := \big\{ y \in X : (x,y) \in \operatorname{Dom}(d) \text{ and } d(x,y) < r \big\} \subseteq O. \tag{2.180}$$

In the context of Definition 2.69, it may be readily verified that for each $x \in X$ and $r \in (0, +\infty)$,

$$\text{the } d\text{-ball } \mathscr{B}_d(x,r) \text{ is open in } \tau_d \text{ and contains its center, } x. \tag{2.181}$$

It follows from this and Proposition 2.65 that τ_d is the largest topology on X with the property that for any point $x \in X$ a fundamental system of neighborhoods is given by $\{\mathscr{B}_d(x,r)\}_{r>0}$.

We next record a lemma that will be useful later on (in the proof of Theorem 3.46). Recall that given two topological spaces (X_j, τ_j), $j = 1, 2$, the `product topology` $\tau_1 \times \tau_2$ on $X_1 \times X_2$ is characterized by the property that Cartesian products of open sets in X_1 with open sets in X_2 form a base for $\tau_1 \times \tau_2$ (cf., e.g., [87, p. 86]).

Lemma 2.70. *(i) For $j = 1, 2$ let d_j be a partially defined pseudodistance on a nonempty set X_j with domain* $\mathrm{Dom}\,(d_j)$. *Consider*

$$\begin{aligned}\mathrm{Dom}\,(d_1 \otimes d_2) := \big\{\big((x_1,x_2),(y_1,y_2)\big) \in (X_1 \times X_2) \times (X_1 \times X_2) :\\ (x_j,y_j) \in \mathrm{Dom}\,(d_j) \text{ for } j = 1,2\big\}\end{aligned} \tag{2.182}$$

and

$$\begin{aligned}&d_1 \otimes d_2 : \mathrm{Dom}\,(d_1 \otimes d_2) \longrightarrow [0,+\infty] \text{ defined}\\ &\text{for each } \big((x_1,x_2),(y_1,y_2)\big) \in \mathrm{Dom}\,(d_1 \otimes d_2) \text{ by}\\ &(d_1 \otimes d_2)\big((x_1,x_2),(y_1,y_2)\big) := d_1(x_1,y_1) + d_2(x_2,y_2).\end{aligned} \tag{2.183}$$

Then

$$\begin{aligned}&d_1 \otimes d_2 \text{ is a partially defined pseudodistance on} X_1 \times X_2,\\ &\text{whose domain is} \mathrm{Dom}\,(d_1 \otimes d_2), \text{and} \tau_{d_1 \otimes d_2}, \text{ the topology}\\ &\text{induced by} d_1 \otimes d_2 \text{ on } X_1 \times X_2, \text{coincides with} \tau_{d_1} \times \tau_{d_2}.\end{aligned} \tag{2.184}$$

(ii) If $d : X \times X \to [0, +\infty)$ is a pseudodistance, then d is $(d \otimes d)$-Lipschitz with constant ≤ 1, in the sense that

$$\begin{aligned}&\big|d(x_1,x_2) - d(y_1,y_2)\big| \leq (d \otimes d)\big((x_1,x_2),(y_1,y_2)\big)\\ &\text{for every } (x_1,x_2),(y_1,y_2) \in X \times X.\end{aligned} \tag{2.185}$$

As a consequence,

$$d : \big(X \times X, \tau_d \times \tau_d\big) \longrightarrow [0,+\infty) \text{ is continuous.} \tag{2.186}$$

Proof. All the claims are routine consequences of definitions given previously (with (2.181) also playing a role in the proof of (2.184)). □

Moving on, we now review the concept of uniform space. This notion originates in A. Weil's monograph [128], and elegant, modern, accounts may be found in, e.g., [44, 71].

Definition 2.71. A `uniform space` is a pair $(X,\mathcal{U})$ consisting of a nonempty set X and a nonempty family $\mathcal{U}$ of subsets of $X \times X$ satisfying the following system of axioms:

(1) Each set $U \in \mathcal{U}$ contains the diagonal $\mathrm{diag}\,(X)$.
(2) If $U \in \mathcal{U}$ and $V \subseteq X \times X$ is such that $U \subseteq V$, then $V \in \mathcal{U}$.
(3) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(4) If $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ with the property that

$$V^2 := \{(x,y) \in X \times X : \exists\, z \in X \text{ so that } (x,z), (z,y) \in V\} \subseteq U. \tag{2.187}$$

(5) If $U \in \mathcal{U}$, then $U^{-1} := \{(y,x) : (x,y) \in U\}$ is also in $\mathcal{U}$.

The set $\mathcal{U}$ is called the `uniform structure` (or uniformity) of X, and its elements are referred to as `entourages`. Call an entourage $U \in \mathcal{U}$ `symmetric` if $U = U^{-1}$. Finally, for each $U \in \mathcal{U}$ and $x \in X$ define

$$U[x] := \{y \in X : (x,y) \in U\}. \tag{2.188}$$

If $(X,\mathcal{U})$ is a uniform space and $U \in \mathcal{U}$, then $x, y \in X$ are called U-close provided $(x,y) \in U$. In this terminology, $U[x]$ in (2.188) can be described as the collection of all points in X which are U-close to x.

Definition 2.72. If $(X,\mathcal{U})$ is a uniform space, a `fundamental system of entourages` of the uniformity $\mathcal{U}$ is any subset $\mathcal{V}$ of $\mathcal{U}$ such that every entourage of $\mathcal{U}$ contains a set belonging to $\mathcal{V}$.

Clearly, by axiom (2) in Definition 2.71, a fundamental system of entourages specifies the uniformity of a uniform space in a unique fashion. Also, every uniform space has a fundamental system of entourages consisting of symmetric entourages. Every uniformity induces a canonical topology, as described in the next definition.

Definition 2.73. Let $(X,\mathcal{U})$ be a uniform space. Then $\tau_{\mathcal{U}}$, the topology induced by the uniformity $\mathcal{U}$, is the unique topology on X with the property that for each $x \in X$ the family $\{U[x] : U \in \mathcal{U}\}$ is a neighborhood filter for x.

Hence, every uniform space $(X,\mathcal{U})$ becomes a topological space in a canonical fashion and

$$O \subseteq X \text{ is open in } \tau_{\mathcal{U}} \iff \forall\, x \in O \;\; \exists\, U \in \mathcal{U} \text{ such that } U[x] \subseteq O. \tag{2.189}$$

Also,

$$(X, \tau_{\mathcal{U}}) \text{ is Hausdorff} \iff \bigcap_{U \in \mathcal{U}} U = \mathrm{diag}\,(X). \tag{2.190}$$

Compared to a general topological space the existence of a uniform structure on X makes it possible to compare the sizes of neighborhoods of points in X. Indeed, heuristically, given $x, y \in X$ and $U \in \mathcal{U}$, the sets $U[x]$ and $U[y]$ should be thought of as having the "same size."

A convenient way to think about the uniformity $\mathcal{U}$ of a uniform space $(X,\mathcal{U})$ is to consider X to be a reasonable topological space and envision $\mathcal{U}$ as the collection of neighborhoods of the diagonal in $X \times X$. Three basic examples follow.

Example 2.74. Let (X, ρ) be a quasimetric space, and consider then the sets

$$U_\varepsilon := \{(x, y) \in X \times X : \rho(x, y) < \varepsilon\}, \quad \text{where} \quad \varepsilon > 0. \tag{2.191}$$

Define $\mathcal{U}_\rho$ to be the collection of subsets of $X \times X$ with the property that each contains a set U_ε for some $\varepsilon > 0$. Then $\mathcal{U}_\rho$ is a uniform structure on X, referred to as the `uniformity canonically associated with` ρ, and the family of sets described in (2.191) is a fundamental system of entourages for this uniformity.

Note that, given a quasimetric space (X, ρ), the topology induced by the uniformity $\mathcal{U}_\rho$ (canonically associated with ρ) coincides with the topology τ_ρ induced by the quasidistance ρ on X (for more on the nature of τ_ρ see item (10) of Theorem 3.46, and the discussion in Sect. 4.1).

Example 2.75. Let $\mathscr{P} = (p_i)_{i\in I}$ be a family of pseudometrics on a set X (recall that a pseudometric on X is a real-valued, nonnegative, symmetric function that satisfies the triangle inequality and vanishes on the diagonal on $X \times X$). Define $\mathcal{U}_{\mathscr{P}}$ to be the collection of subsets of $X \times X$ with the property that each contains a set of the form

$$\Big\{\bigcap_{i\in I_o} p_i^{-1}([0, r_i]) : I_o \subseteq I \text{ finite}, r_i > 0 \text{ for } i \in I_o\Big\}. \tag{2.192}$$

Then $\mathcal{U}_{\mathscr{P}}$ is a uniform structure on X, called the `uniformity canonically associated with the family of pseudo-metrics` $\mathscr{P}$, and the collection of sets described in (2.192) is a fundamental system of entourages for this uniformity.

Example 2.76. Let G be a topological group, i.e., a group $(G, *)$ equipped with a topology τ_G that makes the group multiplication $* : G \times G \to G$ and the operation of taking the inverse $(\cdot)^{-1} : G \to G$ continuous functions. Then there is a natural uniform structure on G that induces the topology τ_G. To describe it, let $\{W_\alpha\}_{\alpha\in A}$ be a fundamental system of neighborhoods of the identity element $e \in G$ in this topology. Consider the family of sets

$$U_\alpha := \{(x, y) \in G \times G : x * y^{-1} \in W_\alpha\}, \quad \text{where} \quad \alpha \in A, \tag{2.193}$$

and define the uniform structure $\mathcal{U}^{\mathrm{R}}_{\tau_G}$ on G, called the `right uniformity` of the topological group G, as the collection of all subsets of $G \times G$ with the property that each contains a set U_α for some $\alpha \in A$ (that the half-neighborhood axiom (4) from Definition 2.71 is satisfied is a consequence of the fact that, since $*$ is continuous, for each $\alpha \in A$ there exists $\beta \in A$ such that $W_\beta * W_\beta \subseteq W_\alpha$).

Note that the family of sets described in (2.193) is a fundamental system of entourages for this uniformity and the topology induced by this uniformity (in the sense of Definition 2.73) is precisely τ_G.

Of course, there is a natural variant of this construction, yielding what is called the `left uniformity` $\mathcal{U}^{\mathrm{L}}_{\tau_G}$ on G (generally speaking, the left and right uniformities are different), which once again induces the topology τ_G on G.

It should be noted that, in concert with Theorem 1.1, the discussion in Example 2.76 yields the following classical metrization theorem for topological groups.

Theorem 2.77 (Birkhoff–Kakutani [19, 65]). *Let G be a Hausdorff topological group. Then the topology on G is metrizable if and only if the identity element in G has a countable fundamental system of neighborhoods. Furthermore, in such a case, the distance function may be taken to be either left-invariant or right-invariant.*

For a related discussion, the reader is also referred to [71, p. 210], where it is shown that the demand of metrizing a topological group via a distance that is simultaneously left-invariant and right-invariant is rather stringent and cannot, generally speaking, be accommodated. A generalization of Theorem 2.77 is discussed in Sect. 6.2 (cf. Theorem 6.33).

We continue by making several definitions and discussing some purely topological results that will be useful later on, in Sect. 3.2.

Definition 2.78. A topological space (X, τ) is called `quasiregular` provided for each nonempty open set A in (X, τ) there exists some nonempty open set B in (X, τ) whose closure, $\overline{B}$, is contained in A.

Definition 2.79. Let (X, τ) be a topological space. A collection $\mathscr{B}$ of nonempty open sets of (X, τ) is called a `pseudobase` (for (X, τ)) provided for each nonempty open set A in (X, τ) there exists some $B \in \mathscr{B}$ such that $B \subseteq A$.

Definition 2.80. A topological space (X, τ) is called `pseudocomplete` provided it is quasiregular and there exists a sequence $\big(\mathscr{B}_n\big)_{n\in\mathbb{N}}$ of pseudobases (for (X, τ)) with the property that

$$\left.\begin{array}{l}\text{for each sequence } (B_n)_{n\in\mathbb{N}} \text{ of subsets of } X \text{ such that} \\ B_n \in \mathscr{B}_n \ \text{ and } \ \overline{B_{n+1}} \subseteq B_n \ \text{ for every number } n \in \mathbb{N}\end{array}\right\} \Longrightarrow \bigcap_{n\in\mathbb{N}} B_n \neq \emptyset. \tag{2.194}$$

Definition 2.81. A subset of a topological space is called `nowhere dense` if the interior of its closure is empty. Also, a subset Y of a topological space (X, τ) is said to be of `second Baire category` provided Y may not be written as the union of countably many nowhere dense subsets (relative to (X, τ)).

A useful practical criterion ensuring that a given topological space is of second Baire category then reads as follows (cf. [92]):

$$(X,\tau)\ \text{ pseudocomplete topological space}$$
$$\Longrightarrow\ (X,\tau)\ \text{ is of second Baire category.} \tag{2.195}$$

For completeness, we indicate how the standard proof of the fact that a complete pseudometric space is of second Baire category may be adapted to justify (2.195).

Consider a topological space (X,τ) that is pseudocomplete, i.e., (X,τ) is quasiregular, and there exists a sequence $(\mathscr{B}_n)_{n\in\mathbb{N}}$ of pseudobases for (X,τ) satisfying (2.194). For (X,τ), being of second Baire category is equivalent to the property that the intersection of any sequence of dense open sets in (X,τ) is dense in (X,τ). With this in mind, let $(O_n)_{n\in\mathbb{N}}$ be a countable collection of open dense subsets in (X,τ). The goal is to show that

$$A := \bigcap_{n\in\mathbb{N}} O_n \text{ is dense in } (X,\tau), \tag{2.196}$$

i.e., that every nonempty open subset of (X,τ) intersects A.

Pick a nonempty open set W in (X,τ). Since O_1 is open and dense in (X,τ), it follows that $W\cap O_1\in\tau\setminus\{\emptyset\}$. Granted this and given that $\mathscr{B}_1$ is a pseudobase for (X,τ), it follows that

$$\text{there exists } B_1\in\mathscr{B}_1 \text{ such that } B_1\subseteq W\cap O_1. \tag{2.197}$$

Recall that (X,τ) is assumed to be quasiregular and, since the membership of B_1 to $\mathscr{B}_1$ guarantees that $B_1\in\tau\setminus\{\emptyset\}$, we deduce that

$$\text{there exists } W_1\in\tau\setminus\{\emptyset\} \text{ such that } \overline{W_1}\subseteq B_1. \tag{2.198}$$

Since O_2 is open and dense in (X,τ), it follows that $W_1\cap O_2\in\tau\setminus\{\emptyset\}$. Granted this and given that $\mathscr{B}_2$ is a pseudobase for (X,τ), it follows that

$$\text{there exists } B_2\in\mathscr{B}_2 \text{ such that } B_2\subseteq W_1\cap O_2. \tag{2.199}$$

Proceeding inductively, we therefore obtain two sequences $(W_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}}$ of subsets of X (where $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$) satisfying the following properties:

$$W_0=W,\quad W_n\in\tau\setminus\{\emptyset\}\ \text{ and }\ \overline{W_n}\subseteq B_n\ \ \forall\,n\in\mathbb{N}, \tag{2.200}$$

$$B_n\in\mathscr{B}_n\ \text{ and }\ B_n\subseteq W_{n-1}\cap O_n\ \text{ for every }\ n\in\mathbb{N}. \tag{2.201}$$

In particular, these entail

$$\overline{B_{n+1}}\subseteq B_n\ \text{ and }\ W_n\subseteq W_{n-1},\qquad \forall\,n\in\mathbb{N}. \tag{2.202}$$

As such, (2.194) may be invoked, and we obtain

$$B := \bigcap_{n\in\mathbb{N}} B_n \neq \emptyset. \tag{2.203}$$

Since $B_n \subseteq W \cap O_n$ for every $n \in \mathbb{N}$, by (2.201) and (2.202), we finally arrive at the conclusion that $\emptyset \neq B \subseteq W \cap A$. This justifies (2.196) and completes the proof of (2.195).

In closing, we make two comments pertaining to the nature of the first two metrization results stated in Sect. 1.

Comment 2.82. Here we attempt to place the Alexandroff–Urysohn theorem for uniform spaces, stated in Sect. 1 as Theorem 1.1, in a proper, broader perspective. The starting point is the observation that both the formulation and the proof of the Alexandroff–Urysohn theorem have undergone significant developments since the initial publication of [4]. A version closer to the manner in which Theorem 1.1 is stated was established by Chittenden in [31], and his proof has been substantially simplified by Frink in [49], by Aronszajn in [8], and by Bourbaki in [20]. In its current format, Theorem 1.1 first appeared in Weil's monograph [128], where the role of uniformity was brought to the forefront. Presently, this classical theorem is an integral part of the landscape of modern topology, and excellent expositions may be found in, e.g., the monographs [71, p. 186] by Kelley, [20, pp. 15–17, 35] by Bourbaki, and [74, pp. 29–30, 45–47] by Köthe. Up to a point, in the historical development of this subject, there has been a pervasive belief that there are three distinct types of metrization results pertaining, respectively, to topological spaces (as in [18, 30, 88, 90, 105, 112, 115, 124, 125]), proximity spaces (as in [43, 76]), and uniform spaces (as in [4, 71]). See the comments in [71, Note 14, p. 186], where J.L. Kelley opinionated that the Alexandroff–Urysohn result (stated as in Theorem 1.1) was a "rather unsatisfactory solution to the topological metrization problem." More recently, however, it has been realized that such distinctions are somewhat artificial and that such a criticism is largely unwarranted, as the vast majority of these results can be fairly easily deduced from the Alexandroff–Urysohn theorem.

Consider, for instance, the Nagata–Smirnov metrization theorem (cf. [88, 112] and the more timely presentations in [71, Theorem 18, p. 127], [87, Theorem 40.3, p. 250]), asserting that a topological space is metrizable if and only if it is Hausdorff, is regular (i.e., any closed set and any point not belonging to it have disjoint neighborhoods), and has a countably locally finite base (i.e., a base for the given topology that is the union of a countable family of subsets of the space in question, each of which is locally finite). In this case, the connection with the Alexandroff–Urysohn theorem stems from the observation that every completely regular space can (typically in a variety of ways) be equipped with a uniform structure that induces (in the sense of Definition 2.73) the same topology as the original one. A more detailed discussion, which also indicates how the Alexandroff–Urysohn theorem implies the metrization results from [18, 30, 43, 76, 88, 90, 105, 112, 115, 124, 125], can be found in the informative survey[40]. ■

Comment 2.83. This further expands on the discussion initiated just prior to the statement of Theorem 1.2. Specifically, quasimetric spaces are uniform spaces whose uniformity has a countable base; hence, by Theorem 1.1, they are metrizable. This is a classical result that has been understood for a long time. As far as the issue of metrization of quasimetric spaces is concerned, the main contribution by Macías and Segovia in [79] was to quantitatively relate the distance d given by Theorem 1.1 to the original quasidistance ρ on the set X. Concretely, following the approach in [79], consider the entourages

$$U_i := \{(x, y) \in X \times X : \rho(x, y) < M^{-i}\}, \qquad i \in \mathbb{N}, \tag{2.204}$$

where the constant $c \geq 1$ is as in (1.1) and $M > 1$ is to be chosen shortly. Then, clearly,

$$\text{for every } i \in \mathbb{N} \text{ the set } U_i \text{ is symmetric, and } \bigcap_{i\in\mathbb{N}} U_i = \operatorname{diag}(X). \tag{2.205}$$

The idea is now to choose $M > 1$ in a manner that ensures that

$$U_i^3 \subseteq U_{i-1} \text{ for each } i \geq 2. \tag{2.206}$$

Unraveling definitions, this condition comes down to checking that for every quartet $x, y, z, w \in X$ and for each $i \in \mathbb{N}$ with $i \geq 2$ the following implication holds:

$$\rho(x, y) < M^{-i},\ \rho(y, z) < M^{-i},\ \rho(z, w) < M^{-i} \Longrightarrow \rho(x, w) < M^{-i+1}. \tag{2.207}$$

On a separate track, applying the quasitriangle inequality listed at the end of (1.1) twice yields $\rho(x, w) \leq c\rho(x, y)+c^2\rho(y, z)+c^2\rho(z, w)$; hence, the optimal selection of M for which (2.207) is valid is $M := c(2c + 1)$. For this choice, the metrization lemma from [71, p. 185] (cf. also [44, Theorem 8.1.10, p. 430]), which is closely related to Theorem 1.1, guarantees the existence of a distance d on X with the property that

$$U_i \subseteq \{(x, y) \in X \times X : d(x, y) < 2^{-i}\} \subseteq U_{i-1}, \qquad \forall\, i \geq 2. \tag{2.208}$$

Since $2^{-i} = \big([c(2c+1)]^{-i}\big)^{\alpha}$ precisely when α is as in (1.2), it follows from (2.208) that $\rho \approx d^{1/\alpha}$. Thus, if one defines $\rho_* := d^{1/\alpha}$, then it follows that ρ_* is equivalent to ρ, and (1.3) holds.

Inherently, this approach has the basic drawback that, in the process of reconciling the aforementioned topological result with the quantitative features of ρ (cf. (1.1)), the exponent α for which the claims in assertion (1) of Theorem 1.2 hold is the nonoptimal value given in (1.2). This explains why a conceptually new approach is needed if the aim is to identify the sharp value of the exponent α. ■

Chapter 3
Quantitative Metrization Theory

This chapter contains the bulk of our work pertaining to metrization results for semigroupoids and groupoids, dealt with, respectively, in Sects. 3.2 and 3.3. Then, in Sect. 3.4, we will specialize these results to obtain a sharp metrization theorem for quasimetric spaces, which considerably strengthens the work of Macías and Segovia cited earlier (cf. discussion in Sect. 1). As a preamble, we begin by studying the regularization of quasisubadditive functions in the next section.

3.1 Regularizing Quasisubadditive Functions

To set the stage, we first recall the following definition.

Definition 3.1. (i) A `monoid` is an algebraic structure $(\mathcal{M}, +)$ consisting of a nonempty set $\mathcal{M}$ and an associative binary operation $\mathcal{M} \times \mathcal{M} \ni (x, y) \mapsto x + y \in \mathcal{M}$ that has a null element, denoted by $0 \in \mathcal{M}$ (i.e., $0 \in \mathcal{M}$ is such that $x + 0 = 0 + x = x$ for all $x \in \mathcal{M}$).

(ii) Call $(\mathcal{M}, +, \leq)$ an `ordered monoid` if $(\mathcal{M}, +)$ is a monoid and $\leq$ satisfies the following conditions:

(1) $\leq$ is a total order relation on $\mathcal{M}$, i.e., and for any $x, y, z \in \mathcal{M}$

$$\begin{aligned} & x \leq y \text{ and } y \leq z \Rightarrow x \leq z, \\ & \text{either } x \leq y \text{ or } y \leq x, \\ & x = y \Leftrightarrow x \leq y \text{ and } y \leq x. \end{aligned} \tag{3.1}$$

(2) $\leq$ is compatible with the monoid binary operation, in the sense that

$$x, y \in \mathcal{M} \text{ and } x < y \Longrightarrow z + x < z + y \text{ and } x + z < y + z \text{ for each } z \in \mathcal{M}, \tag{3.2}$$

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where $x < y$ means that $x \leq y$ and $x \neq y$.

In such a setting, denote by $\mathcal{M}^+$ the ordered monoid of nonnegative elements in $\mathcal{M}$, i.e.,

$$\mathcal{M}^+ := \{x \in \mathcal{M} : 0 \leq x\}. \tag{3.3}$$

Also, given $x, y \in \mathcal{M}$, write $x \geq y$ (or $x > y$) if and only if $y \leq x$ (or $y < x$).

Two simple observations to be used later on are as follows. If $(\mathcal{M}, +, \leq)$ is an ordered monoid, then for each $x_j, y_j \in \mathcal{M}$, $j = 1, 2$, we have that

$$x_j \leq y_j \quad \text{for} \quad j = 1, 2 \implies x_1 + x_2 \leq y_1 + y_2. \tag{3.4}$$

Also, if we set

$$nx := \underbrace{x + \cdots + x}_{n \text{ times}} \qquad \text{for any } x \in \mathcal{M} \text{ and any } n \in \mathbb{N}, \tag{3.5}$$

then for any $x, y \in \mathcal{M}$ there holds

$$2x \leq 2y \iff x \leq y, \tag{3.6}$$

which further implies

$$2 \min\{x, y\} \leq x + y, \quad \forall\, x, y \in \mathcal{M}^+, \tag{3.7}$$

$$2 \min\{x, y\} = \min\{2x, 2y\}, \quad \forall\, x, y \in \mathcal{M}^+. \tag{3.8}$$

In particular,

$$\min\{2x, 2y\} \leq x + y, \qquad \forall\, x, y \in \mathcal{M}^+. \tag{3.9}$$

We are now prepared to state and prove an iteration lemma of a purely abstract, algebraic nature, which is of paramount importance for all our subsequent results. We wish to stress that the context in which we work is, generally speaking, highly noncommutative and lacks any type of cancellation.[1]

Lemma 3.2. *Let $(G, *)$ be a semigroupoid, and assume that $(\mathcal{M}, +, \leq)$ is an ordered monoid. Suppose that $\Lambda : \mathcal{M}^+ \to \mathcal{M}^+$ is a function satisfying*

$$x \leq \Lambda(x) \leq 2x, \quad \textit{for all} \;\; x \in \mathcal{M}^+. \tag{3.10}$$

Also, consider a function $\psi : G \to \mathcal{M}^+$ with the property that

$$\psi(a * b) \leq \Lambda\left(\max\{\psi(a), \psi(b)\}\right), \quad \textit{for all } (a, b) \in G^{(2)}. \tag{3.11}$$

[1]The issue of proving metrization results in the spirit of the Aoki–Rolewicz theorem in the noncommutative context has been occasionally raised in other works, such as in [28, Problem 5.2, p. 47].

Then for every $N \in \mathbb{N}$ *and every* $(a_1, \ldots, a_N) \in G^{(N)}$ *there holds*

$$\psi(a_1 * a_2 * \cdots * a_N) \leq \Lambda(\psi(a_1)) + \sum_{i=2}^{N-1} 2\Lambda(\psi(a_i)) + \Lambda(\psi(a_N)), \tag{3.12}$$

with the convention that the sum on the right-hand side of (3.12) *is omitted if* $N \leq 2$.

In particular, if $\psi : G \to \mathcal{M}^+$ *is a function satisfying*

$$\psi(a * b) \leq 2 \max\{\psi(a), \psi(b)\}, \quad \textit{for all } (a, b) \in G^{(2)}, \tag{3.13}$$

then for every $N \in \mathbb{N}$ *and every* $(a_1, \ldots, a_N) \in G^{(N)}$ *there holds*

$$\psi(a_1 * a_2 * \cdots * a_N) \leq 2\psi(a_1) + \sum_{i=2}^{N-1} 4\psi(a_i) + 2\psi(a_N), \tag{3.14}$$

with the convention that the sum on the right-hand side of (3.14) *is disregarded if* $N \leq 2$.

Proof. The proof of (3.12) proceeds by induction on the parameter $N \in \mathbb{N}$. Given that $\psi(a) \geq 0$ for every $a \in G$, estimate (3.12) corresponding to $N = 1$ holds thanks to (3.10), while (3.12) corresponding to $N = 2$ is a consequence of (3.11) since, trivially,

$$\Lambda(\max\{x, y\}) \leq \Lambda(x) + \Lambda(y), \qquad \forall\, x, y \in \mathcal{M}^+. \tag{3.15}$$

Assume now that $N \in \mathbb{N}$, with $N \geq 2$, is such that

$$\begin{aligned} &\text{estimate (3.12) holds with } N \text{ replaced by } K \\ &\text{if } K \in \{1, \ldots, N\} \text{ and } (a_1, \ldots, a_K) \in G^{(K)}. \end{aligned} \tag{3.16}$$

Next, fix an arbitrary $(a_1, \ldots, a_{N+1}) \in G^{(N+1)}$, with the goal of proving that

$$\psi(a_1 * a_2 * \cdots * a_{N+1}) \leq \Lambda(\psi(a_1)) + \sum_{i=2}^{N} 2\Lambda(\psi(a_i)) + \Lambda(\psi(a_{N+1})). \tag{3.17}$$

For $M \in \{1, \ldots, N+1\}$ consider the inequality

$$\psi(a_1 * \cdots * a_{N+1}) \leq \Lambda\left(\psi\left(a_M * \cdots * a_{N+1}\right)\right). \tag{3.18}$$

Denote by M_0 the largest number $M \in \{1, \ldots, N+1\}$ for which (3.18) holds. Since (3.18) is verified for the choice $M = 1$ granted the inequalities (3.10), it follows that $M_0 \in \{1, \dots, N+1\}$ is well defined. If $M_0 = N+1$, then (3.17) holds as well (since ψ is $\mathcal{M}^+$-valued). Hence, it suffices to analyze the case when $1 \le M_0 \le N$. By design,

$$\psi(a_1 * \cdots * a_{N+1}) \le \Lambda\left(\psi\left(a_{M_0} * \cdots * a_{N+1}\right)\right) \tag{3.19}$$

and

$$\psi(a_1 * \cdots * a_{N+1}) > \Lambda\left(\psi\left(a_{M_0+1} * \cdots * a_{N+1}\right)\right). \tag{3.20}$$

On the other hand, by (3.11), we have

$$\psi(a_1 * \cdots * a_{N+1}) \le \Lambda\left(\max\left\{\psi(a_1 * \cdots * a_{M_0}), \psi(a_{M_0+1} * \cdots * a_{N+1})\right\}\right). \tag{3.21}$$

In light of (3.20), we deduce from (3.21) that

$$\psi(a_1 * \cdots * a_{N+1}) \le \Lambda\left(\psi\left(a_1 * \cdots * a_{M_0}\right)\right). \tag{3.22}$$

If $M_0 = 1$, given that ψ is $\mathcal{M}^+$-valued, it follows from (3.22) that (3.17) holds, so we are done. Thus, there remains to study the case when $2 \le M_0 \le N$. Under this assumption, from (3.19), (3.22), and (3.10) we deduce that

$$\psi(a_1 * \cdots * a_{N+1}) \le 2\psi\left(a_{M_0} * \cdots * a_{N+1}\right), \tag{3.23}$$

$$\psi(a_1 * \cdots * a_{N+1}) \le 2\psi\left(a_1 * \cdots * a_{M_0}\right). \tag{3.24}$$

Having established (3.23) and (3.24), we conclude from these and (3.9) that

$$\begin{aligned}\psi(a_1 * \cdots * a_{N+1}) &\le \min\left\{2\psi(a_1 * \cdots * a_{M_0}), 2\psi(a_{M_0} * \cdots * a_{N+1})\right\} \\ &\le \psi(a_1 * \cdots * a_{M_0}) + \psi(a_{M_0} * \cdots * a_{N+1}).\end{aligned} \tag{3.25}$$

On the other hand, by the induction hypothesis (3.16), we have

$$\psi(a_1 * \cdots * a_{M_0}) \le \Lambda(\psi(a_1)) + \sum_{i=2}^{M_0-1} 2\Lambda(\psi(a_i)) + \Lambda(\psi(a_{M_0})), \tag{3.26}$$

$$\psi(a_{M_0} * \cdots * a_{N+1}) \le \Lambda(\psi(a_{M_0})) + \sum_{i=M_0+1}^{N} 2\Lambda(\psi(a_i)) + \Lambda(\psi(a_{N+1})). \tag{3.27}$$

Summing up (3.26) and (3.27) and then returning with this to (3.25), it follows in view of (3.4), that (3.17) also holds when $2 \le M_0 \le N$. This completes the proof of the claim made in the first part of the statement of the lemma.

Finally, the claim made in the second part of the statement of the lemma becomes a consequence of what we have proved so far if we take $\Lambda(x) := 2x$ for each $x \in \mathcal{M}^+$. □

Note that when equipped with the canonical addition and order (on the extended real line), $[0, +\infty]$ can be naturally regarded as an ordered monoid (in the sense of Definition 3.1). This scenario is considered separately below.

Theorem 3.3. *Let $(G, *)$ be a semigroupoid, and assume that a function $\Lambda : [0, +\infty] \to [0, +\infty]$ is given having at most linear growth, i.e., there exists $C \in [0, +\infty)$ such that*

$$\Lambda(x) \le Cx, \quad \textit{for all} \ \ x \in [0, +\infty]. \tag{3.28}$$

Consider a function $\psi : G \to \mathcal{M}^+$ with the property that

$$\psi(a * b) \le \Lambda\left(\max\{\psi(a), \psi(b)\}\right) \quad \textit{for all } (a, b) \in G^{(2)}. \tag{3.29}$$

Then, for every $N \in \mathbb{N}$ and every $(a_1, \dots, a_N) \in G^{(N)}$, whenever $C > 1$, there holds

$$\begin{aligned}\psi(a_1 * a_2 * \cdots * a_N) \le \Big[&\max\left\{\psi(a_1), \Lambda(\psi(a_1))\right\}^{(\log_2 C)^{-1}} \\ &+ 2\sum_{i=2}^{N-1} \max\left\{\psi(a_i), \Lambda(\psi(a_i))\right\}^{(\log_2 C)^{-1}} \\ &+ \max\left\{\psi(a_N), \Lambda(\psi(a_N))\right\}^{(\log_2 C)^{-1}}\Big]^{\log_2 C},\end{aligned} \tag{3.30}$$

with the convention that the sum on the right-hand side of (3.30) is omitted if $N \le 2$. In particular, if $C > 1$, then for every $N \in \mathbb{N}$ and every $(a_1, \dots, a_N) \in G^{(N)}$ one has

$$\psi(a_1 * a_2 * \cdots * a_N) \le C\left[\sum_{i=1}^{N} \max\left\{\psi(a_i), \Lambda(\psi(a_i))\right\}^{(\log_2 C)^{-1}}\right]^{\log_2 C}. \tag{3.31}$$

Finally, in the case when $C \le 1$, for every $N \in \mathbb{N}$ and every $(a_1, \dots, a_N) \in G^{(N)}$ there holds

$$\begin{aligned}&\psi(a_1 * a_2 * \cdots * a_N) \\ &\quad \le \max\left\{C\psi(a_1), C^2\psi(a_2), \dots, C^{N-1}\psi(a_{N-1}), C^{N-1}\psi(a_N)\right\}.\end{aligned} \tag{3.32}$$

Proof. In the case when $C \le 1$, estimate (3.32) follows by simply iterating the inequality $\psi(a * b) \le \max\{C\psi(a), C\psi(b)\}$, which holds for all $(a, b) \in G^{(2)}$. Consider now the case when $C \in (1, +\infty)$. In this scenario, we bring in the function $\widetilde{\psi} : G \to [0, +\infty]$ given by

$$\widetilde{\psi}(a) := [\psi(a)]^{(\log_2 C)^{-1}}, \qquad \forall\, a \in G, \tag{3.33}$$

and define $\widetilde{\Lambda} : [0, +\infty] \to [0, +\infty]$ by setting

$$\widetilde{\Lambda}(x) := \max\left\{x, \Lambda(x^{\log_2 C})^{(\log_2 C)^{-1}}\right\}, \qquad \forall\, x \in [0, +\infty]. \tag{3.34}$$

Since for every $x \in [0, +\infty]$ we have

$$\Lambda(x^{\log_2 C})^{(\log_2 C)^{-1}} \leq \left(C x^{\log_2 C}\right)^{(\log_2 C)^{-1}} = 2x, \tag{3.35}$$

it follows that

$$x \leq \widetilde{\Lambda}(x) \leq 2x \quad \text{ for every } x \in [0, +\infty]. \tag{3.36}$$

Furthermore, condition (3.29) entails

$$\begin{aligned}\psi(a * b)^{(\log_2 C)^{-1}} &\leq \Lambda\left(\left(\max\{\psi(a)^{(\log_2 C)^{-1}}, \psi(b)^{(\log_2 C)^{-1}}\}\right)^{\log_2 C}\right)^{(\log_2 C)^{-1}} \\ &\leq \widetilde{\Lambda}\left(\max\{\psi(a)^{(\log_2 C)^{-1}}, \psi(b)^{(\log_2 C)^{-1}}\}\right) \end{aligned} \tag{3.37}$$

for each $(a, b) \in G^{(2)}$. Hence,

$$\widetilde{\psi}(a * b) \leq \widetilde{\Lambda}\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right) \quad \text{ for all } (a, b) \in G^{(2)}, \tag{3.38}$$

and (3.30) follows from Lemma 3.2, granted (3.36) and (3.38). □

Corollary 3.4. *Let $(G, *)$ be a semigroupoid, and assume that $\psi : G \to [0, +\infty]$ is a function with the property that there exists a constant $C \in (1, +\infty)$ such that*

$$\psi(a * b) \leq C \max\{\psi(a), \psi(b)\} \quad \textit{for all } (a, b) \in G^{(2)}. \tag{3.39}$$

Then for every $N \in \mathbb{N}$ and every $(a_1, \ldots, a_N) \in G^{(N)}$ there holds

$$\begin{aligned}\psi(a_1 * a_2 * \cdots * a_N) &\leq C\left\{\psi(a_1)^{(\log_2 C)^{-1}} + 2\sum_{i=2}^{N-1}\psi(a_i)^{(\log_2 C)^{-1}} + \psi(a_N)^{(\log_2 C)^{-1}}\right\}^{\log_2 C} \\ &\leq C^2\left\{\sum_{i=1}^{N}\psi(a_i)^{(\log_2 C)^{-1}}\right\}^{\log_2 C}, \end{aligned} \tag{3.40}$$

with the convention that the sum in the middle term of (3.40) is omitted if $N \leq 2$.

In particular, if $C \in (1, 2]$ for every $N \in \mathbb{N}$ and every $(a_1, \ldots, a_N) \in G^{(N)}$, then one has

$$\psi(a_1 * a_2 * \cdots * a_N) \leq C\left(\psi(a_1) + 2\sum_{i=2}^{N-1}\psi(a_i) + \psi(a_N)\right), \tag{3.41}$$

with the convention that the sum on the right-hand side of (3.41) is omitted if $N \leq 2$.

Proof. The first inequality in (3.40) follows from Theorem 3.3 used in the case when Λ is the linear function

$$\Lambda : [0,+\infty] \to [0,+\infty], \quad \Lambda(x) := Cx \text{ for each } x \in [0,+\infty]. \tag{3.42}$$

In turn, the second inequality in (3.40) is an immediate consequence of the first upon observing that $2^{\log_2 C} = C$. Finally, whenever $C \in (1,2]$, estimate (3.41) readily follows from the first inequality in (3.40) (alternatively, (3.41) is directly implied by the first part of Lemma 3.2 by observing that the function Λ considered in (3.42) satisfies (3.10)). □

In the following theorem we present a related version of Theorem 3.3. For a discussion on the relationship between these two results the reader is referred to Remark 3.6.

Theorem 3.5. *Let $(G,*)$ be a semigroupoid, and fix two constants $C_0 \in [1,+\infty)$ and $K \in [0,+\infty]$. In this context, let $\psi : G \to [0,K]$ and $\Lambda : [0,K] \to [0,K]$ be two functions satisfying (with* id *denoting the identity function)*

$$\mathrm{id} \le \Lambda, \quad \Lambda \circ \Lambda \le C_0 \cdot \mathrm{id}, \quad \Lambda \ \textit{nondecreasing}, \tag{3.43}$$

as well as

$$\psi(a * b) \le \Lambda\left(\max\{\psi(a), \psi(b)\}\right), \qquad \forall\, (a,b) \in G^{(2)}. \tag{3.44}$$

Introduce

$$\alpha := \frac{1}{\log_2 C_0} \in (0,+\infty]. \tag{3.45}$$

Then for each $N \in \mathbb{N}$ and each $(a_1,\dots,a_N) \in G^{(N)}$ there holds

$$\psi(a_1 * \cdots * a_N) \le \left\{\sum_{i=1}^{N} [(\Lambda \circ \Lambda)(\psi(a_i))]^{\alpha}\right\}^{1/\alpha} \quad \textit{if } \alpha \in (0,+\infty) \tag{3.46}$$

and

$$\psi(a_1 * \cdots * a_N) \le \max\{\psi(a_1), \dots, \psi(a_N)\} \quad \textit{if } \alpha = +\infty. \tag{3.47}$$

As a consequence, for each $N \in \mathbb{N}$ and each $(a_1,\dots,a_N) \in G^{(N)}$,

$$\psi(a_1 * \cdots * a_N) \le C_0 \left\{\sum_{i=1}^{N} \psi(a_i)^{\alpha}\right\}^{1/\alpha} \quad \textit{if } \alpha \in (0,+\infty). \tag{3.48}$$

In particular, if $C_0 = 2$, i.e., if the function ψ satisfies (3.44) where Λ is such that

$$\mathrm{id} \le \Lambda, \quad \Lambda \circ \Lambda \le 2\mathrm{id}, \quad \Lambda \ \textit{nondecreasing}, \tag{3.49}$$

then for each $N \in \mathbb{N}$ *there holds*

$$\psi(a_1 * \cdots * a_N) \leq \sum_{i=1}^{N} (\Lambda \circ \Lambda)(\psi(a_i)), \qquad \forall\, (a_1, \ldots, a_N) \in G^{(N)}; \tag{3.50}$$

hence, further,

$$\psi(a_1 * \cdots * a_N) \leq 2 \sum_{i=1}^{N} \psi(a_i), \qquad \forall\, (a_1, \ldots, a_N) \in G^{(N)}. \tag{3.51}$$

Proof. We will first treat the case $C_0 = 2$, i.e., when the functions $\psi : G \to [0, K]$ and $\Lambda : [0, K] \to [0, K]$ satisfy (3.44) and (3.49). Our goal is to show that estimate (3.50) holds, and we will prove by induction on $N \in \mathbb{N}$ that

$$\begin{gathered} \forall\, M \in \{1, \ldots, N\}, \quad \forall\, (a_1, \ldots, a_N) \in G^{(N)} \ \text{ one has} \\ \psi(a_1 * \cdots * a_N) \leq \sum_{i=1}^{M} (\Lambda \circ \Lambda)(\psi(a_i)). \end{gathered} \tag{3.52}$$

The case $N = 1$ is clear from the observation that $\mathrm{id} \leq \Lambda \circ \Lambda$, which in turn follows by writing $x \leq \Lambda(x) \leq \Lambda(\Lambda(x))$ for each $x \in [0, K]$ using the first inequality in (3.49) twice.

Assume next that (3.52) holds for some $N \in \mathbb{N}$, with the goal of proving that (3.52) holds with N replaced by $N + 1$. Using the induction hypothesis, this of course reduces to showing that

$$\psi(a_1 * \cdots * a_{N+1}) \leq \sum_{i=1}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)), \quad \forall\, (a_1, \ldots, a_{N+1}) \in G^{(N+1)}. \tag{3.53}$$

To this end, fix $(a_1, \ldots, a_{N+1}) \in G^{(N+1)}$ and introduce

$$\lambda := \sum_{i=1}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)) \in [0, +\infty]. \tag{3.54}$$

Given that $\mathrm{Im}(\psi) \subseteq [0, K]$, inequality (3.53) is obvious if $\lambda \geq K$ since, whenever $(a_1, \ldots, a_{N+1}) \in G^{(N+1)}$, we may write

$$\psi(a_1 * \cdots * a_{N+1}) \leq K \leq \lambda = \sum_{i=1}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)). \tag{3.55}$$

Thus, without loss of generality we may work under the assumption that $\lambda \in [0, K)$. In particular, we may assume that $\lambda < +\infty$ in what follows. Consider first the case when

$$(\Lambda \circ \Lambda)(\psi(a_1)) \geq \lambda/2. \tag{3.56}$$

Then

$$\psi(a_2 * \cdots * a_{N+1}) \le \sum_{i=2}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2, \tag{3.57}$$

where the first inequality follows from the induction hypothesis and the second one is a consequence of (3.54), (3.56), and the fact that $\lambda < +\infty$. Thus,

$$\begin{aligned} \psi(a_1 * \cdots * a_{N+1}) &\le \Lambda\left(\psi(a_1), \max\{\psi(a_2 * \cdots * a_{N+1})\}\right) \\ &\le \Lambda\left(\max\{\psi(a_1), \lambda/2\}\right) \\ &\le (\Lambda \circ \Lambda)\left(\max\{\psi(a_1), \lambda/2\}\right) \\ &= \max\{(\Lambda \circ \Lambda)(\psi(a_1)), (\Lambda \circ \Lambda)(\lambda/2)\} \le \lambda. \end{aligned} \tag{3.58}$$

The first of the preceding inequalities follows from (3.44), the second is a consequence of (3.57) and the fact that Λ is Nondecreasing, the third inequality follows from the bound from below on Λ from (3.49), and the equality is again a consequence of the monotonicity of Λ. Finally, the last inequality in (3.58) follows from the bound from above for $\Lambda \circ \Lambda$ from (3.49), along with the fact that, using (3.54), we have $(\Lambda \circ \Lambda)(\psi(a_1)) \le \lambda$. Given the significance of λ, estimate (3.58) proves (3.53) in the case when (3.56) is satisfied.

Consider next the case when

$$(\Lambda \circ \Lambda)(\psi(a_{N+1})) \ge \lambda/2. \tag{3.59}$$

Then, as before, based on the induction hypothesis, the definition of λ from (3.54), (3.59), and the fact that $\lambda < +\infty$, it follows that

$$\psi(a_1 * \cdots * a_N) \le \sum_{i=1}^{N} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2. \tag{3.60}$$

Again, similarly to our reasoning for establishing (3.58) in the previous case, this further implies

$$\begin{aligned} \psi(a_1 * \cdots * a_{N+1}) &\le \Lambda\left(\max\{\psi(a_1 * \cdots * a_N), \psi(a_{N+1})\}\right) \\ &\le \Lambda\left(\max\{\lambda/2, \psi(a_{N+1})\}\right) \\ &\le (\Lambda \circ \Lambda)\left(\max\{\lambda/2, \psi(a_{N+1})\}\right) \\ &= \max\left\{(\Lambda \circ \Lambda)(\lambda/2), (\Lambda \circ \Lambda)(\psi(a_{N+1}))\right\} \le \lambda. \end{aligned} \tag{3.61}$$

This establishes (3.53) in the case when (3.59) holds.

At this stage, there remains to consider the situation when

$$(\Lambda \circ \Lambda)(\psi(a_1)) < \lambda/2 \quad \text{and} \quad (\Lambda \circ \Lambda)(\psi(a_{N+1})) < \lambda/2. \tag{3.62}$$

Assuming that this is the case, define

$$N_o := \max \left\{ n \in \{1, \dots, N+1\} : \sum_{i=1}^{n} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2 \right\}. \tag{3.63}$$

Then the first inequality in (3.62) entails that the number N_o is well defined and that $N_o \geq 1$. Also, the second inequality in (3.62), together with the definition of λ from (3.54), and the fact that $\lambda < +\infty$ further imply that $\sum_{i=1}^{N} (\Lambda \circ \Lambda)(\psi(a_i)) \geq \lambda/2$, which further forces $N_o \leq N-1$. Moreover, by design,

$$\sum_{i=1}^{N_o} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2 \quad \text{and} \quad \sum_{i=1}^{N_o+1} (\Lambda \circ \Lambda)(\psi(a_i)) \geq \lambda/2. \tag{3.64}$$

From the latter estimate, (3.54), and given that $\lambda < +\infty$ and $N_o \leq N-1$, we deduce that

$$\sum_{i=N_o+2}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2. \tag{3.65}$$

Hence, using (3.65) and the induction hypothesis we obtain

$$\psi(a_{N_o+2} * \cdots * a_{N+1}) \leq \sum_{i=N_o+2}^{N+1} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2. \tag{3.66}$$

Also, using the fact that $N_o \leq N-1$, the induction hypothesis, and the first inequality in (3.64), we conclude

$$\psi(a_1 * \cdots * a_{N_o}) \leq \sum_{i=1}^{N_o} (\Lambda \circ \Lambda)(\psi(a_i)) < \lambda/2. \tag{3.67}$$

Consequently,

$$\begin{aligned}
\psi(a_1 * \cdots * a_{N+1}) &\leq \Lambda\left(\max\{\psi(a_1 * \cdots * a_{N_o}), \psi(a_{N_o+1} * \cdots * a_{N+1})\}\right) \\
&\leq \Lambda\left(\max\left\{\lambda/2, \Lambda\left(\max\{\psi(a_{N_o+1}), \psi(a_{N_o+2} * \ldots * a_{N+1})\}\right)\right\}\right) \\
&\leq \Lambda\left(\max\left\{\lambda/2, \Lambda\left(\max\{\psi(a_{N_o+1}), \lambda/2\}\right)\right\}\right) \\
&= \Lambda(\max\{\lambda/2, \Lambda(\psi(a_{N_o+1})), \Lambda(\lambda/2)\}) \\
&= \Lambda(\max\{\Lambda(\psi(a_{N_o+1})), \Lambda(\lambda/2)\}) \\
&= \max\{(\Lambda \circ \Lambda)(\psi(a_{N_o+1})), (\Lambda \circ \Lambda)(\lambda/2)\} \\
&\leq \lambda.
\end{aligned} \tag{3.68}$$

The first inequality above follows from (3.44), the second one is a consequence of (3.67), and the third one is implied by (3.66). Also, the first equality in (3.68) uses the fact that the function Λ is nondecreasing, the second equality is a consequence of the lower bound on Λ from (3.49), while the last equality in (3.68) uses again the monotonicity of Λ. Finally, the last inequality in (3.68) relies on the upper bound property on Λ from (3.49) along with the fact that, as seen from (3.54), we have $(\Lambda \circ \Lambda)\,(\psi(a_{N_o+1})) \leq \lambda$. Given the significance of λ, this completes the proof of (3.53) in the case when (3.62) is satisfied. Hence, the proof of the estimate in (3.50) is complete. Since (3.51) follows immediately from (3.50) using the upper bound on Λ from (3.49), this concludes the analysis of the case $C_0 = 2$.

Next, we will treat the case $C_0 = 1$. In this scenario, thanks to (3.45), we have $\alpha = +\infty$ and using the two inequalities from (3.43) we obtain

$$x \leq \Lambda(x) \leq \Lambda(\Lambda(x)) \leq x, \qquad \forall\, x \in [0, K]. \tag{3.69}$$

Thus, $\Lambda = \mathrm{id}$. Hence, in this case, (3.44) becomes

$$\psi(a * b) \leq \max\{\psi(a), \psi(b)\}, \qquad \forall\, (a, b) \in G^{(2)}, \tag{3.70}$$

and estimate (3.47) follows by repeated applications of (3.70).

For the remainder of the proof we may suppose that $C_0 \in (1, +\infty)$, thus $\alpha \in (0, +\infty)$. Assuming that this is the case, introduce the function

$$\widetilde{\Lambda} : [0, K^\alpha] \to [0, K^\alpha], \quad \widetilde{\Lambda}(x) := \left[\Lambda(x^{1/\alpha})\right]^\alpha, \qquad \forall\, x \in [0, K^\alpha]. \tag{3.71}$$

Note that since Λ is a nondecreasing function, then

$$\widetilde{\Lambda} \text{ is nondecreasing on } [0, K^\alpha]. \tag{3.72}$$

In addition, using the first inequality from (3.43), we may write

$$\widetilde{\Lambda}(x) = \left[\Lambda(x^{1/\alpha})\right]^\alpha \geq \left[x^{1/\alpha}\right]^\alpha = x, \qquad \forall\, x \in [0, K^\alpha], \tag{3.73}$$

i.e.,

$$\mathrm{id} \leq \widetilde{\Lambda}. \tag{3.74}$$

Going further, based on the definition of the function $\widetilde{\Lambda}$, the upper bound on $\Lambda \circ \Lambda$ from (3.43), and the definition of α from (3.45), we write

$$\begin{aligned}(\widetilde{\Lambda} \circ \widetilde{\Lambda})(x) &= \widetilde{\Lambda}\left([\Lambda(x^{1/\alpha})]^\alpha\right) = \left[\Lambda\left(\Lambda(x^{1/\alpha})\right)\right]^\alpha \\ &\leq \left(C_0 x^{1/\alpha}\right)^\alpha = C_0^\alpha x = 2x, \qquad \forall\, x \in [0, K^\alpha],\end{aligned} \tag{3.75}$$

which shows that

$$\widetilde{\Lambda} \circ \widetilde{\Lambda} \leq 2\mathrm{id}. \tag{3.76}$$

Thus, based on (3.72), (3.74), and (3.76), we may conclude that (3.49) holds with Λ replaced by $\widetilde{\Lambda}$.

Introduce now the function $\widetilde{\psi} : G \to [0, K^\alpha]$ by setting

$$\widetilde{\psi}(a) := [\psi(a)]^\alpha, \qquad \forall\, a \in G. \tag{3.77}$$

Then, using (3.77), (3.44), and the definition of $\widetilde{\Lambda}$ from (3.71), for every $(a, b) \in G^{(2)}$ we have

$$\begin{aligned}
\widetilde{\psi}(a * b) &= [\psi(a * b)]^\alpha \le [\Lambda\left(\max\{\psi(a), \psi(b)\}\right)]^\alpha \\
&= \left[\Lambda\left(\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right)^{1/\alpha}\right)\right]^\alpha \\
&= \widetilde{\Lambda}\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right),
\end{aligned} \tag{3.78}$$

i.e., (3.44) holds with ψ replaced by $\widetilde{\psi}$ and Λ replaced by $\widetilde{\Lambda}$. Consequently, based on what we have proved so far in the case $C_0 = 2$, it follows that for each $N \in \mathbb{N}$ there holds

$$\widetilde{\psi}(a_1 * \cdots * a_N) \le \sum_{i=1}^{N} (\widetilde{\Lambda} \circ \widetilde{\Lambda})(\widetilde{\psi}(a_i)), \qquad \forall\, (a_1, \dots, a_N) \in G^{(N)}. \tag{3.79}$$

Unraveling definitions, for each $N \in \mathbb{N}$, estimate (3.79) may be written as

$$[\psi(a_1 * \cdots * a_N)]^\alpha \le \sum_{i=1}^{N} [\Lambda\left(\Lambda(\psi(a_i))\right)]^\alpha, \qquad \forall\, (a_1, \dots, a_N) \in G^{(N)}, \tag{3.80}$$

which proves (3.46) (recall that in the current scenario $\alpha \in (0, +\infty)$). Finally, (3.80), in concert with the upper bound on $\Lambda \circ \Lambda$ from (3.43), immediately implies the estimate in (3.48). The proof of the theorem is now complete. □

Remark 3.6. If $\Lambda = C_1\,\mathrm{id}$ for some $C_1 \in [1, +\infty)$, then estimate (3.48) from Theorem 3.5 works with

$$C_0 := C_1^2 \quad \text{and} \quad \alpha := \frac{1}{\log_2(C_1^2)} = \frac{1}{2\log_2 C_1}, \tag{3.81}$$

whereas Theorem 3.3 yields an estimate of the same type as (3.48) but for the choices

$$C_0 := C_1 \quad \text{and} \quad \alpha := \frac{1}{\log_2 C_1}. \tag{3.82}$$

The latter is better on both counts, i.e., it uses a smaller multiplicative constant and a larger exponent on the right-hand side. That being said, estimate (3.51) from Theorem 3.5 is sharper than what (3.30) yields in this case. This is because in the context of Theorem 3.5, stronger assumptions have been made on the function Λ (compared with the requirements on Λ from Theorem 3.3). Thus, for this portion of our results we have a natural phenomenon in which stronger assumptions imply stronger conclusions.

A relevant consequence of Theorem 3.5 is singled out below.

Corollary 3.7. *Let $(G, *)$ be a semigroupoid, and let $\psi : G \to [0, +\infty]$ satisfy*

$$\psi(a * b) \leq \eta\left(\max\{\psi(a), \psi(b)\}\right), \qquad \forall\, (a, b) \in G^{(2)}, \tag{3.83}$$

for some nondecreasing function $\eta : [0, +\infty] \to [0, +\infty]$. In addition, assume that

$$\text{id} \leq \eta \quad \textit{and} \quad \varphi \circ \eta \circ \eta \leq 2\varphi \quad \textit{on} \;\; [0, +\infty], \tag{3.84}$$

where $\varphi : [0, +\infty] \to [0, +\infty]$ is a continuous function that is strictly increasing and satisfies $\varphi(0) = 0$. Then for each $N \in \mathbb{N}$ there holds

$$\varphi(\psi(a_1 * \cdots * a_N)) \leq 2 \sum_{i=1}^{N} \varphi(\psi(a_i)), \qquad \forall\, (a_1, \ldots, a_N) \in G^{(N)}. \tag{3.85}$$

Proof. Consider the functions

$$\begin{aligned} &\widetilde{\psi} : G \to [0, +\infty], \quad \widetilde{\psi} := \varphi \circ \psi \quad \text{and} \\ &\Lambda : [0, K] \to [0, K], \quad \Lambda := \varphi \circ \eta \circ \varphi^{-1}, \end{aligned} \tag{3.86}$$

where

$$K := \lim_{x \to +\infty} \varphi(x). \tag{3.87}$$

Here φ^{-1} stands for the inverse of the bijective function $\varphi : [0, +\infty] \to [0, K]$. We claim that the following properties hold:

$$\text{Im}(\widetilde{\psi}) \subseteq [0, K], \tag{3.88}$$

$$\Lambda \;\text{ is nondecreasing}, \tag{3.89}$$

$$\text{id} \leq \Lambda \leq \Lambda \circ \Lambda \leq 2\text{id}, \tag{3.90}$$

$$\widetilde{\psi}(a * b) \leq \Lambda\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right), \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.91}$$

Indeed, the inclusion in (3.88) follows from the definition of $\widetilde{\psi}$ in (3.86) and the fact that $\text{Im}(\varphi) = [0, K]$. Next, (3.89) is an immediate consequence of the definition of Λ in (3.86) and the monotonicity properties of φ and η. Turning to (3.90) we first notice that, due to the bound from below on η from (3.84), for each $x \in [0, K]$ we have $\varphi^{-1}(x) \leq \eta(\varphi^{-1}(x))$. Since φ is nondecreasing, this further implies that

$$x = \varphi(\varphi^{-1}(x)) \leq \varphi(\eta(\varphi^{-1}(x))) = \Lambda(x), \qquad \forall\, x \in [0, K], \tag{3.92}$$

establishing the first inequality in (3.90). In particular, $\Lambda(x) \leq (\Lambda \circ \Lambda)(x)$ for any $x \in [0, K]$, i.e., the second estimate in (3.90) also holds. Next we write

$$\begin{aligned}\Lambda \circ \Lambda &= \left(\varphi \circ \eta \circ \varphi^{-1}\right) \circ \left(\varphi \circ \eta \circ \varphi^{-1}\right)\\ &= \varphi \circ \eta \circ \eta \circ \varphi^{-1} \leq 2\,\mathrm{id} \quad \text{on } [0, K], \end{aligned} \tag{3.93}$$

where the inequality in (3.93) follows by appealing to the second estimate in (3.84). This completes the proof of (3.90). Finally, fix an arbitrary pair $(a, b) \in G^{(2)}$ and notice that

$$\begin{aligned}\widetilde{\psi}(a * b) = \varphi(\psi(a * b)) &\leq (\varphi \circ \eta)\,(\max\{\psi(a), \psi(b)\})\\ &= (\varphi \circ \eta \circ \varphi^{-1})\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right)\\ &= \Lambda\left(\max\{\widetilde{\psi}(a), \widetilde{\psi}(b)\}\right), \end{aligned} \tag{3.94}$$

as desired. In (3.94), the first equality follows from the definition of $\widetilde{\psi}$, the first inequality is a consequence of (3.83) and the fact that the function φ is nondecreasing, the second equality uses again the definition of $\widetilde{\psi}$, along with the fact that φ^{-1} is nondecreasing, while the last equality follows immediately from the definition of Λ. This completes the proof of (3.91).

There remains to observe that, by virtue of (3.88)–(3.91), the hypotheses of Theorem 3.5 are satisfied with $\widetilde{\psi}$ in place of ψ and $C_0 = 2$. The main estimate in Theorem 3.5 then yields (3.85). □

Comment 3.8. In the particular case when the semigroupoid $(G, *)$ is given by

$$G := \{(x, y) : x, y \in X\}, \tag{3.95}$$

where X is some fixed nonempty set, with $G^{(2)} := \{((x, y), (y, z)) : x, y, z \in X\}$ and $(x, y) * (y, z) := (x, z)$ for any $((x, y), (y, z)) \in G^{(2)}$, estimate (3.85) has established in [2] under the additional assumptions that η is convex and continuous, $\eta > 2\,\mathrm{id}$, $\eta(0) = 0$, and φ is continuous and concave, and $\varphi(0) = 0$. In [2] it was also shown that the second functional inequality from (3.84) has a solution φ, given any such η. ■

Next we present a second corollary of Theorem 3.5, this time in the spirit of the Aoki–Rolewicz theorem.

Corollary 3.9. *Let $(X, +)$ be a semigroup and assume that $\|\cdot\| : X \to [0, +\infty]$ and $\Lambda : [0, +\infty] \to [0, +\infty]$ are functions satisfying*

$$\Lambda \ \textit{is nondecreasing}, \quad \mathrm{id} \leq \Lambda, \quad \textit{and} \quad \Lambda \circ \Lambda \leq C_0 \cdot \mathrm{id}, \tag{3.96}$$

for some $C_0 \in (1, +\infty)$, and

$$\|x + y\| \leq \Lambda(\|x\|), \qquad \forall\, x, y \in X \ \textit{such that} \ \|x\| \geq \|y\|. \tag{3.97}$$

Then, for each $N \in \mathbb{N}$ and $\{x_i\}_{1 \le i \le N} \subseteq X$ there holds

$$\|x_1 + \cdots + x_N\| \le C_0 \left\{ \sum_{i=1}^{N} \|x_i\|^\alpha \right\}^{1/\alpha}, \quad \textit{where} \quad \alpha := \frac{1}{\log_2 C_0}. \tag{3.98}$$

Proof. This is a direct consequence of Theorem 3.5. □

Going further, the aim is to describe a regularization procedure suggested by our earlier considerations for a given, arbitrary (real-valued) nonnegative function defined on a semigroupoid. To set the stage, we first make the following definition.

Definition 3.10. Assume that $(G, *)$ is a semigroupoid. A function $\psi : G \to [0, +\infty]$ is said to be α-`quasisubadditive` for some $\alpha \in (0, +\infty]$ provided there exists a finite constant $C \ge 2^{-1/\alpha}$ such that

$$\psi(a * b) \le C \left([\psi(a)]^\alpha + [\psi(b)]^\alpha \right)^{\frac{1}{\alpha}}, \qquad \forall\, (a, b) \in G^{(2)}, \tag{3.99}$$

if $\alpha < +\infty$ and, corresponding to $\alpha = +\infty$ (in which scenario it is assumed that $C \ge 1$),

$$\psi(a * b) \le C \, \max\{\psi(a)\,,\, \psi(b)\}, \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.100}$$

Furthermore, it is agreed that the notion "∞-quasisubadditive" may be abbreviated simply as `quasisubadditive`.

Finally, say that $\psi : G \to [0, +\infty]$ is α-`subadditive` for some $\alpha \in (0, +\infty]$ if one may take $C = 1$ in (3.99) when $\alpha < +\infty$ and in (3.100) when $\alpha = +\infty$.

Remark 3.11. Note that given a semigroupoid $(G, *)$, a function $\psi : G \to [0, +\infty]$ is α-quasisubadditive for some $\alpha \in (0, +\infty]$ if and only if ψ is quasisubadditive (what varies is the nature of the constant involved in the definitions of these properties). Moreover, if $\psi : G \to [0, +\infty]$ is α-quasisubadditive for some $\alpha \in (0, +\infty]$, then ψ is β-quasisubadditive for any $\beta \in (0, \alpha]$.

Remark 3.12. If $(G, *)$ is a semigroupoid and $\psi : G \to [0, +\infty]$ is a quasisubadditive function, then $G \ni a \mapsto \min\{\psi(a), 1\} \in [0, +\infty)$ is also a quasisubadditive function. Moreover, if $\psi_1, \psi_2 : G \to [0, +\infty]$ are quasisubadditive functions, then so are the functions $\max\{\psi_1, \psi_2\}$ and $\psi_1 + \psi_2$.

The following definition, describing the α-subadditive regularization of a real-valued, nonnegative function defined on a semigroupoid, plays a key role in much of our subsequent work.

Definition 3.13. Given a semigroupoid $(G, *)$, a function $\psi : G \to [0, +\infty]$, and an exponent $\alpha \in (0, +\infty)$, define $\psi_\alpha : G \to [0, +\infty]$ by setting for each $a \in G$

$$\psi_\alpha(a) := \inf \left\{ \left(\sum_{i=1}^{N} \psi(a_i)^\alpha \right)^{\frac{1}{\alpha}} : N \in \mathbb{N},\ (a_1, \ldots, a_N) \in G^{(N)},\ a = a_1 * \cdots * a_N \right\}. \tag{3.101}$$

Also, corresponding to $\alpha = +\infty$, define $\psi_\infty : G \to [0, +\infty]$ by setting for each $a \in G$

$$\psi_\infty(a) := \inf\left\{ \max_{1\leq i\leq N} \psi(a_i) : N \in \mathbb{N}, (a_1, \dots, a_N) \in G^{(N)}, \ a = a_1 * \cdots * a_N \right\}. \tag{3.102}$$

Finally, call the function ψ_α the α-`subadditive regularization` of ψ.

The following lemma substantiates the heuristic principle that while ψ_α continues to be closely related to ψ, in general it possesses better functional analytic properties. In fact, the reason for using the terminology "α-subadditive regularization" for the function ψ_α, defined according to the regularization scheme in Definition 3.13, becomes clear from part (3) of the lemma.

Lemma 3.14. *Given a semigroupoid $(G, *)$, the following properties hold.*

(1) For each $\alpha \in (0, +\infty]$ and any $\psi : G \to [0, +\infty]$ the function $\psi_\alpha : G \to [0, +\infty]$ is well defined and has the property that for every $C \in [0, +\infty)$ and $\beta \in (0, \alpha]$

$$\psi_\alpha \leq \psi_\beta, \qquad (C\,\psi)_\alpha = C\,\psi_\alpha \quad \text{and} \quad \psi_\alpha \leq \psi \qquad \text{on } G. \tag{3.103}$$

(2) For any $\alpha \in (0, +\infty]$ and any two functions $\psi_1, \psi_2 : G \to [0, +\infty]$

$$\psi_1 \leq \psi_2 \ \text{ on } G \Longrightarrow (\psi_1)_\alpha \leq (\psi_2)_\alpha \ \text{ on } G. \tag{3.104}$$

(3) For each $\alpha \in (0, +\infty]$, each $\psi : G \to [0, +\infty]$, and each $\beta \in (0, \alpha]$ the function ψ_α is β-subadditive. That is, if β is finite, then there holds

$$\psi_\alpha(a * b) \leq \big([\psi_\alpha(a)]^\beta + [\psi_\alpha(b)]^\beta\big)^{\frac{1}{\beta}}, \qquad \forall\,(a, b) \in G^{(2)}, \tag{3.105}$$

and, corresponding to the case $\beta = \alpha = +\infty$,

$$\psi_\infty(a * b) \leq \max\{\psi_\infty(a), \psi_\infty(b)\}, \qquad \forall\,(a, b) \in G^{(2)}. \tag{3.106}$$

(4) For each index $\alpha \in (0, +\infty]$ and each function $\psi : G \to [0, +\infty]$ the function ψ_α is quasisubadditive, in the precise sense that

$$\psi_\alpha(a * b) \leq 2^{1/\alpha} \max\{\psi_\alpha(a), \psi_\alpha(b)\}, \qquad \forall\,(a, b) \in G^{(2)}. \tag{3.107}$$

(5) For any function $\psi : G \to [0, +\infty]$ and any exponent $\alpha \in (0, +\infty]$ one has

$$\psi = \psi_\alpha \iff \psi \text{ is } \alpha\text{-subadditive}. \tag{3.108}$$

As a consequence of this, as well as part (3)*, for each exponent* $\alpha \in (0, +\infty]$ *and* $\beta \in (0, \alpha]$ *and each function* $\psi : G \to [0, +\infty]$ *one has* $(\psi_\alpha)_\beta = \psi_\alpha$*. In particular,*

$$(\psi_\alpha)_\alpha = \psi_\alpha. \tag{3.109}$$

(6) Assume that the function $\psi : G \to [0, +\infty]$ *and the index* $\alpha \in (0, +\infty]$ *are given. Then* ψ_α*, the* α*-subadditive regularization of* ψ*, can be characterized as the largest nonnegative function defined on* G *with the property that it is both* α*-subadditive and* $\leq \psi$ *on* G*.*

(7) Assume that $\alpha \in (0, +\infty)$ *and that* $\psi_i : G \to [0, +\infty]$*,* $i \in \mathbb{N}$*, are* α*-subadditive functions. Then for each* $\beta \in [\alpha, +\infty)$ *the function*

$$\psi : G \longrightarrow [0, +\infty], \qquad \psi := \left(\sum_{i=1}^{\infty} \psi_i^\beta \right)^{1/\beta}, \tag{3.110}$$

is also α*-subadditive.*

(8) The regularization process described in Definition 3.13 is invariant under semigroupoid isomorphisms in the following sense. Suppose that $\alpha \in (0, +\infty]$ *and* $\psi : G \to [0, +\infty]$ *are given,* $\widetilde{G}$ *is another semigroupoid, and* $\phi : \widetilde{G} \to G$ *is a semigroupoid homomorphism. Then* $\psi_\alpha \circ \phi \leq (\psi \circ \phi)_\alpha$*. In particular,*

$$\phi : \widetilde{G} \to G \text{ semigroupoid isomorphism} \Longrightarrow (\psi \circ \phi)_\alpha = \psi_\alpha \circ \phi. \tag{3.111}$$

(9) For each $\alpha \in (0, +\infty]$ *and* $\beta \in (0, +\infty)$ *and any function* $\psi : G \to [0, +\infty]$ *one has* $(\psi^\beta)_\alpha = (\psi_{\alpha\beta})^\beta$*.*

Proof. The claims in (1) and (2) are straightforward consequences of definitions. To prove (3), fix $\alpha \in (0, +\infty)$ and $(a, b) \in G^{(2)}$. Also, assume that $M, N \in \mathbb{N}$ and $(a_1, \dots, a_N) \in G^{(N)}$, $(a_{N+1}, \dots, a_{N+M}) \in G^{(M)}$, are such that $a = a_1 * \cdots * a_N$ and $b = a_{N+1} * \cdots * a_{N+M}$. Then $(a_1, \dots, a_{N+M}) \in G^{(N+M)}$ and $a_1 * \cdots * a_{N+M} = a * b$. Consequently, by (3.101),

$$[\psi_\alpha(a * b)]^\alpha \leq \sum_{i=1}^{N+M} [\psi(a_i)]^\alpha = \sum_{i=1}^{N} [\psi(a_i)]^\alpha + \sum_{i=1}^{M} [\psi(a_{N+i})]^\alpha. \tag{3.112}$$

Taking the infimum over all such N, M and $a_1, \dots, a_{N+M}$, we obtain that

$$\psi_\alpha(a * b) \leq \left([\psi_\alpha(a)]^\alpha + [\psi_\alpha(b)]^\alpha\right)^{\frac{1}{\alpha}}. \tag{3.113}$$

Let now $\beta \in (0, \alpha]$ be arbitrary, fixed. Then, since $\alpha/\beta \geq 1$, we have

$$\begin{aligned} \left([\psi_\alpha(a)]^\alpha + [\psi_\alpha(b)]^\alpha\right)^{\frac{1}{\alpha}} &= \left(\left([\psi_\alpha(a)]^\beta\right)^{\frac{\alpha}{\beta}} + \left([\psi_\alpha(b)]^\beta\right)^{\frac{\alpha}{\beta}} \right)^{\frac{1}{\alpha}} \\ &\leq \left([\psi_\alpha(a)]^\beta + [\psi_\alpha(b)]^\beta \right)^{\frac{1}{\beta}}, \end{aligned} \tag{3.114}$$

so that (3.105) is proved in the case when $\alpha \in (0, +\infty)$ by combining (3.113) and (3.114). Finally, the case $\alpha = +\infty$ of (3.105) is treated similarly [using (3.102)], and the same type of reasoning also proves (3.106). This completes the proof of (3).

Consider next the claim made in (4). The case $\alpha = +\infty$ is contained in (3.106), so it suffices to treat the situation when $\alpha \in (0, +\infty)$. In this scenario, the inequality proved in (3) written for $\beta := \alpha$ gives that for each $(a, b) \in G^{(2)}$ we have

$$\begin{aligned}\psi_\alpha(a * b) &\leq ([\psi_\alpha(a)]^\alpha + [\psi_\alpha(b)]^\alpha)^{\frac{1}{\alpha}} \leq (2 \max \{[\psi_\alpha(a)]^\alpha , [\psi_\alpha(b)]^\alpha\})^{\frac{1}{\alpha}} \\ &= 2^{\frac{1}{\alpha}} \max \{\psi_\alpha(a) , \psi_\alpha(b)\} ,\end{aligned} \tag{3.115}$$

and hence (3.107) is proved.

As far as (5) is concerned, the right-pointing implication in (3.108) is a direct consequence of (3.105) and (3.106). To prove the opposite implication, consider first the case $\alpha < +\infty$, and note that if $a \in G$, $N \in \mathbb{N}$ and $(a_1, \ldots, a_N) \in G^{(N)}$ are such that $a = a_1 * \cdots * a_N$, then an inductive argument (on the parameter N) shows that

$$\psi(a)^\alpha \leq \sum_{i=1}^{N} \psi(a_i)^\alpha. \tag{3.116}$$

Fixing $a \in G$ and taking the infimum of both sides of (3.116) over all representations of a as a product $a = a_1 * \cdots * a_N$ with $N \in \mathbb{N}$ and $(a_1, \ldots, a_N) \in G^{(N)}$ then yields $\psi(a)^\alpha \leq \psi_\alpha(a)^\alpha$ for every $a \in G$, i.e., $\psi \leq \psi_\alpha$. Since the opposite inequality is contained in (3.103), we eventually conclude that $\psi = \psi_\alpha$, as desired. Finally, the reasoning in the case $\alpha = +\infty$ is similar except that, this time, one has

$$\psi(a) \leq \max_{1 \leq i \leq N} \psi(a_i) \tag{3.117}$$

in place of (3.116). The proof of (5) is therefore complete.

Concerning the maximality claim made in (6), thanks to (3) and the last inequality in (1), it suffices to show that if the functions $\psi, \phi : G \to [0, +\infty]$ and the exponent $\alpha \in (0, +\infty]$ are such that $\phi \leq \psi$ on G and ϕ is α-subadditive, then necessarily $\phi \leq \psi_\alpha$ on G. This, however, follows from (3.108) and (2), which allow us to write that $\phi = \phi_\alpha \leq \psi_\alpha$ on G.

The claim in (7) is readily implied by the special case when all but finitely many ψ_i's are identically zero, via a limiting argument. In turn, the latter scenario can be handled by induction (on the number of nonzero functions involved) as soon as we have established the corresponding assertion for two arbitrary α-subadditive functions $\psi_1, \psi_2 : G \to [0, +\infty]$. To this end, pick $(a, b) \in G^{(2)}$ and, based on the α-subadditivity of ψ_1, ψ_2, estimate

$$\begin{aligned}\big(\psi_1(a * b)^\beta + \psi_2(a*b)^\beta\big)^{1/\beta} &\leq \Big[(\psi_1(a)^\alpha + \psi_1(b)^\alpha)^{\beta/\alpha} + (\psi_2(a)^\alpha + \psi_2(b)^\alpha)^{\beta/\alpha}\Big]^{1/\beta} \\ &= [(x_1 + y_1)^p + (x_2 + y_2)^p]^{1/\beta} ,\end{aligned} \tag{3.118}$$

where we have set $x_1 := \psi_1(a)^\alpha$, $x_2 := \psi_2(a)^\alpha$, $y_1 := \psi_1(b)^\alpha$, $y_2 := \psi_2(b)^\alpha$, and $p := \beta/\alpha$. Since $p \geq 1$ and $x_1, x_2, y_1, y_2 \geq 0$, the discrete version of Minkowski's inequality gives $[(x_1 + y_1)^p + (x_2 + y_2)^p]^{1/p} \leq (x_1^p + x_2^p)^{1/p} + (y_1^p + y_2^p)^{1/p}$. Using this back in (3.118) and unraveling notation then yields

$$\big(\psi_1(a * b)^\beta + \psi_2(a * b)^\beta\big)^{\frac{1}{\beta}} \leq \Big[\big(\psi_1(a)^\beta + \psi_2(a)^\beta\big)^{\frac{\alpha}{\beta}} + \big(\psi_1(b)^\beta + \psi_2(b)^\beta\big)^{\frac{\alpha}{\beta}}\Big]^{\frac{1}{\alpha}}, \tag{3.119}$$

which shows that $(\psi_1^\beta + \psi_2^\beta)^{1/\beta} : G \to [0, +\infty]$ is an α-subadditive function. This completes the proof of (7).

As regards the claim made in (8), note that if $\phi : \widetilde{G} \to G$ is a semigroupoid homomorphism, then for any $N \in \mathbb{N}$ and $(a_1, \dots, a_N) \in \widetilde{G}^{(N)}$, $a \in \widetilde{G}$, such that $a = a_1 * \cdots * a_N$, it follows that $(\phi(a_1), \dots, \phi(a_N)) \in G^{(N)}$ and $\phi(a) = \phi(a_1) * \cdots * \phi(a_N)$. This shows that $(\psi \circ \phi)_\alpha(a) \geq \psi_\alpha(\phi(a))$ for every $a \in \widetilde{G}$, hence, ultimately, $\psi_\alpha \circ \phi \leq (\psi \circ \phi)_\alpha$. If, on the other hand, $\phi : \widetilde{G} \to G$ is actually a semigroupoid isomorphism, then writing the inequality just proved with G and $\widetilde{G}$ intertwined, ψ replaced by $\psi \circ \phi$, and ϕ replaced by ϕ^{-1} yields

$$(\psi \circ \phi)_\alpha \circ \phi^{-1} \leq \big((\psi \circ \phi) \circ \phi^{-1}\big)_\alpha = \psi_\alpha, \tag{3.120}$$

which further implies that $(\psi \circ \phi)_\alpha \leq \psi_\alpha \circ \phi$. All together this proves (3.111) and completes the proof of (8).

Finally, the claim in (9) is a straightforward consequence of Definition 3.13. □

For the purpose of comparing various functions we now introduce the following equivalence relation.

Definition 3.15. Let G be a fixed, arbitrary, nonempty set. Call two given functions $\psi_1, \psi_2 : G \to [0, +\infty]$ `equivalent` and write $\psi_1 \approx \psi_2$ if there exist two constants $C', C'' \in (0, +\infty)$ with the property that

$$C'\psi_1(a) \leq \psi_2(a) \leq C''\psi_1(a), \qquad \forall\, a \in G. \tag{3.121}$$

Remark 3.16. (i) For any two functions $\psi_1, \psi_2 : G \to [0, +\infty]$ we have that $\psi_1 \approx \psi_2$ if and only if there exists $\eta : G \to (C', C'')$, where $C', C'' \in (0, +\infty)$, with the property that

$$\psi_2(a) = \eta(a)\psi_1(a), \qquad \forall\, a \in G. \tag{3.122}$$

It is natural to think of any function $\eta : G \to (0, +\infty)$ with the property that

$$0 < \inf_{a \in G} \eta(a) \leq \sup_{a \in G} \eta(a) < +\infty \tag{3.123}$$

as a `modulation` (or `multiplicative weight`) and to call the product $\eta\psi$ the η-`modulated version of` ψ (or the η-`multiplicatively weighted version of` ψ).

(ii) Given a semigroupoid $(G, *)$, for any index $\alpha \in (0, +\infty]$ and any two functions $\psi_1, \psi_2 : G \to [0, +\infty]$, it follows from (3.104) and the second formula in (3.103) that

$$\psi_1 \approx \psi_2 \text{ on } G \Longrightarrow (\psi_1)_\alpha \approx (\psi_2)_\alpha \text{ on } G. \tag{3.124}$$

Our next theorem, which should be contrasted to (3.108), shows that if ψ is a real-valued, nonnegative function defined on a semigroupoid G that, for some $\alpha \in (0, +\infty]$, is equivalent (in the sense of Definition 3.15) with its α-subadditive regularization ψ_α (cf. Definition 3.13), then ψ is also α-quasisubadditive or, equivalently, quasisubadditive (cf. Remark 3.11). Also, in the converse direction, if ψ is quasisubadditive, then ψ is equivalent with its α-subadditive regularization ψ_α for a judicious choice of the index α (depending on the constant involved in the quasisubadditivity condition satisfied by the function ψ). The latter result is of paramount importance for our work.

Theorem 3.17. *Let $(G, *)$ be a semigroupoid.*

(I) Assume that $\psi : G \to [0, +\infty]$ is a function that is quasisubadditive on G in the sense that there exists a finite constant $C_1 \geq 1$ with the property that

$$\psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\}, \quad \textit{for all } (a, b) \in G^{(2)}. \tag{3.125}$$

Introduce

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty], \tag{3.126}$$

and define the function $\psi_\alpha : G \to [0, +\infty]$ as in Definition 3.13.

Then $\psi_\alpha \approx \psi$. More specifically, with C_1 the same constant as in (3.125),

$$C_1^{-2}\psi \leq \psi_\alpha \leq \psi \qquad \textit{on } G. \tag{3.127}$$

In particular,

$$(\psi_\alpha)^{-1}(\{0\}) = \psi^{-1}(\{0\}). \tag{3.128}$$

More generally, if $\beta \in (0, \alpha]$, then

$$2^{-2/\beta}\psi \leq \psi_\beta \leq \psi \qquad \textit{on } G. \tag{3.129}$$

(II) Conversely, if $\psi : G \to [0, +\infty]$ is a function for which there exist some finite constant $C \geq 1$ and some $\alpha \in (0, +\infty]$ with the property that

$$\psi \leq C\,\psi_\alpha \quad \textit{on } G \tag{3.130}$$

(hence $\psi \approx \psi_\alpha$ since the estimate $\psi_\alpha \leq \psi$ is always true), then ψ satisfies estimate (3.125) for the choice $C_1 := C2^{1/\alpha}$ (hence ψ is quasisubadditive).

(III) In the setting of part (I), consider the constant $C_\psi \in [0, +\infty]$ defined by $C_\psi := 1$ if the function ψ is either identically zero or identically infinity on G and, otherwise,

$$C_\psi := \sup \left(\frac{\psi(a_1 * \cdots * a_N)}{\left(\sum_{i=1}^{N} \psi(a_i)^\alpha\right)^{1/\alpha}} \right) \qquad \text{if } \alpha < +\infty, \tag{3.131}$$

where the supremum is taken over all $N \in \mathbb{N}$ and all $(a_1, \ldots, a_N) \in G^{(N)}$ with $0 < \max_{1 \leq i \leq N} \psi(a_i) < +\infty$, and, corresponding to $\alpha = +\infty$,

$$C_\psi := \sup \left(\frac{\psi(a_1 * \cdots * a_N)}{\max\limits_{1 \leq i \leq N} \psi(a_i)} \right), \tag{3.132}$$

with the same conventions regarding the supremum. Then the following improvement of (3.127) holds:

$$1 \leq C_\psi \leq C_1^2 \quad \text{and} \quad C_\psi^{-1}\psi \leq \psi_\alpha \leq \psi \ \text{ on } G. \tag{3.133}$$

Also,

$$C_\psi = 1 \iff \psi \ \text{ is } \alpha\text{-subadditive}. \tag{3.134}$$

Proof. Consider the claims made in part (I) of the theorem. The case $C_1 = 1$, corresponding to $\alpha = +\infty$, is immediate from Lemma 3.14; therefore, we will assume in what follows that $\alpha < +\infty$. From (3.103) we know that $\psi_\alpha(a) \leq \psi(a)$ for every $a \in G$. Corresponding to the inequality in the opposite direction, the idea is to apply Lemma 3.2 (taking $\mathcal{M}$ to be $[0, +\infty]$ with its natural structure of ordered monoid) to the function $(\psi)^\alpha$. Note that the choice (3.126) ensures that $C_1^\alpha = 2$, and hence $(\psi)^\alpha$ satisfies an inequality as described in (3.13). This allows us to conclude that

$$[\psi(a_1 * \cdots * a_N)]^\alpha \leq 4 \sum_{i=1}^{N} [\psi(a_i)]^\alpha \tag{3.135}$$

each time $N \in \mathbb{N}$ and $(a_1, \ldots, a_N) \in G^{(N)}$. In particular, this implies that

$$\psi(a) \leq 4^{\frac{1}{\alpha}} \left(\sum_{i=1}^{N} [\psi(a_i)]^\alpha \right)^{\frac{1}{\alpha}} \tag{3.136}$$

if $a \in G$, $N \in \mathbb{N}$, and $(a_1, \dots, a_N) \in G^{(N)}$ are such that $a = a_1 * \cdots * a_N$. Starting with an arbitrary $a \in G$ and then taking the infimum in (3.136) over all $N \in \mathbb{N}$ and $(a_1, \dots, a_N) \in G^{(N)}$ satisfying $a = a_1 * \cdots * a_N$ allows us to conclude that

$$\psi(a) \leq 4^{\frac{1}{\alpha}} \psi_\alpha(a) = C_1^{\,2} \psi_\alpha(a), \qquad \forall\, a \in G. \tag{3.137}$$

This completes the proof of (3.127). Note that (3.128) readily follows from (3.127). To see (3.129), observe that if $\beta \in (0, \alpha]$, then $C_1 \leq 2^{1/\beta}$. Hence,

$$\psi(a * b) \leq 2^{1/\beta} \max\{\psi(a), \psi(b)\}, \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.138}$$

Consequently, (3.127) may be used with α replaced by β and C_1 replaced by $2^{1/\beta}$. This yields (3.129) and completes the proof of (I).

As regards part (II), note that based on (3.130), (3.107), and (3.103), for each pair $(a, b) \in G^{(2)}$ we may write

$$\begin{aligned} \psi(a * b) &\leq C \psi_\alpha(a * b) \leq C 2^{1/\alpha} \max\{\psi_\alpha(a), \psi_\alpha(b)\} \\ &\leq C 2^{1/\alpha} \max\{\psi(a), \psi(b)\}, \end{aligned} \tag{3.139}$$

and hence ψ satisfies a quasisubadditivity condition with $C_1 := C 2^{1/\alpha}$.

Finally, concerning part (III), first observe that, by design, $C_\psi \geq 1$ (as seen by taking $N = 1$ in (3.131) and (3.132)). Next, if $\alpha = +\infty$, then we have $C_1 = 1$ and, hence, $\psi(a * b) \leq \max\{\psi(a), \psi(b)\}$ for every $(a, b) \in G^{(2)}$. In turn, this and (3.132) imply that $C_\psi \leq 1$; thus, ultimately, $C_\psi = 1$ in this case, and all desired conclusions follow (from earlier results). Consider now the case $\alpha < +\infty$. Then inequality (3.135) entails $C_\psi \leq 4^{1/\alpha} = C_1^2$. This proves the first double inequality in (3.133). In addition, from definitions we have that, for every $a \in G$,

$$\psi(a) \leq C_\psi \left(\sum_{i=1}^{N} \psi(a_i)^\alpha \right)^{1/\alpha} \tag{3.140}$$

for all $N \in \mathbb{N}$ and all $(a_1, \dots, a_N) \in G^{(N)}$ with $a_1 * \cdots * a_N = a$. Taking the infimum over all such choices we arrive at the conclusion that $\psi \leq C_\psi \psi_\alpha$ on G. Since the inequality $\psi_\alpha \leq \psi$ on G has already been noted, this completes the proof of (3.133). Lastly, if $C_\psi = 1$, then the fact that ψ is α-subadditive follows from (3.140), whereas the converse is a consequence of (3.131) and (3.132). □

Remark 3.18. In the context of the first part of Theorem 3.17, if in place of the quasisubadditivity condition (3.125) the function $\psi : G \to [0, +\infty]$ is assumed to have the property that there exist $p \in (0, +\infty)$ and $C \geq 2^{-1/p}$ such that

$$\psi(a * b) \leq C(\psi(a)^p + \psi(b)^p)^{1/p}, \quad \text{for all } (a, b) \in G^{(2)}, \tag{3.141}$$

i.e., ψ satisfies a p-quasisubadditivity condition, then

$$\begin{aligned}\psi(a * b) &\leq C\,([\psi(a)]^p + [\psi(b)]^p)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}} C\ \max\{\psi(a), \psi(b)\}, \quad \forall\,(a, b) \in G^{(2)},\end{aligned} \tag{3.142}$$

which means that estimate (3.125) holds for the choice $C_1 := 2^{\frac{1}{p}} C$. Note that we have $\left(\log_2[2^{\frac{1}{p}} C]\right)^{-1} = p/(1 + p \log_2 C)$. Hence, part (I) in Theorem 3.17 gives that $\psi \approx \psi_\alpha$, where, instead of (3.126), this time we take α to be

$$\frac{p}{1 + p\ \log_2 C} \in (0, +\infty]. \tag{3.143}$$

At this stage, the goal is to define a notion of symmetry and quasisymmetry for a function ψ defined on a semigroupoid G with respect to a given involution of G and study how the regularizations of ψ (in the sense of Definition 3.13) relate to this property. We begin by making the following definition.

Definition 3.19. (i) Let G be an arbitrary, nonempty set. Call a given function $\iota : G \to G$ an `involution` if

$$\iota(\iota(a)) = a, \qquad \forall\, a \in G. \tag{3.144}$$

(ii) If $(G, *)$ is a semigroupoid, then call ι a `multiplicative involution` of G if $\iota : G \to G$ is an involution and if

$$(a, b) \in G^{(2)} \implies (\iota(b), \iota(a)) \in G^{(2)} \text{ and } \iota(a * b) = \iota(b) * \iota(a). \tag{3.145}$$

We are now ready to define the notions of symmetry and quasisymmetry alluded to earlier.

Definition 3.20. Let G be an arbitrary, nonempty set, and assume that $\iota : G \to G$ is an involution. Call a given function $\psi : G \to \mathbb{R}$ `symmetric with respect to` ι provided

$$\psi(\iota(a)) = \psi(a), \qquad \forall\, a \in G, \tag{3.146}$$

and call ψ `quasisymmetric with respect to` ι provided

$$\exists\, C_0 \geq 0 \text{ such that } \psi(\iota(a)) \leq C_0\, \psi(a), \qquad \forall\, a \in G. \tag{3.147}$$

Parenthetically we wish to note that, in the context of (3.147), if the function ψ is not identically zero, then necessarily $C_0 \geq 1$. Indeed, if $a \in G$ is such that $\psi(a) > 0$, then $\psi(a) = \psi(\iota(\iota(a))) \leq C_0\, \psi(\iota(a)) \leq C_0^2\, \psi(a)$, and the desired conclusion follows.

In the next lemma we introduce a certain symmetrization procedure of an arbitrary function and study some of its most basic properties.

Lemma 3.21. *Suppose that G is an arbitrary, nonempty set and that $\iota : G \to G$ is an involution. Given a function $\psi : G \to [0, +\infty]$, consider $\psi_\iota : G \to [0, +\infty]$, its max symmetrization relative to ι, defined by*

$$\psi_\iota(a) := \max\{\psi(a), \psi(\iota(a))\}, \quad \forall\, a \in G. \tag{3.148}$$

Then the following properties hold.

(1) The function ψ_ι is symmetric with respect to ι.
(2) The function ψ is symmetric with respect to ι if and only if $\psi = \psi_\iota$.
(3) One has $\psi \approx \psi_\iota$ if and only if ψ is quasisymmetric with respect to ι.
(4) The function ψ is quasisymmetric with respect to ι if and only if there exists $\psi' : G \to [0, +\infty]$ that is symmetric with respect to ι and such that $\psi \approx \psi'$.
(5) The function ψ_ι may characterized as the smallest $[0, +\infty]$-valued function defined on G that is symmetric with respect to ι and that is $\geq \psi$ pointwise on G.
(6) Assume that $(G, *)$ is a semigroupoid and that ι is a multiplicative involution on G. Then, if ψ is quasisubadditive on G, then it follows that so is ψ_ι, and with the same constant. More precisely, if ψ satisfies (3.125) for some finite constant $C_1 \geq 1$, then also

$$\psi_\iota(a * b) \leq C_1 \max\{\psi_\iota(a)\psi_\iota(b)\}, \quad \textit{for all} \ \ (a, b) \in G^{(2)}. \tag{3.149}$$

Proof. Since ι is an involution, for every $a \in G$ we may write

$$\begin{aligned} \psi_\iota(\iota(a)) &= \max\,\{\psi(\iota(a)), \psi(\iota(\iota(a)))\} \\ &= \max\,\{\psi(\iota(a)), \psi(a)\} = \psi_\iota(a). \end{aligned} \tag{3.150}$$

Hence ψ_ι is symmetric with respect to ι, proving (1). Then (2) is an immediate consequence of (1) and (3.148). As regards (3), note that if ψ satisfies the quasisymmetry condition (3.147), then

$$\psi(a) \leq \psi_\iota(a) \leq \max\{1, C_0\}\,\psi(a), \qquad \forall\, a \in G, \tag{3.151}$$

which shows that $\psi \approx \psi_\iota$. In the opposite direction, if $\psi \approx \psi_\iota$, then, since ψ_ι is symmetric with respect to ι, there exist two finite constants $C, C' \geq 0$ such that

$$\psi(\iota(a)) \leq C\,\psi_\iota(\iota(a)) = C\,\psi_\iota(a) \leq C'\psi(a), \quad \forall\, a \in G. \tag{3.152}$$

Hence ψ is quasisymmetric with respect to ι, completing the proof of (3). Going further, the right-pointing implication in (4) follows from (1) and (3), whereas the left-pointing implication in (4) is handled much as in (3.152). This justifies (4).

To prove the claim in part (5), observe that, on the one hand, ψ_ι is symmetric with respect to ι and satisfies $\psi_\iota \geq \psi$ on G. On the other hand, given a function $\psi' : G \to [0, +\infty]$ that is symmetric with respect to ι and satisfies $\psi'(a) \geq \psi(a)$ for every $a \in G$, we have $\psi'(a) = \psi'(\iota(a)) \geq \psi(\iota(a))$ for every $a \in G$. Thus, ultimately, $\psi'(a) \geq \max\{\psi(a), \psi(\iota(a))\} = \psi_\iota(a)$ for all $a \in G$. Thus, $\psi' \geq \psi_\iota$ on G, as desired.

Finally, as concerns (6), if ψ satisfies the quasisubadditivity condition (3.125) for some finite constant $C_1 \geq 1$, then for every $(a, b) \in G^{(2)}$ we may write

$$\begin{aligned}\psi_\iota(a * b) &= \max\{\psi(a * b), \psi(\iota(a * b))\} = \max\{\psi(a * b), \psi(\iota(b) * \iota(a))\}\\ &\leq C_1 \max\{\max\{\psi(a), \psi(b)\}, \max\{\psi(\iota(b)), \psi(\iota(a))\}\}\\ &= C_1 \max\{\max\{\psi(a), \psi(\iota(a))\}, \max\{\psi(b), \psi(\iota(b))\}\}\\ &= C_1 \max\{\psi_\iota(a), \psi_\iota(b)\}. \end{aligned} \tag{3.153}$$

As a result, ψ_ι satisfies the same estimate as ψ in (3.125) and, as claimed, with the same constant C_1 as in (3.125). This completes the proof of the lemma. □

We next discuss how the regularization procedure from Definition 3.13 interacts with the notions of symmetry and quasisymmetry from Definition 3.20.

Lemma 3.22. *Assume that $(G, *)$ is a semigroupoid and that $\iota : G \to G$ is a multiplicative involution on G. Also, fix an arbitrary index $\alpha \in (0, +\infty]$. Then for any function $\psi : G \to [0, +\infty]$ the following claims are valid.*

(1) There holds $\psi_\alpha \circ \iota = (\psi \circ \iota)_\alpha$ on G. In particular, if ψ is symmetric with respect to ι, then so is ψ_α.
(2) If ψ is quasisymmetric with respect to ι, then so is ψ_α. More precisely, if $C_0 \geq 0$ is such that $\psi \circ \iota \leq C_0\,\psi$ on G, then $\psi_\alpha \circ \iota \leq C_0\,\psi_\alpha$ on G.

Proof. We will treat in detail the case when $\alpha < +\infty$ since the argument for $\alpha = +\infty$ is similar. Thus, assuming that $\alpha < +\infty$, for every $a \in G$ we have

$$\begin{aligned}\psi_\alpha(\iota(a)) &= \inf\left\{\left(\sum_{i=1}^N \psi(a_i)^\alpha\right)^{\frac{1}{\alpha}} : N \in \mathbb{N},\ (a_1, \dots, a_N) \in G^{(N)},\ \iota(a) = a_1 * \dots * a_N\right\}\\ &= \inf\left\{\left(\sum_{i=1}^N (\psi \circ \iota)(\iota(a_i))^\alpha\right)^{\frac{1}{\alpha}} : N \in \mathbb{N},\ (a_1, \dots, a_N) \in G^{(N)},\ a = \iota(a_N) * \dots * \iota(a_1)\right\}\\ &= \inf\left\{\left(\sum_{i=1}^N (\psi \circ \iota)(b_i)^\alpha\right)^{\frac{1}{\alpha}} : N \in \mathbb{N},\ (b_1, \dots, b_N) \in G^{(N)},\ a = b_1 * \dots * b_N\right\}\\ &= (\psi \circ \iota)_\alpha(a), \end{aligned} \tag{3.154}$$

proving (1). Next, if ψ is quasisymmetric with respect to ι, then there exists $C_0 \geq 0$ such that $\psi \circ \iota \leq C_0\, \psi$ on G. In light of (3.103) and (3.104), and given what we have proved so far, this entails

$$\psi_\alpha \circ \iota = (\psi \circ \iota)_\alpha \leq (C_0\, \psi)_\alpha = C_0\, \psi_\alpha, \tag{3.155}$$

proving (2). The proof of the lemma is therefore complete. □

Our next result is a version of Theorem 3.17 in which, given a quasisubadditive function ψ defined on a semigroupoid G, we seek to construct an equivalent, regularized version of it that is symmetric with respect to some background multiplicative involution on G. As property (4) of Lemma 3.21 shows, for this type of conclusion it is actually necessary to assume that ψ is quasisymmetric with respect to the named involution.

Theorem 3.23. *Let $(G, *)$ be a semigroupoid, and suppose that ι is a multiplicative involution on G. Also, assume that $\psi : G \to [0, +\infty]$ is a function that is quasisymmetric with respect to the involution ι and has the property that there exists a finite constant $C_1 \geq 1$ such that (3.125) holds. Finally, define the exponent $\alpha \in (0, +\infty]$ as in (3.126).*

Then the function $(\psi_\iota)_\alpha$, defined according to the recipe from Definition 3.13 (but using ψ_ι in place of ψ), satisfies the following properties.

(i) *$(\psi_\iota)_\alpha$ is symmetric with respect to the involution ι.*
(ii) *$(\psi_\iota)_\alpha \approx \psi$. More specifically, with C_1 the same constant as in (3.125) and with C_0 the constant appearing in (3.147), one has*

$$C_1^{-2}\psi \leq (\psi_\iota)_\alpha \leq \max\{1, C_0\}\,\psi \qquad \text{on } G. \tag{3.156}$$

Consequently,

$$(\psi_\iota)_\alpha^{-1}(\{0\}) = \psi^{-1}(\{0\}). \tag{3.157}$$

(iii) *For each $\beta \in (0, \alpha]$ the function $(\psi_\iota)_\alpha$ is β-subadditive. That is, one has (with a natural interpretation when $\beta = \alpha = +\infty$)*

$$(\psi_\iota)_\alpha(a * b) \leq \big([(\psi_\iota)_\alpha(a)]^\beta + [(\psi_\iota)_\alpha(b)]^\beta\big)^{\frac{1}{\beta}}, \ \forall\, (a, b) \in G^{(2)}. \tag{3.158}$$

(iv) *The function $(\psi_\iota)_\alpha$ is quasisubadditive. More precisely, for the same constant C_1 as in (3.125) one has*

$$(\psi_\iota)_\alpha(a * b) \leq C_1 \max\{(\psi_\iota)_\alpha(a), (\psi_\iota)_\alpha(b)\}, \ \forall\, (a, b) \in G^{(2)}. \tag{3.159}$$

Proof. From (1) and (6) in Lemma 3.21 we know that ψ_ι is symmetric with respect to ι and that it satisfies the same estimate as ψ in (3.125), with the same constant

C_1. Based on this, claim (1) in Lemma 3.22 then gives that $(\psi_\iota)_\alpha$ is also symmetric with respect to ι, as claimed in (i). Also,

$$C_1^{-2}\psi \leq C_1^{-2}\psi_\iota \leq (\psi_\iota)_\alpha \leq \psi_\iota \leq \max\{1, C_0\}\,\psi \quad \text{on } G, \tag{3.160}$$

by (3.127) and (3.151). This proves (3.156) in (ii). Finally, (iii) is a direct consequence of property (3) in Lemma 3.14 while (iv) is clear from property (4) in Lemma 3.14. □

For the remainder of this subsection we will strengthen the working algebraic assumptions by considering the case when the semigroupoid G is actually a groupoid (cf. Definition 2.17). In this scenario, G is equipped with a natural multiplicative involution, namely, the function defined by $\iota(a) := a^{-1}$, $a \in G$ [cf. (2.64)], with respect to which we will adapt our earlier notions of symmetry and quasisymmetry. Most importantly, however, the "cancellation" condition implicit in (2.67) permits us to manufacture a partially defined distance (cf. Definition 2.68) on G that is left- (or right-) invariant under the groupoid multiplication, each time we are given a nonnegative function ψ on G that is symmetric with respect to ι, satisfies a nondegeneracy condition (cf. (3.188)) and is quasisubadditive (cf. (3.186)).

Theorem 3.24. *Let* $(G, *, (\cdot)^{-1})$ *be a groupoid, and suppose that* $\psi : G \to [0, +\infty]$ *is a function for which there exists* $C_1 \geq 1$ *such that* (3.125) *holds. Define* α *as in* (3.126)*, and construct* ψ_α *as in Definition 3.13. Finally, recall* $\mathcal{G}^{\mathrm{R}}$ *from* (2.69)*, and fix a finite number* $\beta \in (0, \alpha]$. *Then the function*

$$d^{\mathrm{R}}_{\psi,\beta} : \mathcal{G}^{\mathrm{R}} \to [0, +\infty], \qquad d^{\mathrm{R}}_{\psi,\beta}(a, b) := \left[\psi_\alpha(a * b^{-1})\right]^\beta, \ \forall\, (a, b) \in \mathcal{G}^{\mathrm{R}}, \tag{3.161}$$

satisfies the following properties.

(i) $d^{\mathrm{R}}_{\psi,\beta}$ *satisfies the triangle inequality (in the sense of* (3) *in Definition 2.68).*
(ii) $d^{\mathrm{R}}_{\psi,\beta}$ *is right-invariant in the following sense:*

$$\begin{aligned} &(a, b) \in \mathcal{G}^{\mathrm{R}} \text{ and } c \in G \text{ such that } (a, c), (b, c) \in G^{(2)} \\ &\Longrightarrow (a * c, b * c) \in \mathcal{G}^{\mathrm{R}} \text{ and } d^{\mathrm{R}}_{\psi,\beta}(a * c, b * c) = d^{\mathrm{R}}_{\psi,\beta}(a, b). \end{aligned} \tag{3.162}$$

(iii) If ψ *is quasisymmetric with respect to the multiplicative involution* $\iota := (\cdot)^{-1}$ *on* G*, i.e., there exists* $C_0 \geq 0$ *for which*

$$\psi(a^{-1}) \leq C_0\, \psi(a), \qquad \forall\, a \in G, \tag{3.163}$$

then

$$d^{\mathrm{R}}_{\psi,\beta}(b, a) \leq C_0^\beta\, d^{\mathrm{R}}_{\psi,\beta}(a, b) \quad \textit{for all } (a, b) \in \mathcal{G}^{\mathrm{R}}. \tag{3.164}$$

In fact,

$$\textit{if } \psi \textit{ is symmetric with respect to } \iota, \textit{ then } d^{\rm R}_{\psi,\beta} \textit{ is symmetric} \tag{3.165}$$

in the sense of (2) *in Definition 2.68.*

(iv) *One has*

$$C_1^{-2}\psi(a*b^{-1}) \le \Big[d^{\rm R}_{\psi,\beta}(a,b)\Big]^{1/\beta} \le \psi(a*b^{-1}), \quad \forall\,(a,b)\in\mathcal{G}^{\rm R}. \tag{3.166}$$

(v) *With $G^{(0)}$ denoting the unit space of the groupoid G, one has*

$$\begin{gathered}\psi^{-1}(\{0\}) = G^{(0)} \Longrightarrow d^{\rm R}_{\psi,\beta} \textit{ is nondegenerate} \Longrightarrow G^{(0)} \subseteq \psi^{-1}(\{0\}),\\ \textit{and}\quad G^{(0)} \subseteq \psi^{-1}(\{0\}) \Longleftrightarrow d^{\rm R}_{\psi,\beta} \textit{ is pseudo-nondegenerate,}\end{gathered} \tag{3.167}$$

with the notions of nondegeneracy and pseudo-nondegeneracy understood, respectively, in the sense of (1) *and* (1′) *in Definition 2.68.*

(vi) *If ψ is symmetric with respect to $\iota := (\cdot)^{-1}$ and $G^{(0)} \subseteq \psi^{-1}(\{0\})$, then $d^{\rm R}_{\psi,\beta}$ is a partially defined pseudodistance on G, with domain $\mathcal{G}^{\rm R}$, and has the property that the topology induced by $d^{\rm R}_{\psi,\beta}$ on G (in the sense of Definition 2.69) coincides with $\tau^{\rm R}_\psi$ (the right-topology induced by ψ on G, defined in Definition 2.62). Furthermore, if in fact $G^{(0)} = \psi^{-1}(\{0\})$, then $d^{\rm R}_{\psi,\beta}$ is actually a partially defined distance on G.*

Finally, analogous results are valid for

$$d^{\rm L}_{\psi,\beta} : \mathcal{G}^{\rm L} \to [0,+\infty], \qquad d^{\rm L}_{\psi,\beta}(a,b) := \big[\psi_\alpha(a^{-1}*b)\big]^\beta, \ \forall\,(a,b)\in\mathcal{G}^{\rm L}. \tag{3.168}$$

Proof. Assume that $(a,c),(c,b) \in \mathcal{G}^{\rm R}$, and note that, from (2.71), this entails $(a,b)\in\mathcal{G}^{\rm R}$. Moreover, $a*b^{-1} = (a*c^{-1})*(c*b^{-1})$ by (2.67). Consequently, using this, (3.161), and (3.105) we may write

$$\begin{aligned} d^{\rm R}_{\psi,\beta}(a,b) &= \big[\psi_\alpha(a*b^{-1})\big]^\beta = \big[\psi_\alpha\big((a*c^{-1})*(c*b^{-1})\big)\big]^\beta \\ &\le \big[\psi_\alpha(a*c^{-1})\big]^\beta + \big[\psi_\alpha(c*b^{-1})\big]^\beta \\ &= d^{\rm R}_{\psi,\beta}(a,c) + d^{\rm R}_{\psi,\beta}(c,b). \end{aligned} \tag{3.169}$$

This proves (i). Now let $(a,b)\in\mathcal{G}^{\rm R}$ and $c\in G$ such that $(a,c),(b,c)\in G^{(2)}$. Then, by (2.65) and (2.67), it follows that $(a*c,b*c)\in\mathcal{G}^{\rm R}$ and $(a*c)*(b*c)^{-1} = a*b^{-1}$, so that $d^{\rm R}_{\psi,\beta}(a*c,b*c) = d^{\rm R}_{\psi}(a,b)$, proving (ii).

Suppose next that ψ satisfies (3.163). If $(a, b) \in \mathcal{G}^{\mathrm{R}}$, then $(b, a) \in \mathcal{G}^{\mathrm{R}}$ by (2.71) and

$$\begin{aligned} d^{\mathrm{R}}_{\psi,\beta}(b, a) &= \left[\psi_\alpha(b * a^{-1})\right]^\beta = \left[(\psi_\alpha \circ \iota)(a * b^{-1})\right]^\beta \\ &\leq C_0^\beta \left[\psi_\alpha(a * b^{-1})\right]^\beta = C_0^\beta \, d^{\mathrm{R}}_{\psi,\beta}(a, b), \end{aligned} \tag{3.170}$$

where the inequality uses claim (2) in Lemma 3.22. This proves (3.164). If ψ is actually symmetric with respect to ι, then from claim (1) in Lemma 3.22 we deduce that ψ_α is also symmetric with respect to ι. Granted this and keeping in mind (2.65), for every $(a, b) \in \mathcal{G}^{\mathrm{R}}$ we may write

$$\begin{aligned} d^{\mathrm{R}}_{\psi,\beta}(b, a) &= \left[\psi_\alpha(b * a^{-1})\right]^\beta = \left[(\psi_\alpha \circ \iota)(a * b^{-1})\right]^\alpha \\ &= \left[\psi_\alpha(a * b^{-1})\right]^\beta = d^{\mathrm{R}}_{\psi,\beta}(a, b), \end{aligned} \tag{3.171}$$

completing the proof of (iii). Going further, (iv) follows directly from (3.127).

As regards (v), note that if $(a, b) \in \mathcal{G}^{\mathrm{R}}$, then by (3.166),

$$d^{\mathrm{R}}_{\psi,\beta}(a, b) = 0 \iff \psi(a * b^{-1}) = 0 \iff a * b^{-1} \in \psi^{-1}(\{0\}). \tag{3.172}$$

Hence, if $\psi^{-1}(\{0\}) \subseteq G^{(0)}$, then the last condition in (3.172) forces $a = b$, by virtue of (2.73). On the other hand, the double inequality in (3.166) gives that

$$C_1^{-2}\psi(a * a^{-1}) \leq \left[d^{\mathrm{R}}_{\psi,\beta}(a, a)\right]^{1/\beta} \leq \psi(a * a^{-1}), \quad \forall\, a \in G, \tag{3.173}$$

$$C_1^{-2}\psi(a) \leq \left[d^{\mathrm{R}}_{\psi,\beta}(a, a)\right]^{1/\beta} \leq \psi(a), \quad \forall\, a \in G^{(0)}. \tag{3.174}$$

In addition, $a * a^{-1} \in G^{(0)}$ for every $a \in G$. With these in hand, all implications displayed in (3.167) follow easily.

Finally, the claims in (vi) are direct consequences of (i), (iii), and (v), as well as (3.166), Definitions 2.62, and 2.69 (which, collectively, show that the topology induced by $d^{\mathrm{R}}_{\psi,\beta}$ on G coincides with τ^{R}_ψ). □

Our last result in this section pertains to the (Hölder-type) regularity of the regularization (in the sense of Definition 3.13) of a nonnegative function defined on a groupoid that is symmetric with respect to the groupoid inversion and is quasisubadditive and nondegenerate as well.

Theorem 3.25. *Let $(G, *, (\cdot)^{-1})$ be a groupoid, and recall the set $\mathcal{G}^{\mathrm{R}}$ introduced in* (2.69)*. Also, assume that $\psi : G \to [0, +\infty)$ is a function satisfying the following two properties:*

$$\textit{There exists } C_1 \in [1, +\infty) \textit{ such that } (3.125) \textit{ holds, and} \tag{3.175}$$

$$\textit{the function } \psi \textit{ is symmetric with respect to } \iota := (\cdot)^{-1}. \tag{3.176}$$

Finally, define α as in (3.126). Then for each exponent

$$\beta \in (0, \min\{1, \alpha\}] \tag{3.177}$$

the function ψ_α, constructed as in Definition 3.13, satisfies the following local Hölder regularity condition of order β: for each $r > 0$ and $(a,c),(c,b) \in \mathcal{G}^{\mathrm{R}}$ such that

$$\max\left\{\psi_\alpha(a * b^{-1}), \psi_\alpha(a * c^{-1})\right\} \leq r, \tag{3.178}$$

one has

$$\left|\psi_\alpha(a * b^{-1}) - \psi_\alpha(a * c^{-1})\right| \leq \tfrac{1}{\beta}\, r^{1-\beta} \left[\psi_\alpha(c * b^{-1})\right]^\beta. \tag{3.179}$$

If, on the other hand, $1 < \beta \leq \alpha$, then (3.179) holds whenever $(a,c),(c,b) \in \mathcal{G}^{\mathrm{R}}$ and $r > 0$ are such that

$$\min\left\{\psi_\alpha(a * b^{-1}), \psi_\alpha(a * c^{-1})\right\} \geq r. \tag{3.180}$$

Proof. Fix α as in (3.126), and assume that β is as in (3.177). Let $(a,c),(c,b) \in \mathcal{G}^{\mathrm{R}}$ be arbitrary. Then, by (2.71), we also have that $(a,b),(b,c) \in \mathcal{G}^{\mathrm{R}}$. The triangle inequality and the symmetry condition for $d^{\mathrm{R}}_{\psi,\beta}$ then yield

$$\begin{aligned} &d^{\mathrm{R}}_{\psi,\beta}(a,b) \leq d^{\mathrm{R}}_{\psi,\beta}(a,c) + d^{\mathrm{R}}_{\psi,\beta}(c,b) \text{ and} \\ &d^{\mathrm{R}}_{\psi,\beta}(a,c) \leq d^{\mathrm{R}}_{\psi,\beta}(a,b) + d^{\mathrm{R}}_{\psi,\beta}(b,c) = d^{\mathrm{R}}_{\psi,\beta}(a,b) + d^{\mathrm{R}}_{\psi,\beta}(c,b). \end{aligned} \tag{3.181}$$

In concert, the inequalities in (3.181) further imply

$$\left|d^{\mathrm{R}}_{\psi,\beta}(a,b) - d^{\mathrm{R}}_{\psi,\beta}(a,c)\right| \leq d^{\mathrm{R}}_{\psi,\beta}(c,b). \tag{3.182}$$

An elementary fact that is useful in this context is the estimate

$$|x^\gamma - y^\gamma| \leq \gamma\, |x - y|\, [\max\{x, y\}]^{\gamma - 1} \quad \text{if } x, y \in (0, +\infty) \text{ and } \gamma \geq 1. \tag{3.183}$$

Writing (3.183) for $x := \left[\psi_\alpha(a * b^{-1})\right]^\beta$, $y := \left[\psi_\alpha(a * c^{-1})\right]^\beta$ (analyzing separately the case when one of these numbers vanishes) and $\gamma := 1/\beta \geq 1$ yields

$$\begin{aligned} &\left|\psi_\alpha(a * b^{-1}) - \psi_\alpha(a * c^{-1})\right| \\ &\quad\leq \tfrac{1}{\beta}\left|d^{\mathrm{R}}_{\psi,\beta}(a,b) - d^{\mathrm{R}}_{\psi,\beta}(a,c)\right| \left[\max\left\{d^{\mathrm{R}}_{\psi,\beta}(a,b), d^{\mathrm{R}}_{\psi,\beta}(a,c)\right\}\right]^{\frac{1}{\beta}-1} \\ &\quad\leq \tfrac{1}{\beta}\, d^{\mathrm{R}}_{\psi,\beta}(c,b) \left[\max\left\{\psi_\alpha(a * b^{-1}), \psi_\alpha(a * c^{-1})\right\}\right]^{1-\beta} \\ &\quad= \tfrac{1}{\beta}\left[\psi_\alpha(c * b^{-1})\right]^\beta \left[\max\left\{\psi_\alpha(a * b^{-1}), \psi_\alpha(a * c^{-1})\right\}\right]^{1-\beta}, \end{aligned} \tag{3.184}$$

where for the second inequality in (3.184) we have used (3.182). Based on this, (3.179) follows whenever (3.178) holds. Finally, the last claim in the statement of the theorem is proved in a similar fashion, this time making use of the fact that

$$|x^\gamma - y^\gamma| \leq \gamma\, |x-y|\, [\min\{x,y\}]^{\gamma-1} \qquad \text{if } x,y \in (0,+\infty) \text{ and } \gamma \in (0,1). \tag{3.185}$$

This completes the proof of the theorem. □

3.2 Main Groupoid Metrization Theorem

This section is devoted to presenting the most complete statement of the groupoid metrization theorem (GMT) (summarily reviewed in Theorem 1.5) and to providing the proof of this basic result. Later on (Sect. 3.3), we will indicate how a significant portion of this GMT continues to hold in the more general setting of semigroupoids.

3.2.1 Formulation of Groupoid Metrization Theorem

Strictly speaking, the notation employed in the formulation of the GMT below was already introduced in Sects. 2.1 and 3.1. Nonetheless, in an effort to make its statement as self-contained as possible, we will occasionally incorporate definitions of such notions/symbols once more into the actual body of the theorem.

Theorem 3.26 (Groupoid metrization theorem). *Let $(G, \ast, (\cdot)^{-1})$ be a groupoid, with partial multiplication $\ast$, inverse operation $(\cdot)^{-1}$, and unit space $G^{(0)}$. Furthermore, introduce $\mathcal{G}^{\mathrm{R}} := \{(a,b) \in G \times G : (a, b^{-1}) \in G^{(2)}\}$.*

Next, assume that $\psi : G \to [0,+\infty)$ is a function for which there exist two finite constants $C_0 \geq 0$ and $C_1 \geq 1$ such that the following properties hold:

- *quasisubadditivity:* $\psi(a \ast b) \leq C_1 \max\{\psi(a), \psi(b)\}$ *for all* $(a,b) \in G^{(2)}$; (3.186)
- *quasisymmetry:* $\psi(a^{-1}) \leq C_0\, \psi(a)$ *for every* $a \in G$; (3.187)
- *nondegeneracy:* $a \in G$ *and* $\psi(a) = 0 \Leftrightarrow a \in G^{(0)}$, *i.e.,* $\psi^{-1}(\{0\}) = G^{(0)}$. (3.188)

Denote by τ^{R}_ψ the right-topology induced by ψ on G, defined as the largest topology on G with the property that for any element $a \in G$ a fundamental system of neighborhoods is given by $\{B^{\mathrm{R}}_\psi(a,r)\}_{r>0}$, where for each $r \in (0,+\infty)$

$$B^{\mathrm{R}}_\psi(a,r) := \left\{ b \in G : (a,b) \in \mathcal{G}^{\mathrm{R}} \text{ and } \psi(a \ast b^{-1}) < r \right\}. \tag{3.189}$$

Also, with $C_1 \in [1, +\infty)$ as in (3.186), let

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty]. \tag{3.190}$$

Finally, introduce a symmetrized version of ψ by setting

$$\psi_{\rm sym}(a) := \max\{\psi(a), \psi(a^{-1})\}, \quad \forall\, a \in G, \tag{3.191}$$

and define the canonical regularization $\psi_{\rm reg} : G \to [0, +\infty)$ of ψ by considering for each $a \in G$, in the case $\alpha < +\infty$,

$$\psi_{\rm reg}(a) := \inf\left\{\left(\sum_{i=1}^{N} \psi_{\rm sym}(a_i)^{\alpha}\right)^{\frac{1}{\alpha}} : N \in \mathbb{N}, \right. \\ \left. (a_1, \dots, a_N) \in G^{(N)}, \quad a = a_1 * \cdots * a_N \right\} \tag{3.192}$$

and, corresponding to $\alpha = +\infty$,

$$\psi_{\rm reg}(a) := \inf\left\{ \max_{1 \le i \le N} \psi_{\rm sym}(a_i) : N \in \mathbb{N}, \right. \\ \left. (a_1, \dots, a_N) \in G^{(N)}, \quad a = a_1 * \cdots * a_N \right\}. \tag{3.193}$$

Then the following conclusions hold.

(1) The function $\psi_{\rm reg}$ is symmetric, in the sense that

$$\psi_{\rm reg}(a^{-1}) = \psi_{\rm reg}(a) \quad \textit{for every} \quad a \in G, \tag{3.194}$$

and $\psi_{\rm reg}$ is quasisubadditive, in the precise sense that, with C_1 denoting the same constant as in (3.186), one has

$$\psi_{\rm reg}(a * b) \le C_1 \max\{\psi_{\rm reg}(a), \psi_{\rm reg}(b)\} \quad \textit{for all} \quad (a, b) \in G^{(2)}. \tag{3.195}$$

In other words, $\psi_{\rm reg}$ also satisfies (3.186) and (3.187) with $C_0 = 1$ and the same constant C_1 as for the original function ψ.

(2) One has $\psi_{\rm reg} \approx \psi$ on G so that, in particular,

$$\psi_{\rm reg}^{-1}(\{0\}) = G^{(0)}. \tag{3.196}$$

More precisely, with C_0 and C_1 as in (3.186) and (3.187), there holds

$$C_1^{-2}\psi \le \psi_{\rm reg} \le \max\{1, C_0\}\,\psi \qquad \textit{on} \quad G. \tag{3.197}$$

Furthermore,

$$\alpha = +\infty \ (\textit{i.e.}, C_1 = 1) \implies \psi_{\rm reg} = \psi_{\rm sym}. \tag{3.198}$$

Moreover, α *from* (3.190) *is the optimal exponent ensuring the validity of* $\psi_{\rm reg} \approx \psi$ *on* G.

(3) *For each* $\beta \in (0, \alpha]$ *the function* $\psi_{\rm reg}$ *is* β*-subadditive in the sense that one has (with a natural interpretation when* $\beta = \alpha = +\infty$*)*

$$\psi_{\rm reg}(a * b) \leq \left(\psi_{\rm reg}(a)^\beta + \psi_{\rm reg}(b)^\beta\right)^{\frac{1}{\beta}}, \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.199}$$

In particular, if $C_1 \in [1, 2]$, *then* $\psi_{\rm reg}$ *is subadditive in the sense that*

$$\psi_{\rm reg}(a * b) \leq \psi_{\rm reg}(a) + \psi_{\rm reg}(b), \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.200}$$

Moreover, if for any $\beta \in (0, +\infty]$ *one considers*

$$\mathfrak{R}^{(\beta)}(G) := \{\phi : G \to [0, +\infty] : \phi \textit{ is symmetric and } \beta\textit{-subadditive}\}, \tag{3.201}$$

then the following alternative description of the regularization of ψ *holds:*

$$\psi_{\rm reg} = \sup\left\{\phi : \phi \in \mathfrak{R}^{(\alpha)}(G) \ \textit{ and } \ \phi \leq \psi \ \textit{ on } \ G\right\}. \tag{3.202}$$

(4) *The process that associates to a function* $\psi : G \to [0, +\infty)$ *satisfying* (3.186)–(3.188) *the function* $\psi_{\rm reg}$ *defined as in* (3.190)–(3.192) *(which, as* (1) *and* (2) *above show, can be iterated) is idempotent in the sense that*

$$\left(\psi_{\rm reg}\right)_{\rm reg} = \psi_{\rm reg}. \tag{3.203}$$

Furthermore, this regularization process is invariant under groupoid isomorphisms in the sense that if $\widetilde{G}$ *is another groupoid and* $\phi : \widetilde{G} \to G$ *is a groupoid isomorphism, then* $\psi \circ \phi : \widetilde{G} \to [0, +\infty)$ *is also quasisubadditive, quasisymmetric, and nondegenerate, and*

$$(\psi \circ \phi)_{\rm reg} = \psi_{\rm reg} \circ \phi. \tag{3.204}$$

(5) *For each finite number* $\beta \in (0, \alpha]$ *the function* $\psi_{\rm reg}$ *satisfies the following Hölder-type regularity condition of order* β*:*

$$\left|\psi_{\rm reg}(a) - \psi_{\rm reg}(b)\right| \leq \frac{1}{\beta} \max\left\{\psi_{\rm reg}(a)^{1-\beta}, \psi_{\rm reg}(b)^{1-\beta}\right\} \left[\psi_{\rm reg}(a * b^{-1})\right]^\beta \tag{3.205}$$

whenever $(a, b) \in \mathcal{G}^{\rm R}$ *(with the understanding that when* $\beta \geq 1$, *one also imposes the condition that* $a, b \notin G^{(0)}$*).*

(6) The function $\psi_{\rm reg} : \left(G, \tau_\psi^{\rm R}\right) \to [0, +\infty)$ *is continuous and for every* $a \in G$ *and* $r > 0$ *the right* $\psi_{\rm reg}$*-ball*

$$B^{\rm R}_{\psi_{\rm reg}}(a, r) := \{b \in G : (a, b) \in \mathcal{G}^{\rm R} \text{ and } \psi_{\rm reg}(a * b^{-1}) < r\}$$

is open in the topology $\tau_\psi^{\rm R}$*. As a consequence of this and (3.197),*

$$B^{\rm R}_\psi(a, r) \text{ is a neighborhood of } a \in G \text{ in } \tau_\psi^{\rm R} \text{ for every } r > 0. \tag{3.206}$$

In fact, if $A := 1$ *when* G *reduces to* $G^{(0)}$ *and, otherwise,*

$$A := \sup_{\substack{(a,b)\in G^{(2)} \\ \text{not both } a,b \text{ in } G^{(0)}}} \frac{\psi(a * b)}{\psi(a) + \psi(b)}, \tag{3.207}$$

then it follows that A *is a well-defined number that belongs to* $[1, C_1]$ *and satisfies*

$$B^{\rm R}_\psi(a, r/A) \subseteq \left(B^{\rm R}_\psi(a, r)\right)^\circ, \qquad \forall\, a \in G, \ \forall\, r > 0, \tag{3.208}$$

$$\overline{B^{\rm R}_\psi(a, r/A_o)} \subseteq B^{\rm R}_\psi(a, r) \ \text{ if } \ A_o > C_0^2 A, \quad \forall\, a \in G, \ \forall\, r > 0, \tag{3.209}$$

where the interior $(\ldots)^\circ$ *and closure* $\overline{(\ldots)}$ *are taken with respect to the topology* $\tau_\psi^{\rm R}$.

Furthermore, while $\psi : \left(G, \tau_\psi^{\rm R}\right) \to [0, +\infty)$ *may not be continuous, for any sequence* $\{a_n\}_{n\in\mathbb{N}} \subseteq G$ *that converges to some* $a \in G$ *in the topology* $\tau_\psi^{\rm R}$ *one nonetheless has*

$$\begin{aligned} C_1^{-2} \min\left\{1, C_0^{-1}\right\} \psi(a) &\le \liminf_{n\to\infty} \psi(a_n) \\ &\le \limsup_{n\to\infty} \psi(a_n) \le C_1^2 \max\{1, C_0\}\, \psi(a) \end{aligned} \tag{3.210}$$

and

$$\sup_{n\in\mathbb{N}} \psi(a_n) < +\infty. \tag{3.211}$$

(7) For each finite number $\beta \in (0, \alpha]$ *define the function*

$$d^{\rm R}_{\psi,\beta} : \mathcal{G}^{\rm R} \to [0, +\infty), \qquad d^{\rm R}_{\psi,\beta}(a, b) := \left[\psi_{\rm reg}(a * b^{-1})\right]^\beta, \quad \forall\, (a, b) \in \mathcal{G}^{\rm R}. \tag{3.212}$$

Then $d^{\rm R}_{\psi,\beta}$ *is a partially defined distance on* G *with domain* $\mathcal{G}^{\rm R}$*, i.e., it satisfies the following properties:*

$$\begin{aligned}
&\textit{for any } (a,b) \in \mathcal{G}^{\rm R} \textit{ one has } d^{\rm R}_{\psi,\beta}(a,b) = 0 \textit{ if and only if } a = b;\\
&d^{\rm R}_{\psi,\beta}(a,b) = d^{\rm R}_{\psi,\beta}(b,a) \textit{ for every } (a,b)\in\mathcal{G}^{\rm R};\\
&\textit{if } (a,c),(c,b)\in\mathcal{G}^{\rm R}, \textit{ then } (a,b)\in\mathcal{G}^{\rm R} \textit{ and } d^{\rm R}_{\psi,\beta}(a,b)\leq d^{\rm R}_{\psi,\beta}(a,c)+d^{\rm R}_{\psi,\beta}(c,b).
\end{aligned} \tag{3.213}$$

In the particular case when G is a group, $\mathcal{G}^{\rm R} = G \times G$, and hence $d^{\rm R}_{\psi,\beta}$ becomes a genuine distance on G.

(8) *The partially defined distance $d^{\rm R}_{\psi,\beta}$ introduced in (3.212) is right-invariant in the sense that*

$$(a,b) \in \mathcal{G}^{\rm R} \textit{ and } c \in G \textit{ such that } (a,c),(b,c) \in G^{(2)}$$

$$\Longrightarrow (a*c, b*c) \in \mathcal{G}^{\rm R} \textit{ and } d^{\rm R}_{\psi,\beta}(a*c,b*c) = d^{\rm R}_{\psi,\beta}(a,b). \tag{3.214}$$

Thus, when G is a group, the function $d^{\rm R}_{\psi,\beta}$ is a genuine right-invariant distance on G.

(9) *The topology induced by the partially defined distance $d^{\rm R}_{\psi,\beta}$ on G is precisely $\tau^{\rm R}_\psi$ in the sense that a set $O \subseteq G$ is open in $\tau^{\rm R}_\psi$ if and only if for every element $a \in O$ there exists some finite number $r > 0$ with the property that $\{b \in G : (a,b) \in \mathcal{G}^{\rm R}$ and $d^{\rm R}_{\psi,\beta}(a,b) < r\} \subseteq O$.*

(10) *If $\mathcal{G}^{\rm L} := \{(a,b) \in G \times G : (a^{-1},b) \in G^{(2)}\}$ and for each finite number $\beta \in (0,\alpha]$ one defines the function*

$$d^{\rm L}_{\psi,\beta} : \mathcal{G}^{\rm L} \to [0,+\infty), \qquad d^{\rm L}_{\psi,\beta}(a,b) := \big[\psi_{\rm reg}(a^{-1}*b)\big]^\beta, \quad \forall\,(a,b) \in \mathcal{G}^{\rm L}, \tag{3.215}$$

then $d^{\rm L}_{\psi,\beta}$ is a partially defined distance on G with domain $\mathcal{G}^{\rm L}$, and is left-invariant in the sense that

$$(a,b) \in \mathcal{G}^{\rm L} \textit{ and } c \in G \textit{ such that } (c,a),(c,b) \in G^{(2)}$$

$$\Longrightarrow (c*a, c*b) \in \mathcal{G}^{\rm L} \textit{ and } d^{\rm L}_{\psi,\beta}(c*a,c*b) = d^{\rm L}_{\psi,\beta}(a,b). \tag{3.216}$$

Furthermore, the topology induced by the partially defined distance $d^{\rm L}_{\psi,\beta}$ on G is, in a sense analogous to item (9) above, precisely $\tau^{\rm L}_\psi$, which is the left-topology induced by ψ on G, defined as the largest topology on G with the property that a fundamental system of neighborhoods for any element $a \in G$ is given by $\{B^{\rm L}_\psi(a,r)\}_{r>0}$, where for each $r \in (0,+\infty)$

$$B^{\rm L}_\psi(a,r) := \big\{b \in G : (a,b) \in \mathcal{G}^{\rm L} \textit{ and } \psi(a^{-1}*b) < r\big\}. \tag{3.217}$$

Also, the function $\psi_{\rm reg} : \left(G, \tau_\psi^{\rm L}\right) \to [0,+\infty)$ *is continuous, and for every* $a \in G$ *and* $r > 0$ *the left* $\psi_{\rm reg}$*-ball*

$$B^{\rm L}_{\psi_{\rm reg}}(a,r) := \{b \in G : (a,b) \in \mathcal{G}^{\rm L} \text{ and } \psi_{\rm reg}(a^{-1} * b) < r\}$$

is open in the topology $\tau_\psi^{\rm L}$. *Consequently,* $B^{\rm L}_\psi(a,r)$ *is a neighborhood of* $a \in G$ *in* $\tau_\psi^{\rm R}$ *for every* $r > 0$*, and if* A *is as in* (3.207)*, then*

$$B^{\rm L}_\psi(a, r/A) \subseteq \left(B^{\rm L}_\psi(a,r)\right)^\circ, \qquad \forall\, a \in G,\ \ \forall\, r > 0, \tag{3.218}$$

$$\overline{B^{\rm L}_\psi(a, r/A_o)} \subseteq B^{\rm L}_\psi(a,r) \ \ \text{if}\ \ A_o > C_0^2 A, \quad \forall\, a \in G,\ \ \forall\, r > 0, \tag{3.219}$$

where the interior $(\dots)^\circ$ *and closure* $\overline{(\dots)}$ *are taken with respect to the topology* $\tau_\psi^{\rm L}$. *In addition,* (3.210) *holds whenever* $\{a_n\}_{n\in\mathbb{N}} \subseteq G$ *is a sequence that converges to* $a \in G$ *in the topology* $\tau_\psi^{\rm L}$. *Finally, in the particular case when* G *is a group, the function* $d^{\rm L}_{\psi,\beta}$ *becomes a genuine left-invariant distance on* G.

(11) *The function* $\psi_{\rm reg} : \left(G, \tau_\psi^{\rm L} \vee \tau_\psi^{\rm R}\right) \to [0,+\infty)$ *is continuous, where* $\tau_\psi^{\rm L} \vee \tau_\psi^{\rm R}$ *is the topology on* G *characterized by the property that a fundamental system of neighborhoods for a point* $a \in G$ *is given by* $\{B^{\rm L}_\psi(a,r) \cup B^{\rm R}_\psi(a,r)\}_{r>0}$.

(12) *The Hölder-type regularity result described in part* (5) *above is sharp in the following precise sense. Given* $C_1 > 1$*, there exist a groupoid* $(G, *, (\cdot)^{-1})$ *and a function* $\psi : G \to [0,+\infty)$ *satisfying* (3.186)–(3.188) *for the given* C_1 *and with* $C_0 := 1$*, that has the property that if* $\psi' : G \to [0,+\infty)$ *is such that* $\psi' \approx \psi$ *and there exist* $\beta \in (0,+\infty)$ *and* $C \in [0,+\infty)$ *for which*

$$\left|\psi'(a) - \psi'(b)\right| \le C\ \max\left\{\psi'(a)^{1-\beta}, \psi'(b)^{1-\beta}\right\}\left[\psi'(a * b^{-1})\right]^\beta \tag{3.220}$$

whenever $(a,b) \in \mathcal{G}^{\rm R}$ *[and also* $a, b \notin G^{(0)}$ *if* $\beta \ge 1$*], then necessarily*

$$\beta \le \frac{1}{\log_2 C_1}. \tag{3.221}$$

(13) *If in lieu of* (3.186) *the function* ψ *satisfies the* p*-quasisubadditivity condition*

$$\psi(a * b) \le C_2\left([\psi(a)]^p + [\psi(b)]^p\right)^{1/p} \quad \text{for all}\ \ (a,b) \in G^{(2)} \tag{3.222}$$

for some $p \in (0,+\infty)$ *and some* $C_2 \ge 2^{-1/p}$*, then the same conclusions as in* (1)–(12) *above hold if in place of* (3.190) *one takes* α *to be*

$$\frac{p}{1 + p\ \log_2 C_2} \in (0,+\infty]. \tag{3.223}$$

(14) *With the topologies $\tau^{\mathrm{R}}_{\psi}, \tau^{\mathrm{L}}_{\psi}$ induced by ψ on G being defined as before, the following equivalences hold:*

$$
\begin{aligned}
(G, \tau^{\mathrm{R}}_{\psi}) \ \textit{is a topological groupoid} \iff \tau^{\mathrm{R}}_{\psi} = \tau^{\mathrm{L}}_{\psi} \\
\iff (G, \tau^{\mathrm{L}}_{\psi}) \ \textit{is a topological groupoid.}
\end{aligned}
\tag{3.224}
$$

Thus, if $\tau^{\mathrm{R}}_{\psi} = \tau^{\mathrm{L}}_{\psi} =: \tau_{\psi}$, then (G, τ_{ψ}) is a topological groupoid whose topology may be given by either a left-invariant or a right-invariant partially defined distance. In particular, if G is a group and $\tau^{\mathrm{R}}_{\psi} = \tau^{\mathrm{L}}_{\psi} =: \tau_{\psi}$, then (G, τ_{ψ}) becomes a topological group whose topology may be given by either a left-invariant or a right-invariant (genuine) distance.

Furthermore, if the groupoid G is actually a group and, in addition to the previous hypotheses made on ψ, this function also has the property that there exists a finite constant $C \geq 0$ such that

$$
\psi(a * b) \leq C\,\psi(b * a), \qquad \forall\, a, b \in G, \tag{3.225}
$$

then necessarily $\tau^{\mathrm{L}}_{\psi} = \tau^{\mathrm{R}}_{\psi}$. In particular, if the groupoid G is an Abelian group, then (3.225) always holds (with $C = C_0$, the constant from (3.187)), and hence $(G, \tau^{\mathrm{R}}_{\psi})$ and $(G, \tau^{\mathrm{L}}_{\psi})$ are identical topological groups whose common topology may be given by either a left-invariant or a right-invariant (genuine) distance.

(15) *Continue to assume that $\psi : G \to [0, +\infty)$ satisfies the quasisubadditivity and quasisymmetry properties listed in (3.186) and (3.187), and weaken the nondegeneracy condition (3.188) to just*

$$
\psi(a) = 0 \ \textit{ for each } \ a \in G^{(0)}, \ \textit{i.e., } G^{(0)} \subseteq \psi^{-1}(\{0\}). \tag{3.226}
$$

Then $d^{\mathrm{R}}_{\psi,\beta}$ (defined as in (3.212)) is a partially defined pseudodistance on G with domain $\mathcal{G}^{\mathrm{R}}$ (i.e., (3.213) holds with the first line weakened to $d^{\mathrm{R}}_{\psi,\beta}(a, a) = 0$ for any $a \in G$), which induces the topology τ^{R}_{ψ} on G. Similar properties are valid for $d^{\mathrm{L}}_{\psi,\beta}$.

If, in addition to the background assumptions on ψ imposed so far, one also assumes that G is right-complete with respect to ψ, i.e.,

$$
\left.
\begin{array}{r}
\forall\, (a_n)_{n \in \mathbb{N}} \subseteq G \ \ \textit{with the property that} \\
\forall\, \varepsilon > 0 \ \exists\, N \in \mathbb{N} \ \textit{ such that if } \ n, m \geq N, \\
\textit{then } \ (a_n, a_m) \in \mathcal{G}^{\mathrm{R}} \ \textit{ and } \ \psi(a_n * a_m^{-1}) < \varepsilon
\end{array}
\right\}
\Rightarrow
\left\{
\begin{array}{l}
\exists\, a \in G \ \ \textit{such that} \\
\lim\limits_{n \to \infty} a_n = a \ \ \textit{in} \ \ \tau^{\mathrm{R}}_{\psi},
\end{array}
\right.
\tag{3.227}
$$

then

$$
\textit{the topological space } (G, \tau^{\mathrm{R}}_{\psi}) \textit{ is of second Baire category.} \tag{3.228}
$$

Moreover, if in place of (3.227) it is assumed that G is left-complete with respect to ψ (a concept defined in a similar fashion to (3.227), with natural alterations), then

$$\text{the topological space } (G, \tau_\psi^{\mathrm{L}}) \text{ is of second Baire category.} \tag{3.229}$$

The proof of Theorem 3.26 is presented in Sect. 3.2.2. For now, we wish to comment on the nature of the quantitative assumptions in the formulation of the GMT, specifically on the necessity of the quasisubadditivity estimate (3.186). First, such an assumption is indeed actually necessary in the light of the conclusions in Theorem 3.26. To illustrate this aspect, assume that $(G, *)$ is a group that has a right-invariant distance d^{R}. Then $\psi : G \to [0, +\infty)$ defined at every $a \in G$ by $\psi(a) := d^{\mathrm{R}}(a, e)$, where $e \in G$ is the identity element on G, is subadditive, symmetric, and nondegenerate, and induces (from the right) the same topology as d^{R} on G. More concretely, one has $\tau_\psi^{\mathrm{R}} = \tau_{d^{\mathrm{R}}}$, and, with $(\cdot)^{-1}$ denoting the inverse group operation on G, the following (stronger) counterparts of (3.186)–(3.188) hold:

$$\begin{aligned} &\psi(a * b) \le \psi(a) + \psi(b) \le 2 \max\{\psi(a), \psi(b)\}, \quad \forall\, a, b \in G, \\ &\psi(a^{-1}) = \psi(a), \quad \forall\, a \in G, \quad \text{and} \quad \psi^{-1}(\{0\}) = \{e\}. \end{aligned} \tag{3.230}$$

Second, we wish to emphasize that there are many instances when conditions such as (3.186)–(3.188) arise naturally, and whenever this is the case, Theorem 3.26 yields a useful metrization procedure. Indeed, as demonstrated in later chapters, given a groupoid $(G, *, (\cdot)^{-1})$ along with an arbitrary function $\psi : G \to [0, +\infty)$, its regularization ψ_{reg}, defined as in (3.192) and (3.193) for an arbitrary $\alpha \in [0, +\infty)$, always satisfies (3.186)–(3.188) with $C_0 = 1$ and $C_1 := 2^{1/\alpha}$. The crux of the matter is that in the case when $\psi : G \to [0, +\infty)$ does satisfy (3.186)–(3.188) to begin with, it follows that its canonical regularization ψ_{reg}, defined as in (3.192) and (3.193) for α as in (3.190), is actually equivalent to the original function ψ (in the sense that it is pointwise dominated by a fixed multiple of the other). Thus, in such a scenario, the regularization procedure takes place in such a way that one is able to retain control of the size of the resulting regularized function, relative to the original one (cf. (3.197)). Remarkably, as we will later show in Proposition 3.31, the value of the exponent α in (3.190) is optimal as far as the validity of an estimate like (3.197) is concerned.

3.2.2 Proof of GMT

Here, the goal is to present the proof of our main result pertaining to the metrization of groupoids, i.e., the

Proof of Theorem 3.26 The claims in conclusion (1) are consequences of (i) and (iv) in Theorem 3.23, used in the case when $\iota := (\cdot)^{-1}$. With the exception of (3.198), which is a corollary of (5) in Lemma 3.14 and (6) in Lemma 3.21, conclusion (2) follows from (ii) in Theorem 3.23 (again, when $\iota := (\cdot)^{-1}$). Next, estimate (3.199) in conclusion (3) is a direct consequence of (3.105) and (3.106). Let us also observe here that if $C_1 \in [1, 2]$, then $\alpha \in [1, +\infty]$, and hence we may take $\beta := 1$ in (3.199) to obtain (3.200). To justify (3.202), note that, thanks to what we have proved up to this point, $\psi_{\rm reg} \in \mathfrak{R}^{(\alpha)}(G)$. Thus, on the one hand, $\psi_{\rm reg}$ is pointwise dominated by the supremum on the right-hand side of (3.202). On the other hand, if $\phi \in \mathfrak{R}^{(\alpha)}(G)$, then $\phi \leq \psi \leq \psi_{\rm sym}$ on G, and for any $(a_1, \dots, a_N) \in G^{(N)}$ we may write

$$\phi(a_1 * \cdots * a_N) \leq \left(\sum_{i=1}^{N} \phi(a_i)^\alpha\right)^{1/\alpha} \leq \left(\sum_{i=1}^{N} \psi_{\rm sym}(a_i)^\alpha\right)^{1/\alpha}. \tag{3.231}$$

In turn, this implies that $\phi \leq \psi_\#$ on G; hence the supremum on the right-hand side of (3.202) is pointwise dominated by $\psi_{\rm reg}$. This completes the proof of the claims made in conclusion (3). Going further, the first claim in conclusion (4) is a corollary of (3.194), (3.195) and (3.109). Also, (3.204) is a consequence of (3.192) and (3.193) and the manner in which a groupoid isomorphism commutes with the groupoid operations. The fact that the value of α from (3.190) is optimal vis-a-vis the validity of the equivalence $\psi_{\rm reg} \approx \psi$ follows from Proposition 3.31.

Moving on, from Theorem 3.25 applied to ψ_ι in place of ψ (and again with $\iota := (\cdot)^{-1}$) we can conclude that for each finite number $\beta \in (0, \alpha]$ the function $\psi_{\rm reg}$ (defined in (3.192) and (3.193)) satisfies the Hölder-type regularity condition of order β

$$\begin{aligned}&\left|\psi_{\rm reg}(a * b^{-1}) - \psi_{\rm reg}(a * c^{-1})\right| \\ &\qquad \leq \tfrac{1}{\beta} \max\left\{\psi_{\rm reg}(a * b^{-1})^{1-\beta}, \psi_{\rm reg}(a * c^{-1})^{1-\beta}\right\}\left[\psi_{\rm reg}(c * b^{-1})\right]^\beta\end{aligned} \tag{3.232}$$

whenever $(a, c), (c, b) \in \mathcal{G}^{\rm R}$, with the understanding that when $\beta \geq 1$, one also imposes the condition that $a \notin \{b, c\}$. Now fix a finite number $\beta \in (0, \alpha]$ and $a, b \in G$ such that $(a, b) \in \mathcal{G}^{\rm R}$ and with the property that $a, b \notin G^{(0)}$ if $\beta \geq 1$. It is not difficult to check that $(a^{-1} * a, b), (b, a) \in \mathcal{G}^{\rm R}$ and that, whenever $a, b \notin G^{(0)}$, we have $a^{-1} * a \notin \{a, b\}$. Hence, we can write (3.232) with a replaced by $a^{-1} * a$, b replaced by a, and c replaced by b, to obtain that

$$\begin{aligned}&\left|\psi_{\rm reg}(a^{-1} * a * a^{-1}) - \psi_{\rm reg}(a^{-1} * a * b^{-1})\right| \\ &\quad \leq \tfrac{1}{\beta} \max\left\{\psi_{\rm reg}(a^{-1} * a * a^{-1})^{1-\beta}, \psi_{\rm reg}(a^{-1} * a * b^{-1})^{1-\beta}\right\}\left[\psi_{\rm reg}(b * a^{-1})\right]^\beta.\end{aligned} \tag{3.233}$$

Now (3.205) follows from (3.233) with the help of (2.59) and (3.194). This proves conclusion (5).

Next, take $\beta \in (0, \min\{1, \alpha\}]$, fix $a \in G$, and fix $\varepsilon > 0$. We claim that it is possible to find $r > 0$ such that

$$b \in B^{\mathrm{R}}_{\psi}(a, r) \implies \left|\psi_{\mathrm{reg}}(a) - \psi_{\mathrm{reg}}(b)\right| < \varepsilon. \tag{3.234}$$

For now consider $r > 0$ arbitrary. Starting with (2.59) and, making use of (3.195) and (3.194), for every $b \in B^{\mathrm{R}}_{\psi}(a, r)$ we obtain

$$\begin{aligned}\psi_{\mathrm{reg}}(b) &= \psi_{\mathrm{reg}}(b * a^{-1} * a) \leq C_1 \max\left\{\psi_{\mathrm{reg}}(b * a^{-1}), \psi_{\mathrm{reg}}(a)\right\} \\ &= C_1 \max\left\{\psi_{\mathrm{reg}}(a * b^{-1}), \psi_{\mathrm{reg}}(a)\right\} \leq C \max\left\{\psi(a * b^{-1}), \psi(a)\right\} \\ &= C \max\{r, \psi(a)\},\end{aligned} \tag{3.235}$$

where $C := C_1 \max\{1, C_0\}$ and the second inequality in (3.235) is justified based on (3.197). By combining (3.235) with (3.205) it follows that

$$\left|\psi_{\mathrm{reg}}(a) - \psi_{\mathrm{reg}}(b)\right| \leq C\beta^{-1} \max\left\{r^{1-\beta}, \psi_{\mathrm{reg}}(a)^{1-\beta}\right\} r^{\beta}, \ \forall\, b \in B^{\mathrm{R}}_{\psi}(a, r). \tag{3.236}$$

It is now immediate that the right-hand side of (3.236) converges to 0 as $r \to 0^+$; thus we can choose $r > 0$ small enough to ensure that (3.234) holds. This shows that the function $\psi_{\mathrm{reg}} : \left(G, \tau^{\mathrm{R}}_{\psi}\right) \to [0, +\infty)$ is continuous.

Let us now show that for every $a \in G$ and $r > 0$ the set $B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(a, r)$ is open in the topology τ^{R}_{ψ}. To this end, fix $b \in B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(a, r)$ so that $(a, b) \in \mathcal{G}^{\mathrm{R}}$ and $\psi_{\mathrm{reg}}(a * b^{-1}) < r$. We claim that if $\varepsilon > 0$ is small enough, then

$$B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(b, \varepsilon) \subseteq B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(a, r). \tag{3.237}$$

Indeed, if $c \in B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(b, \varepsilon)$, then $(b, c) \in \mathcal{G}^{\mathrm{R}}$ and $\psi_{\mathrm{reg}}(b * c^{-1}) < \varepsilon$. Consequently, since ψ_{reg} is continuous (when G is endowed with τ^{R}_{ψ}), we have that the quantity $\left|\psi_{\mathrm{reg}}(a * c^{-1}) - \psi_{\mathrm{reg}}(a * b^{-1})\right|$ becomes as small as desired if $\psi_{\mathrm{reg}}(b * c^{-1})$ is taken to be appropriately small. Thus, given that $\psi_{\mathrm{reg}}(a * b^{-1}) < r$, it is possible to select $\varepsilon > 0$ small enough to ensure that $\psi_{\mathrm{reg}}(a * c^{-1}) < r$. For such a choice, (3.237) holds.

Next, given $a \in G$ and $r > 0$, the claim in (3.206) follows from the fact that $B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(a, r)$ is open in the topology τ^{R}_{ψ}, and the inclusion $B^{\mathrm{R}}_{\psi_{\mathrm{reg}}}(a, r/C_1^2) \subseteq B^{\mathrm{R}}_{\psi}(a, r)$, guaranteed by (3.197). As regards (3.208), first it is clear from the quasisubadditivity condition on ψ that the quantity (3.207) is, in the case when G does not reduce to $G^{(0)}$, a well-defined number that belongs to $[1, C_1]$ and satisfies

$$\psi(a * b) \leq A\left(\psi(a) + \psi(b)\right), \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.238}$$

Inequality (3.238) also holds when G reduces to $G^{(0)}$ (in which scenario $A := 1$) since, in this case, $\psi \equiv 0$, by (3.188). Second, the inclusion in (3.208) may be equivalently written as

$$\overline{G \setminus B^{\mathrm{R}}_{\psi}(a, r)} \subseteq G \setminus B^{\mathrm{R}}_{\psi}(a, r/A), \qquad \forall\, a \in G, \ \ \forall\, r > 0, \tag{3.239}$$

where the closure is taken with respect to the topology τ^{R}_{ψ}. To justify this, fix $a \in G$, $r > 0$, along with some $b \in \overline{G \setminus B^{\mathrm{R}}_{\psi}(a, r)}$. Then $b \in G$, and for every $\varepsilon > 0$ there holds $B^{\mathrm{R}}_{\psi}(b, \varepsilon) \setminus B^{\mathrm{R}}_{\psi}(a, r) \neq \emptyset$, thanks to (3.206). Thus, there exists some element $c \in G$ such that $(b, c^{-1}) \in G^{(2)}$, $\psi(b * c^{-1}) < \varepsilon$, and either $(a, c^{-1}) \notin G^{(2)}$ or $(a, c^{-1}) \in G^{(2)}$ and $\psi(a * c^{-1}) \geq r$. Our goal is to show that $b \notin B^{\mathrm{R}}_{\psi}(a, r)$. Note that if $(a, b^{-1}) \notin G^{(2)}$, then this is automatically true, so it remains to study the case $(a, b^{-1}) \in G^{(2)}$. In this scenario, since $(b, c^{-1}) \in G^{(2)}$, it follows that $(a, c^{-1}) \in G^{(2)}$ and, hence, $\psi(a * c^{-1}) \geq r$. Consequently,

$$r \leq \psi(a * c^{-1}) \leq A\left(\psi(a * b^{-1}) + \psi(b * c^{-1})\right) < A(\psi(a * b^{-1}) + \varepsilon), \tag{3.240}$$

which further entails $\psi(a * b^{-1}) \geq r/A$. Thus, once again $b \notin B^{\mathrm{R}}_{\psi}(a, r)$, as desired. This concludes the proof of (3.208). To justify (3.209), fix $a \in G$, $r > 0$, along with $A_o > C_0^2 A$, and select $b \in \overline{B^{\mathrm{R}}_{\psi}(a, r/A_o)}$. Then, thanks to (3.206), for each $\varepsilon > 0$ we have $B^{\mathrm{R}}_{\psi}(b, \varepsilon) \cap B^{\mathrm{R}}_{\psi}(a, r/A_o) \neq \emptyset$, i.e., there exists $c \in G$ such that $(b, c^{-1}) \in G^{(2)}$, $(a, c^{-1}) \in G^{(2)}$, and $\psi(b * c^{-1}) < \varepsilon$, $\psi(a * c^{-1}) < r/A_o$. Then also $(c, a^{-1}) \in G^{(2)}$ and $\psi(a^{-1} * c) \leq C_0 \psi(a * c^{-1}) < (C_0 r)/A_o$. Hence, we have $(b, a^{-1}) \in G^{(2)}$, $(a, b^{-1}) \in G^{(2)}$, and

$$\begin{aligned} \psi(a * b^{-1}) &\leq C_0 \psi(b * a^{-1}) \leq C_0 A\left(\psi(b * c^{-1}) + \psi(c * a^{-1})\right) \\ &\leq C_0 A\left(\varepsilon + (C_0 r)/A_o\right) < r, \end{aligned} \tag{3.241}$$

where the last inequality holds if $\varepsilon > 0$ is sufficiently small, granted the original choice of A_0. Thus, in such a scenario, $b \in B^{\mathrm{R}}_{\psi}(a, r)$, and this completes the proof of (3.209).

The next step in the proof of conclusion (6) is to show that (3.210) and (3.211) hold whenever the sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq G$ converges to $a \in G$ in τ^{R}_{ψ}. With this goal in mind, use (3.197) to estimate

$$(\max\{1, C_0\})^{-1}\, \psi_{\mathrm{reg}}(a_n) \leq \psi(a_n) \leq C_1^2 \psi_{\mathrm{reg}}(a_n) \qquad \text{for each } \ n \in \mathbb{N}, \tag{3.242}$$

then invoke the continuity property of ψ_{reg}, recently established, to conclude that

$$\min\left\{1, C_0^{-1}\right\} \psi_{\mathrm{reg}}(a) \leq \liminf_{n \to \infty} \psi(a_n) \leq \limsup_{n \to \infty} \psi(a_n) \leq C_1^2 \psi_{\mathrm{reg}}(a). \tag{3.243}$$

With this in hand, (3.210) follows by appealing once more to (3.197). Concerning (3.211), observe that $\lim\limits_{n \to \infty} \psi_{\mathrm{reg}}(a_n) = \psi_{\mathrm{reg}}(a) \in [0, +\infty)$ since ψ_{reg} is continuous. Thus, $\{\psi_{\rm reg}(a_n)\}_{n\in\mathbb{N}}$ is a bounded numerical sequence. Then, thanks to (3.197), so is $\{\psi(a_n)\}_{n\in\mathbb{N}}$, and (3.211) follows. Hence, as far as conclusion (6) is concerned, it remains to give an example of a groupoid $(G,*)$ and a function $\psi : G \to [0,+\infty)$ satisfying (3.186)–(3.188) that fails to be continuous when G is equipped with the topology $\tau^{\rm R}_\psi$. Specifically, take $(G,*) := (\mathbb{R},+)$ and, for some fixed $C \in (1,+\infty)$, consider $\psi : G \to [0,+\infty)$ given for each $t \in G = \mathbb{R}$ by

$$\text{either}\quad \psi(t) := \begin{cases} |t|, & \text{if } |t| < 1, \\ C|t|, & \text{if } |t| \geq 1, \end{cases} \qquad \text{or}\quad \psi(t) := \begin{cases} |t|, & \text{if } t \in \mathbb{Q}, \\ C|t|, & \text{if } t \in \mathbb{R}\setminus\mathbb{Q}. \end{cases} \tag{3.244}$$

In each case, ψ is equivalent to the absolute value function, $|\cdot|$, on $\mathbb{R}$, and hence $\tau^{\rm R}_\psi$ is the ordinary topology on $\mathbb{R}$. However, it is obvious that the functions in (3.244) are not continuous in such a context. This completes the proof of conclusion (6).

Going further, conclusions (7)–(10) are covered by the results established in Theorem 3.24 (in the case of the last part of (10), a similar reasoning as in the case of (6) applies). Moreover, the claim in (11) is a direct consequence of the first part in (6) and last part in (10).

Let us now address the sharpness issue as formulated in conclusion (12). To this end, consider the real line with the natural groupoid structure, i.e., $(G,*,(\cdot)^{-1})$ with $G := \mathbb{R}$, $a * b := a + b$ for every $a, b \in G$ and $(a)^{-1} := -a$ for every $a \in \mathbb{R}$. Note that, in this scenario, we have $G^{(0)} = \{0\}$ and $G^{(2)} = \mathcal{G}^{\rm R} = \mathbb{R}\times\mathbb{R}$. Next, fix $C_1 > 1$ and set $s := \log_2 C_1 \in (0,+\infty)$. Finally, define

$$\psi : \mathbb{R} \to [0,+\infty), \qquad \psi(a) := |a|^s, \quad \forall\, a \in \mathbb{R}. \tag{3.245}$$

The choice of s is designed so that this function satisfies (3.186)–(3.188) for the given C_1 and with $C_0 = 1$. Assume now that $\psi' : \mathbb{R} \to [0,+\infty)$ is a function such that $\psi' \approx \psi$ and there exist $\beta \in (0,+\infty)$ and $C \in [0,+\infty)$ for which the version of (3.220) holds in the current setting. Writing this inequality for $a, b \in \mathbb{R}$ arbitrary (with the understanding that we also assume that $a, b \neq 0$ if $\beta \geq 1$) yields

$$\begin{aligned} \big|\psi'(a) - \psi'(b)\big| &\leq C\, \max\left\{\psi'(a)^{1-\beta}, \psi'(b)^{1-\beta}\right\}\left[\psi'(a-b)\right]^\beta \\ &\leq C\, \max\left\{|a|^{s(1-\beta)}, |b|^{s(1-\beta)}\right\}|a-b|^{s\beta}. \end{aligned} \tag{3.246}$$

Note that $s\beta > 1$ would force ψ' to be constant on $(0,+\infty)$, which in turn would contradict the fact that $\psi'(a) \approx |a|^s \to +\infty$ as $a \to +\infty$. Hence, necessarily, $\beta \leq 1/s$, i.e., (3.221) holds. This completes the proof of the claim made in conclusion (12).

As far as conclusion (13) is concerned, if it is assumed that (3.222) holds in place of (3.186), then Remark 3.18 applies and yields the desired conclusion.

We now turn our attention to conclusion (14) in the statement of Theorem 3.26. Recall that $\tau^{\rm R}_\psi$ and $\tau^{\rm L}_\psi$ denote, respectively, the right-topology and left-topology induced by ψ on G, according to Definition 2.62. It suffices to establish the first equivalence in (3.224) since the second one is proved analogously. We start by proving the left-to-right implication in

$$(G, \tau_\psi^{\mathrm{R}}) \text{ is a topological groupoid } \Longleftrightarrow \tau_\psi^{\mathrm{R}} = \tau_\psi^{\mathrm{L}}. \tag{3.247}$$

Hence, assume that $(G, \tau_\psi^{\mathrm{R}})$ is a topological groupoid. Then the inverse operation is continuous, and since it is its own inverse, we may conclude that

$$(\cdot)^{-1} : (G, \tau_\psi^{\mathrm{R}}) \longrightarrow (G, \tau_\psi^{\mathrm{R}}) \quad \text{is a homeomorphism.} \tag{3.248}$$

Composing this on the left with $(\cdot)^{-1} : (G, \tau_\psi^{\mathrm{L}}) \to (G, \tau_\psi^{\mathrm{R}})$, which, from Proposition 2.67, is also known to be a homeomorphism, we conclude (based also on (2.64)) that

$$\mathrm{id}_G : (G, \tau_\psi^{\mathrm{L}}) \longrightarrow (G, \tau_\psi^{\mathrm{R}}) \text{ is a homeomorphism,} \tag{3.249}$$

where id_G denotes the identity function on G. This in turn entails that $\tau_\psi^{\mathrm{R}} = \tau_\psi^{\mathrm{L}}$, as desired. To prove the right-to-left implication in (3.247), assume that

$$\tau_\psi^{\mathrm{R}} = \tau_\psi^{\mathrm{L}} =: \tau_\psi, \tag{3.250}$$

with the goal of showing that (G, τ_ψ) is a topological groupoid. The continuity of the inverse map $(\cdot)^{-1} : (G, \tau_\psi) \to (G, \tau_\psi)$ is clear from (the first part of) Proposition 2.67 and (3.250). Thus, it remains to show that if $\tau_\psi \times \tau_\psi$ is the product topology on $G \times G$, then (with $\tau_\psi \times \tau_\psi\big|_{G^{(2)}}$ denoting the relative topology induced by $\tau_\psi \times \tau_\psi$ on the set $G^{(2)} \subseteq G \times G$)

$$(\cdot) * (\cdot) : \big(G^{(2)}, \tau_\psi \times \tau_\psi\big|_{G^{(2)}}\big) \longrightarrow \big(G, \tau_\psi\big) \text{ is a continuous function.} \tag{3.251}$$

To show this, fix $(a_o, b_o) \in G^{(2)}$ and $r > 0$, with the goal of proving that there exists $\varepsilon = \varepsilon(a_o, b_o, r) > 0$ such that the following implication holds:

$$(a, b) \in G^{(2)} \cap \Big(B_\psi^{\mathrm{L}}(a_o, \varepsilon) \times B_\psi^{\mathrm{L}}(b_o, \varepsilon)\Big) \Longrightarrow a * b \in B_{\psi_{\mathrm{reg}}}^{\mathrm{L}}(a_o * b_o, r), \tag{3.252}$$

where $B_{\psi_{\mathrm{reg}}}^{\mathrm{L}}(a_o * b_o, r)$ is defined analogously to $B_\psi^{\mathrm{L}}(a_o * b_o, r)$ (cf. (2.150)), this time with ψ_{reg} playing the role of ψ. Since, as proved in conclusion (2), $\psi \approx \psi_{\mathrm{reg}}$, (3.252) suffices as far as (3.251) is concerned.

To this end, first notice that for every $\varepsilon > 0$ we have (based on (2.65) and the definition of the ball from (2.150))

$$\begin{aligned}(a, b) \in B_\psi^{\mathrm{L}}(a_o, \varepsilon) \times B_\psi^{\mathrm{L}}(b_o, \varepsilon) &\Longrightarrow (a_o, a), (b_o, b) \in \mathcal{G}^{\mathrm{L}} \\ &\Longleftrightarrow (a^{-1}, a_o), (b^{-1}, b_o) \in G^{(2)}. \end{aligned} \tag{3.253}$$

As a consequence, $(a * b)^{-1} * (a_o * b_o)$ is meaningfully defined and

$$(a * b)^{-1} * (a_o * b_o) = b^{-1} * a^{-1} * a_o * b_o. \tag{3.254}$$

In particular, $(a * b, a_o * b_o) \in \mathcal{G}^{\mathrm{L}}$. Hence, to show that $a * b \in B^{\mathrm{L}}_{\psi_{\mathrm{reg}}}(a_o * b_o, r)$, we need to ensure that, in the context of (3.252), if the number $\varepsilon > 0$ is sufficiently small, then $\psi_{\mathrm{reg}}(b^{-1} * a^{-1} * a_o * b_o) < r$. In turn, this will follow as soon as we establish the following three claims:

$$\begin{gathered}(a_o, b_o) \in G^{(2)} \text{ and } (a, b) \in G^{(2)} \cap \Big(B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \times B^{\mathrm{L}}_{\psi}(b_o, \varepsilon)\Big) \\ \text{imply that } b_o^{-1} * a^{-1} * a_o * b_o \text{ is meaningfully defined;}\end{gathered} \tag{3.255}$$

$$\begin{gathered}\big|\psi_{\mathrm{reg}}(b^{-1} * a^{-1} * a_o * b_o) - \psi_{\mathrm{reg}}(b_o^{-1} * a^{-1} * a_o * b_o)\big| \text{ is as small as desired} \\ \text{if } (a_o, b_o) \in G^{(2)} \text{ and } (a, b) \in G^{(2)} \cap \Big(B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \times B^{\mathrm{L}}_{\psi}(b_o, \varepsilon)\Big) \\ \text{and provided the number } \varepsilon > 0 \text{ is chosen to be appropriately small;}\end{gathered} \tag{3.256}$$

$$\begin{gathered}\psi_{\mathrm{reg}}(b_o^{-1} * a^{-1} * a_o * b_o) \text{ is as small as desired, provided} \\ (a_o, b_o) \in G^{(2)},\ a \in B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \text{ and } \varepsilon > 0 \text{ is sufficiently small.}\end{gathered} \tag{3.257}$$

As far as (3.255) is concerned, thanks to (3.253) and the fact that $(a_o, b_o) \in G^{(2)}$, it suffices to prove that $(b_o^{-1}, a^{-1}) \in G^{(2)}$ or, equivalently, $(a, b_o) \in G^{(2)}$ (cf. (2.65)). However, since we are assuming that $(a, b) \in G^{(2)}$ and, from (3.253), we know that $(b^{-1}, b_o) \in G^{(2)}$, it follows from (2.67) that $(a, b_o) \in G^{(2)}$, as desired. This completes the justification of (3.255).

Turning our attention to (3.256), assume that $\varepsilon \in (0, 1)$, that $(a_o, b_o) \in G^{(2)}$, and that $(a, b) \in G^{(2)} \cap \Big(B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \times B^{\mathrm{L}}_{\psi}(b_o, \varepsilon)\Big)$. From the previous discussions we know that both $b^{-1} * a^{-1} * a_o * b_o$ and $b_o^{-1} * a^{-1} * a_o * b_o$ are meaningfully defined in G. As a preliminary matter, we propose to show that, under the stated background hypotheses,

$$0 \le \psi_{\mathrm{reg}}(b^{-1} * a^{-1} * a_o * b_o),\ \psi_{\mathrm{reg}}(b_o^{-1} * a^{-1} * a_o * b_o) \le M, \tag{3.258}$$

where, for some finite constant $C = C(\psi) \ge 0$,

$$M = M(a_o, b_o) := C \cdot \max\left\{\psi_{\mathrm{reg}}(a_o), \psi_{\mathrm{reg}}(b_o), 1\right\}. \tag{3.259}$$

Indeed, by (3.195) and (3.194), it follows that there exists a finite constant $C = C(\psi) \ge 0$ such that

$$
\begin{aligned}
0 &\le \psi_{\text{reg}}(b^{-1} * a^{-1} * a_o * b_o) \\
&\le C \cdot \max\left\{\psi_{\text{reg}}(b^{-1}), \psi_{\text{reg}}(a^{-1}), \psi_{\text{reg}}(a_o), \psi_{\text{reg}}(b_o)\right\} \\
&= C \cdot \max\left\{\psi_{\text{reg}}(b), \psi_{\text{reg}}(a), \psi_{\text{reg}}(a_o), \psi_{\text{reg}}(b_o)\right\}.
\end{aligned}
\tag{3.260}
$$

On the other hand, since $(b_o^{-1}, b) \in G^{(2)}$ (cf. (3.253) and (2.65)), we may once again employ (3.195) to estimate

$$
\begin{aligned}
\psi_{\text{reg}}(b) = \psi_{\text{reg}}(b_o * b_o^{-1} * b) &\le C \cdot \max\left\{\psi_{\text{reg}}(b_o), \psi_{\text{reg}}(b_o^{-1} * b)\right\} \\
&\le C \cdot \max\left\{\psi_{\text{reg}}(b_o), \psi(b_o^{-1} * b)\right\} \\
&\le C \cdot \max\left\{\psi_{\text{reg}}(b_o), \varepsilon\right\} \le C \cdot \max\left\{\psi_{\text{reg}}(b_o), 1\right\},
\end{aligned}
\tag{3.261}
$$

where in the second inequality we have used $\psi_{\text{reg}} \approx \psi$, for the third inequality the fact that $b \in B_\psi^{\text{L}}(b_o, \varepsilon)$, and for the last inequality the fact that $\varepsilon \in (0, 1)$. Similarly, we also obtain that

$$
\psi_{\text{reg}}(a) \le C \cdot \max\left\{\psi_{\text{reg}}(a_o), 1\right\}, \tag{3.262}
$$

and now (3.258) follows from (3.260)–(3.262).

Having established (3.258), we now observe that

$$
\begin{aligned}
&\left(b^{-1} * a^{-1} * a_o * b_o\,,\, b_o^{-1} * a^{-1} * a_o * b_o\right) \in \mathcal{G}^{\text{R}} \quad \text{and} \\
&\left(b^{-1} * a^{-1} * a_o * b_o\right) * \left(b_o^{-1} * a^{-1} * a_o * b_o\right)^{-1} = b^{-1} * b_o.
\end{aligned}
\tag{3.263}
$$

Consequently, if we now fix some $\beta \in (0, \min\{\alpha, 1\}]$, then writing (3.205) for the preceding pair from $\mathcal{G}^{\text{R}}$ and keeping in mind (3.258) and the last line in (3.263) yields

$$
\begin{aligned}
&\left|\psi_{\text{reg}}(b^{-1} * a^{-1} * a_o * b_o) - \psi_{\text{reg}}(b_o^{-1} * a^{-1} * a_o * b_o)\right| \\
&\qquad \le C\, M^{1-\beta}\left[\psi_{\text{reg}}(b^{-1} * b_o)\right]^{\beta} \le C\, M^{1-\beta}\left[\psi(b^{-1} * b_o)\right]^{\beta} \le C\, M^{1-\beta}\varepsilon^{\beta},
\end{aligned}
\tag{3.264}
$$

where in the second line we have made use of the fact that $\psi_{\text{reg}} \approx \psi$ and $b \in B_\psi^{\text{L}}(b_o, \varepsilon)$. From this, (3.256) follows.

It remains to justify (3.257). Fix $(a_o, b_o) \in G^{(2)}$ along with some $\varepsilon > 0$ and consider the mapping

$$
B_\psi^{\text{L}}(a_o, \varepsilon) \ni a \mapsto \psi_{\text{reg}}(b_o^{-1} * a^{-1} * a_o * b_o) \in [0, +\infty). \tag{3.265}
$$

Since at $a := a_o \in B^{\mathrm{L}}_{\psi}(a_o, \varepsilon)$ this function takes the value

$$\psi_{\mathrm{reg}}(b_o^{-1} * a_o^{-1} * a_o * b_o) = \psi_{\mathrm{reg}}(b_o^{-1} * b_o) = 0, \tag{3.266}$$

given that $b_o^{-1} * b_o \in G^{(0)}$ and (3.196) holds, the claim in (3.257) is true if we show that the assignment in (3.265) is continuous when $B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \subseteq G$ is considered with the topology induced by τ^{L}_{ψ}. To this end, the idea is to regard (3.265) as a composition of the following four mappings, in the order listed below (with the convention that, in each case, the given topology is understood as being appropriately restricted to the set in question):

$$\Big(B^{\mathrm{L}}_{\psi}(a_o, \varepsilon) \cap \Lambda^{\mathrm{R}}(b_o), \tau^{\mathrm{L}}_{\psi}\Big) \ni a \mapsto a^{-1} \in \Big(B^{\mathrm{R}}_{\psi}(a_o^{-1}, C_0\varepsilon) \cap \Lambda^{\mathrm{L}}(b_o^{-1}), \tau^{\mathrm{R}}_{\psi}\Big), \tag{3.267}$$

$$\Big(B^{\mathrm{R}}_{\psi}(a_o^{-1}, C_0\varepsilon) \cap \Lambda^{\mathrm{L}}(b_o^{-1}), \tau^{\mathrm{R}}_{\psi}\Big) \ni c \mapsto c * a_o * b_o \in \Big(\Lambda^{\mathrm{L}}(b_o^{-1}), \tau^{\mathrm{R}}_{\psi}\Big), \tag{3.268}$$

$$\Big(\Lambda^{\mathrm{L}}(b_o^{-1}), \tau^{\mathrm{R}}_{\psi}\Big) = \Big(\Lambda^{\mathrm{L}}(b_o^{-1}), \tau^{\mathrm{L}}_{\psi}\Big) \ni c \mapsto b_o^{-1} * c \in (G, \tau^{\mathrm{L}}_{\psi}), \tag{3.269}$$

and

$$(G, \tau^{\mathrm{L}}_{\psi}) \ni c \mapsto \psi_{\mathrm{reg}}(c) \in [0, +\infty), \tag{3.270}$$

where we have used notation introduced in (2.171). The fact that the mapping (3.267) is well defined and continuous (in fact, a homeomorphism) is a direct consequence of the results proved in Proposition 2.67. Regarding the mapping in (3.267), note that

$$B^{\mathrm{R}}_{\psi}(a_o^{-1}, C_0\varepsilon) \cap \Lambda^{\mathrm{L}}(b_o^{-1}) \subseteq B^{\mathrm{R}}_{\psi}(a_o^{-1}, C_0\varepsilon) \subseteq \Lambda^{\mathrm{R}}(a_o) = \Lambda^{\mathrm{R}}(a_o * b_o), \tag{3.271}$$

and if $c \in B^{\mathrm{R}}_{\psi}(a_o^{-1}, C_0\varepsilon) \cap \Lambda^{\mathrm{L}}(b_o^{-1})$, then $c * a_o * b_o \in \Lambda^{\mathrm{L}}(b_o^{-1})$. Based on these observations and the continuity results proved in Proposition 2.67, we may therefore conclude that the mapping in (3.267) is also well defined and continuous. Going further, the equality in (3.268) is a consequence of the fact that we are currently assuming that $\tau^{\mathrm{L}}_{\psi} = \tau^{\mathrm{R}}_{\psi}$, whereas that the assignment described in (3.268) is well defined is clear from definitions. That this is also continuous is again a consequence of Proposition 2.67. Finally, the mapping in (3.270) has been shown to be continuous in conclusion (6) of Theorem 3.26. This completes the justification of the claim made in (3.257).

All together, this analysis proves that the function (3.251) is continuous, completing the proof of the right-to-left implication in (3.247). This takes care of the double equivalence stated in (3.224). Finally, the remainder of the claims made in conclusion (14) of Theorem 3.26 are clear from what we have proved so far. This concludes the treatment of (14).

The first claim in conclusion (15) of Theorem 3.26, pertaining to the fact that under the weaker condition (3.226) one nonetheless has that $d^{\mathrm{R}}_{\psi,\beta}$ is a partially defined pseudodistance on G with domain $\mathcal{G}^{\mathrm{R}}$ that induces the topology τ_ψ^{R} on G, is seen by inspecting the treatment of conclusion (9). Also, that similar properties are valid for $d_{\psi,\beta}^{\mathrm{L}}$ is seen by inspecting the treatment of conclusion (10).

Turning our attention to the proof of the next claim made in conclusion (15) of Theorem 3.26, we first note that, by (2.195), it suffices to show that (cf. Definition 2.80)

$$\text{the topological space } (G, \tau_\psi^{\mathrm{R}}) \text{ is pseudocomplete.} \tag{3.272}$$

First, the fact that $(G, \tau_\psi^{\mathrm{R}})$ is quasiregular (cf. Definition 2.78) is a direct consequence of (3.208), (3.209), (2.149), and (3.226) (which ensures that every right-ball is nonempty, as it contains its center). Next, for each $n \in \mathbb{N}$ define

$$\mathscr{B}_n := \Big\{\Big(B_\psi^{\mathrm{R}}(a,r)\Big)^\circ : a \in G,\ 0 < r < 1/n\Big\}, \tag{3.273}$$

where the interior is taken with respect to τ_ψ^{R}. Once again based on (3.208), (2.149), and (3.226), we deduce that each $\mathscr{B}_n$ is a pseudobase for $(G, \tau_\psi^{\mathrm{R}})$ (cf. Definition 2.79).

In relation to these, we claim that the implication in (2.194) holds. To prove this, consider a sequence $(B_n)_{n\in\mathbb{N}}$ of subsets of X such that

$$B_n \in \mathscr{B}_n \ \text{ and } \ \overline{B_{n+1}} \subseteq B_n \ \text{ for each } \ n \in \mathbb{N}. \tag{3.274}$$

Hence,

$$B_m \subseteq B_n \ \text{ for all } \ n, m \in \mathbb{N} \ \text{ with } \ m > n, \tag{3.275}$$

and there exist $(a_n)_{n\in\mathbb{N}} \subseteq G$ and $(r_n)_{n\in\mathbb{N}} \subseteq (0, +\infty)$ with the property that

$$B_n = \Big(B_\psi^{\mathrm{R}}(a_n, r_n)\Big)^\circ \ \text{ and } \ r_n \in (0, 1/n) \ \text{ for each } \ n \in \mathbb{N}. \tag{3.276}$$

In particular,

$$a_m \in B_m \subseteq B_\psi^{\mathrm{R}}(a_n, r_n) \ \text{ whenever } \ m > n, \tag{3.277}$$

thanks to the fact that ψ vanishes on $G^{(0)}$ and (3.275). Using this and (3.276) we therefore arrive at the conclusion that

$$(a_n, a_m) \in \mathcal{G}^{\mathrm{R}} \ \text{ and } \ \psi(a_n * a_m^{-1}) < 1/n \ \text{ for all } \ n, m \in \mathbb{N} \ \text{ with } \ m > n. \tag{3.278}$$

However, it is assumed that G is right-complete with respect to ψ, i.e., that (3.227) holds. In concert with (3.278) this proves that

$$\exists\, a \in G \ \text{ such that } \ \lim_{n\to\infty} a_n = a \ \text{ in } \ \tau_\psi^{\mathrm{R}}. \tag{3.279}$$

Observe from (3.275) and (3.277) that $a_m \in B_n$ whenever $m > n$. Together with (3.274) and (3.279), this implies that, for each $n \in \mathbb{N}$ with $n \geq 2$,

$$a \in \overline{B_n} \subseteq B_{n-1}. \tag{3.280}$$

Consequently, $a \in \bigcap_{n\in\mathbb{N}} B_n$, which proves that this intersection is nonempty. This proves the implication in (2.194) for the choice of the sequence $(\mathscr{B}_n)_{n\in\mathbb{N}}$ as in (3.273). Hence, (3.272) holds, and this concludes the proof of the claim made in (3.228). Finally, (3.229) is justified in a similar manner. This completes the treatment of (15) and completes the proof of Theorem 3.26. □

3.2.3 More on the Relationship Between GMT and Macías–Segovia, Aoki–Rolewicz, and Alexandroff–Urysohn Theorems

While it is fairly clear that both Theorems 1.2 and 1.3 contain provisions that conceivably play the role of quasisubadditivity in their respective settings (see the last inequalities in (1.1) and (1.5)), it is perhaps less obvious how such a property manifests itself in the context of Theorems 1.4 and 1.1. As regards the former, the incisive observation is that, in the context of Theorem 1.4, one has (with c denoting the constant from the inclusion $B + B \subseteq cB$)

$$\|x + y\|_B \leq c \cdot \max\{\|x\|_B\,,\ \|y\|_B\}, \qquad \forall\, x, y \in X. \tag{3.281}$$

The manner in which Theorem 3.26 contains the Aoki–Rolewicz result stated as Theorem 1.4 as a special case is made transparent by (3.281), and as such our result can be regarded as a noncommutative version of the Aoki–Rolewicz theorem. This is significant since the proof given in [69] of Theorems 1.3 and 1.4 crucially relies on the fact that the group structure underpinning a vector space is Abelian. In particular, other ideas are required to cope with the present degree of generality.

For the purpose of this discussion, it is actually instructive to record the version of Theorem 1.3 given by our main metrization theorem when applied to the case where the groupoid in question is a vector space and the mapping ψ is a given quasinorm on it. Concretely, in such a setting Theorem 3.26 yields the following more precise version of Theorem 1.3:

Theorem 3.27. *Let $(X, \|\cdot\|)$ be a (nontrivial) quasinormed vector space. Consider*

$$\kappa := \sup_{\substack{x,y\in X \\ \text{not both zero}}} \left(\frac{\|x + y\|}{\max\{\|x\|, \|y\|\}} \right). \tag{3.282}$$

Then $\kappa \in [2, +\infty)$, and, as such,

$$p := \frac{1}{\log_2 \kappa} \in (0, 1]. \tag{3.283}$$

In addition, if for each $x \in X$ one defines

$$\|x\|_{\#} := \inf\left\{\left(\sum_{i=1}^{N}\|x_i\|^p\right)^{\frac{1}{p}} : N \in \mathbb{N},\ x_1, \dots, x_N \in X \text{ with } \sum_{i=1}^{N} x_i = x\right\}, \tag{3.284}$$

then $\|\cdot\|_{\#}$ is a quasinorm on X which is a p-norm, and which is equivalent to $\|\cdot\|$; in particular,

$$\|x+y\| \le 2\max\{\|x\|, \|y\|\}, \quad \forall\, x, y \in X \implies X \text{ is (equivalently) normable.} \tag{3.285}$$

More specifically,

$$\kappa^{-2}\|x\| \le \|x\|_{\#} \le \|x\| \quad \text{and} \quad \|x+y\|_{\#}^p \le \|x\|_{\#}^p + \|y\|_{\#}^p \text{ for all } x, y \in X. \tag{3.286}$$

Consequently, for any exponent $q \in (0, p]$ and any finite family of vectors $x_1, \dots, x_N \in X$ there holds

$$\Big\|\sum_{i=1}^{N} x_i\Big\| \le \kappa^2 \left\{\sum_{i=1}^{N}\|x_i\|^q\right\}^{\frac{1}{q}}. \tag{3.287}$$

In particular, for any exponent $q \in (0, p]$ and any sequence $(x_i)_{i\in\mathbb{N}} \subseteq X$ there holds

$$\sup_{N\in\mathbb{N}}\Big\|\sum_{i=1}^{N} x_i\Big\| \le \kappa^2 \left\{\sum_{i=1}^{\infty}\|x_i\|^q\right\}^{\frac{1}{q}}. \tag{3.288}$$

Moreover, the topologies $\tau^{\mathrm{L}}_{\|\cdot\|}$ and $\tau^{\mathrm{R}}_{\|\cdot\|}$ coincide, and if $\tau_{\|\cdot\|}$ stands for $\tau^{\mathrm{L}}_{\|\cdot\|} = \tau^{\mathrm{R}}_{\|\cdot\|}$, then, while $\|\cdot\| : \big(X, \tau_{\|\cdot\|}\big) \to [0, +\infty)$ may not be continuous, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ that converges to some $x \in X$ in the topology $\tau_{\|\cdot\|}$ one nonetheless has

$$\kappa^{-2}\|x\| \le \liminf_{n\to\infty}\|x_n\| \le \limsup_{n\to\infty}\|x_n\| \le \kappa^2\|x\| \tag{3.289}$$

and

$$\sup_{n\in\mathbb{N}}\|x_n\| < +\infty. \tag{3.290}$$

The exponent p defined in (3.283) is sharp in the context of (3.286). One way to express this is via the claim that *for each $\kappa \in [2, +\infty)$ there exist a nontrivial vector space X and a quasinorm $\|\cdot\|$ on X that satisfies*

$$\kappa = \sup_{\substack{x,y\in X \\ \text{not both zero}}}\left(\frac{\|x+y\|}{\max\{\|x\|, \|y\|\}}\right) \tag{3.291}$$

and has the property that if $|||\cdot|||$ *is a* q*-norm on* X *for some* $q \in (0,1]$ *such that* $|||\cdot||| \approx \|\cdot\|$*, then necessarily* $q \leq (\log_2 \kappa)^{-1}$.

To justify this, fix an arbitrary $\kappa \in [2,+\infty)$ and define $p := (\log_2 \kappa)^{-1} \in (0,1]$. Also, take $X := L^p(\mathbb{R})$ and set $\|f\| := \big(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x\big)^{\frac{1}{p}}$ for each $f \in X$. Then for each $f, g \in X$

$$\begin{aligned}\|f+g\| &\leq \left(\int_{\mathbb{R}} (|f(x)|+|g(x)|)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}} (|f(x)|^p+|g(x)|^p) \,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\leq 2^{\frac{1}{p}-1}\left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + 2^{\frac{1}{p}-1}\left(\int_{\mathbb{R}} |g(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} = 2^{\frac{1}{p}-1}\big(\|f\|+\|g\|\big)\\ &\leq 2^{\frac{1}{p}} \max\{\|f\|,\|g\|\} = \kappa \max\{\|f\|,\|g\|\}. \end{aligned} \tag{3.292}$$

On the other hand, if $f := \mathbf{1}_{[0,1]} \in X$ and $g := \mathbf{1}_{[1,2]} \in X$, then $\|f\| = \|g\| = 1$ and $\|f+g\| = 2^{\frac{1}{p}} = \kappa$. This proves that (3.291) holds. Suppose now that $|||\cdot||| : X \to [0,+\infty)$ is a function with the property that there exist $C \in [1,+\infty)$ and $q \in (0,1]$ for which

$$\begin{aligned} &C^{-1}\|f\| \leq |||f||| \leq C\|f\|, \quad \forall\, f \in X, \quad \text{and}\\ &|||f+g|||^q \leq |||f|||^q + |||g|||^q, \quad \forall\, f, g \in X. \end{aligned} \tag{3.293}$$

Fix an arbitrary $N \in \mathbb{N}$, and consider the functions $f_i := \mathbf{1}_{[i-1,i]} \in X$ for $i = 1, \dots, N$. Then

$$\begin{aligned} C^{-1}N^{\frac{1}{p}} &= C^{-1}\Big\|\sum_{i=1}^N f_i\Big\| \leq \Big|\Big|\Big|\sum_{i=1}^N f_i\Big|\Big|\Big| \leq \left\{\sum_{i=1}^N |||f_i|||^q\right\}^{\frac{1}{q}}\\ &\leq C\left\{\sum_{i=1}^N \|f_i\|^q\right\}^{\frac{1}{q}} = CN^{\frac{1}{q}}. \end{aligned} \tag{3.294}$$

Hence, necessarily, $q \leq p$, as desired. Incidentally, (3.294) also shows that the upper bound $q \leq (\log_2 \kappa)^{-1}$ is sharp in the context of (3.287) as well.

Yet another way of understanding the sharpness of the exponent p from (3.283) in the context of (3.286) (cf. the very first inequality there) is via the following claim: *for each* $\kappa \in [2,+\infty)$ *there exist a nontrivial vector space* X *and a quasinorm* $\|\cdot\|$ *on* X *that satisfies* (3.291) *and has the property that if for some* $q \in (0,+\infty)$ *one has*

$$c\|x\| \leq \inf\left\{\left(\sum_{i=1}^N \|x_i\|^q\right)^{\frac{1}{q}} : N \in \mathbb{N},\ x_1, \dots, x_N \in X \text{ with } \sum_{i=1}^N x_i = x\right\} \tag{3.295}$$

for each $x \in X$ *(for some constant* $c \in (0, +\infty)$ *independent of* x*), then necessarily* $q \leq (\log_2 \kappa)^{-1}$.

This claim may be justified by working in the same setting as before, i.e., given $\kappa \in [2, +\infty)$, define $p := (\log_2 \kappa)^{-1} \in (0, 1]$ and take $X := L^p(\mathbb{R})$ equipped with the quasinorm $\|f\| := \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{\frac{1}{p}}$ for each $f \in X$. In this context, pick an arbitrary $N \in \mathbb{N}$ and consider the functions $f_i := \mathbf{1}_{[i-1,i]} \in X$ for $i = 1, \ldots, N$. Finally, set $f := \sum_{i=1}^{N} f_i = \mathbf{1}_{[0,N]} \in X$. Then, if (3.295) were true for some $q \in (0, +\infty)$, we would be able to estimate

$$cN^{\frac{1}{p}} = c\|f\| \leq \left(\sum_{i=1}^{N} \|f_i\|^q\right)^{1/q} = N^{\frac{1}{q}}. \tag{3.296}$$

Since $N \in \mathbb{N}$ has been arbitrarily chosen, this forces $q \leq p$, as desired.

Below, we wish also to elaborate on the nature of the relationship between Theorems 3.26 and 1.1 and explain how a uniform space X, which is Hausdorff and has a countable fundamental system of entourages, can be equipped with a quasidistance ρ that is compatible with the topology. (Again, for basic definitions the reader is referred to the appropriate chapters.) Then the quasitriangle inequality satisfied by ρ plays the role of our quasisubadditivity condition (3.186), and this is the key connection that make it possible to invoke (a special case of) Theorem 3.26 to provide a conceptually natural proof of the Alexandroff–Urysohn metrization result stated in Theorem 1.1.

Turning to specifics, suppose that $(X, \mathcal{U})$ is a uniform space that is Hausdorff and has a countable fundamental system of entourages $\mathcal{V} = \{V_i\}_{i \in \mathbb{N}}$. Construct a countable fundamental system of symmetric entourages $\{U_i\}_{i \in \mathbb{N}}$ as follows. Set $U_1 := V_1 \cap V_1^{-1}$. Then, inductively, for each $i \in \mathbb{N}$, $i \geq 2$, choose (cf. (4) in Definition 2.71) an entourage $W_i \in \mathcal{U}$ with the property that $W_i^2 \subseteq U_{i-1}$; then define $U_i := W_i \cap W_i^{-1} \cap V_i \cap V_i^{-1} \cap U_{i-1}$. This is a fundamental system of symmetric entourages that also possess the following properties:

$$U_{i+1} \subseteq U_i, \quad U_{i+1}^2 \subseteq U_i, \quad \forall\, i \in \mathbb{N}. \tag{3.297}$$

For related considerations see also [74, p. 46], [53]. The crux of the matter is that if for some fixed $C_1 > 1$ and for each $x, y \in X$ one now sets

$$\rho(x, y) := \begin{cases} \displaystyle\inf_{i \in \mathbb{N}:\, U_i \ni (x,y)} C_1^{-i} & \text{if } (x, y) \in \displaystyle\bigcup_{i=1}^{\infty} U_i, \\ 1 & \text{if } (x, y) \in X \setminus \left(\displaystyle\bigcup_{i=1}^{\infty} U_i\right), \end{cases} \tag{3.298}$$

then ρ is a quasidistance on X (compatible with the topology). Indeed, the symmetry of ρ is ensured by design, while its nondegeneracy (i.e., the fact that $\rho(x, y) = 0$ if and only if $x = y$) is a consequence of the fact that X is Hausdorff. To dispense with the issue at hand, we make the claim that

$$\rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\}, \ \forall\, x, y, z \in X. \tag{3.299}$$

To justify (3.299), observe that if either (x, z) or (z, y) does not belong to $\bigcup_{i=1}^{\infty} U_i$, then $\max\{\rho(x, z), \rho(z, y)\} = 1$, in which case inequality (3.299) holds since $\rho(x, y) \leq 1$ (recall that $C_1 > 1$). If, on the other hand, $(x, z), (z, y) \in \bigcup_{i=1}^{\infty} U_i$ and, say, $\rho(x, z) = C_1^{-k}$ and $\rho(y, z) = C_1^{-\ell}$ with $k \leq \ell$, then $(x, z) \in U_k$ and $(z, y) \in U_\ell \subseteq U_k$. In turn, these imply $(x, y) \in U_k^2 \subseteq U_{k-1}$. As a consequence, we obtain

$$\rho(x, y) \leq C_1^{-k+1} = C_1 \cdot C_1^{-k} = C_1 \max\{\rho(x, z), \rho(z, y)\}, \tag{3.300}$$

as desired. Thus, ρ defined as in (3.298) is a quasidistance. Moreover, the topology induced by the quasidistance τ on the set X coincides with the original topology on the topological space X. It follows, then, from the metrization result for quasimetric spaces proved in Theorem 3.46 that the original topological space X is metrizable.

A word of explanation as to how the Macías–Segovia result (Theorem 1.2) fits into the general framework of Theorem 3.26 is appropriate, particularly since the very formulation of Theorem 1.2 makes no explicit mention of a particular groupoid structure. The idea is that while the set X in the statement of Theorem 1.2 is arbitrary and, as such, is void of any nontrivial algebraic structure, the Cartesian product $X \times X$ has a canonical groupoid structure vis-a-vis the composition $(x, z) * (z, y) := (x, y)$ and inversion $(x, y)^{-1} := (y, x)$ for any $x, y, z \in X$ (see the discussion in Example 2.31). Specializing Theorem 3.26 to this pair groupoid then yields a sharpened version of Theorem 1.2 (see Theorem 3.46 for a precise statement). It is interesting to contrast our proof, which is of a purely algebraic nature, to the original argument in [79] which relies on topological methods (the metrization result described in Theorem 1.1). Indeed, this is the source of the discrepancy between the Macías–Segovia exponent α from (1.2), which is used to formulate the Hölder regularity property (1.4), and our larger exponent α from (3.190), which appears in the formulation of the sharp Hölder regularity condition (3.205). In this regard, the reader is invited to consult the discussion in Comment 2.83 included at the end of Sect. 2.2.

Note that there is an obvious disagreement between the Aoki–Rolewicz theorem on the one hand and the Macías–Segovia theorem on the other hand, in the sense that, if the former result is to be treated based on the latter for the choice of the quasidistance $\rho(x, y) := \|\!|x - y|\!\|_B$ with $\|\!|\cdot|\!\|_B$ as in (1.9) (where B is a bounded and balanced neighborhood of the origin in X), then this would only yield the existence of a p-norm compatible with the topology for the strictly smaller value $p = [\log_2[c(2c+1)]]^{-1}$ than the one predicted by Theorem 1.4 in (1.7). Our Theorem 3.26 corrects this deficiency by producing a value of α that is in line with the exponent p from (1.7) in the formulation of the Aoki–Rolewicz theorem. In our approach, it also becomes apparent that the nondegeneracy condition for the original quasinorm plays no role and that the symmetry condition can be weakened to mere quasisymmetry. We will revisit these issues in Sect. 3.4.

3.2.4 Connections with Homogeneous Groups

Let $n \in \mathbb{N}$, and recall that the `Heisenberg group` $\mathbb{H}_n$ is defined as the set

$$\mathbb{H}_n := \mathbb{C}^n \times \mathbb{R} = \{[\zeta, t] : \zeta \in \mathbb{C}^n,\ t \in \mathbb{R}\} \tag{3.301}$$

endowed with the group inverse operation

$$[\zeta, t]^{-1} := [-\zeta, -t], \quad \forall\, [\zeta, t] \in \mathbb{H}_n, \tag{3.302}$$

and the group multiplication

$$[\zeta, t] \bullet [\eta, s] := [\zeta + \eta, t + s + 2\,\mathrm{Im}\,(\zeta \cdot \overline{\eta})], \quad \forall\, [\zeta, t], [\eta, s] \in \mathbb{H}_n, \tag{3.303}$$

where we have set $\zeta \cdot \overline{\eta} := \sum_{k=1}^{n} \zeta_k \overline{\eta}_k$ for each $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^n$, and where $\overline{z}$ and $\mathrm{Im}(z)$ denote, respectively, the complex conjugate and imaginary part of a complex number $z \in \mathbb{C}$. Then $(\mathbb{H}_n, \bullet, (\cdot)^{-1})$ is a (non-Abelian) group, in which the identity element is $[0, 0] \in \mathbb{C}^n \times \mathbb{R}$. The Heisenberg group is `homogeneous` in the sense that it posseses a family of (anisotropic) dilations $\{\delta_r\}_{r>0}$ that are group isomorphisms satisfying $\delta_r \circ \delta_{r'} = \delta_{rr'}$ for every $r, r' > 0$. Concretely, for each $r > 0$ one may define $\delta_r : \mathbb{H}_n \to \mathbb{H}_n$ by setting $\delta_r([\zeta, t]) := [r\zeta, r^2 t]$ for every $[\zeta, t] \in \mathbb{H}_n$.

Generally speaking, given a group G [with multiplication $\bullet$, inverse $(\cdot)^{-1}$, and identity $e \in G$], a function $\psi : G \to [0, +\infty)$ is called a `group norm` provided ψ is subadditive, symmetric, and nondegenerate, i.e., for any $a, b \in G$ one has

$$\psi(a \bullet b) \le \psi(a) + \psi(b), \quad \psi(a^{-1}) = \psi(a), \ \text{ and } \ \psi^{-1}(\{0\}) = \{e\}. \tag{3.304}$$

If in place of the first condition in (3.304) it is only assumed that

$$\psi(a \bullet b) \le C(\psi(a) + \psi(b)) \ \text{ for all } \ a, b \in G \tag{3.305}$$

for some finite constant $C \ge 1$, we will say that ψ is a `quasinorm` on G.

On the Heisenberg group $\mathbb{H}_n$, frequent use is made of the Koranyi norm (cf., e.g., [36], [114, Sect. 7.12, p. 638])

$$\psi_K : \mathbb{H}_n \to [0, +\infty), \qquad \psi_K([\zeta, t]) := (|\zeta|^4 + |t|^2)^{1/4}, \quad \forall\, [\zeta, t] \in \mathbb{H}_n. \tag{3.306}$$

Then the function $d : \mathbb{H}_n \to [0, +\infty)$ given by

$$d\left([\zeta, t], [\eta, s]\right) := \psi_K\left([\zeta, t] \bullet [\eta, s]^{-1}\right), \ \forall\, [\zeta, t], [\eta, s] \in \mathbb{H}_n, \tag{3.307}$$

is a distance on $\mathbb{H}_n$ called the `Carnot metric`. The Heisenberg group norm (3.306) has the additional property that it is `homogeneous` in the sense that $\psi_K \circ \delta_r = r\psi_K$ on $\mathbb{H}_n$ for every $r > 0$ (whenever a quasinorm satisfies such a property we will refer to it as being a homogeneous quasinorm). Other examples of homogeneous norms ψ on $\mathbb{H}_n$ may be constructed by starting with ψ being defined as identically 1 on the Euclidean sphere (in canonical coordinates in $\mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$), centered at the origin and having a sufficiently small radius, then extending ψ to the entire $\mathbb{H}_n$ via homogeneity. This point of view is explored in [58] in the more general context of homogeneous groups (recalled below).

Consider next the function

$$\psi : \mathbb{H}_n \longrightarrow [0, +\infty), \quad \psi([\zeta, t]) := \max\{|\zeta|, |t|^{\frac{1}{2}}\}, \quad \forall\, [\zeta, t] \in \mathbb{H}_n. \tag{3.308}$$

This function is symmetric with respect to the group inverse operation on $\mathbb{H}_n$, i.e., $\psi\left([\zeta, t]^{-1}\right) = \psi([\zeta, t])$ for every $[\zeta, t] \in \mathbb{H}_n$ is nondegenerate since $\psi([\zeta, t]) = 0$ if and only if $[\zeta, t] = [0, 0]$, and homogeneous, in the sense that $\psi \circ \delta_r = r\psi$ for every $r > 0$. Furthermore, as noted in [114, pp. 541–542], ψ is quasisubadditive, but the proof given there only yields that ψ satisfies the quasisubadditivity condition (3.305) for some $C > 1$. Here we wish to improve upon this statement by showing that ψ is actually subadditive and, hence, is a homogeneous group norm on $\mathbb{H}_n$. Indeed, as far as the subadditivity of ψ is concerned, for every $[\zeta, t], [\eta, s] \in \mathbb{H}_n$ we may estimate

$$\begin{aligned} |\zeta + \eta| &\leq |\zeta| + |\eta| \leq \max\{|\zeta|, |t|^{1/2}\} + \max\{|\eta|, |s|^{1/2}\} \\ &= \psi([\zeta, t]) + \psi([\eta, s]) \end{aligned} \tag{3.309}$$

and

$$\begin{aligned} \left|t + s + 2\,\mathrm{Im}\,(\zeta \cdot \overline{\eta})\right|^{1/2} &\leq \left[|t| + |s| + 2|\zeta||\eta|\right]^{1/2} \\ &\leq \Big[\left(\max\{|\zeta|, |t|^{1/2}\}\right)^2 + \left(\max\{|\eta|, |s|^{1/2}\}\right)^2 \\ &\qquad + 2\,\max\{|\zeta|, |t|^{1/2}\} \cdot \max\{|\eta|, |s|^{1/2}\}\Big]^{1/2} \\ &= \max\{|\zeta|, |t|^{1/2}\} + \max\{|\eta|, |s|^{1/2}\} \\ &= \psi([\zeta, t]) + \psi([\eta, s]). \end{aligned} \tag{3.310}$$

Together, (3.309) and (3.310) prove that

$$\psi([\zeta,t]\bullet[\eta,s]) \le \psi([\zeta,t]) + \psi([\eta,s]), \ \forall\,[\zeta,t],[\eta,s]\in\mathbb{H}_n, \tag{3.311}$$

i.e., ψ is subadditive and, as such, is a homogeneous norm on the group $\mathbb{H}_n$. In addition, using the fact that, for each fixed $\zeta\in\mathbb{C}^n$, the mapping

$$\mathbb{C}^n \ni \eta \mapsto \mathrm{Im}\,(\zeta\cdot\overline{\eta})\in\mathbb{R} \text{ is continuous and vanishes at } \eta=\zeta, \tag{3.312}$$

it can be readily verified that for every $r>0$ and every $[\zeta,t]\in\mathbb{H}_n$ there exists $\varepsilon>0$ such that

$$B^{\mathrm{R}}_{\psi}([\zeta,t],\varepsilon)\subseteq B^{\mathrm{L}}_{\psi}([\zeta,t],r) \quad\text{and}\quad B^{\mathrm{L}}_{\psi}([\zeta,t],\varepsilon)\subseteq B^{\mathrm{R}}_{\psi}([\zeta,t],r) \tag{3.313}$$

(recall the definitions in (3.189) and (3.217)). This implies that if the right-topology and left-topology τ^{R}_{ψ} and τ^{L}_{ψ}, respectively, are constructed on $\mathbb{H}_n$ as in Sect. 2.2 (cf. Definition 2.62), then $\tau^{\mathrm{R}}_{\psi}=\tau^{\mathrm{L}}_{\psi}=:\tau_{\psi}$. By a similar argument, one may also show that τ_{ψ} coincides with the canonical topology on $\mathbb{C}^n\times\mathbb{R}$. Based on this discussion, item (14) of Theorem 3.26 then confirms the conclusion (which may also be verified directly or with the help of Theorem 2.77) that, when equipped with the canonical topology on $\mathbb{C}^n\times\mathbb{R}$, the Heisenberg group $\mathbb{H}_n$ becomes a Hausdorff, locally compact topological group whose topology is metrizable by a left-invariant (or right-invariant) metric (recall that, as was pointed out in the last part of conclusion (7) in Theorem 3.26, in the current setting the functions $d^{\mathrm{R}}_{\psi,\beta}$ and $d^{\mathrm{L}}_{\psi,\beta}$ defined as in (3.212) and (3.215), respectively, are genuine distances on $\mathbb{H}_n$ since $(\mathbb{H}_n,\bullet,(\cdot)^{-1})$ is a group).

These considerations are indicative of a more general phenomenon, sketched below. More specifically, the Heisenberg group belongs to the more general class of `homogeneous groups`, introduced in the 1970s by E.M. Stein. Recall that a homogeneous group is a connected and simply connected nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r\}_{r>0}$. An excellent exposition on this topic, underscoring its significance in the context of harmonic analysis, can be found in the monograph [45] by G.B. Folland and E.M. Stein. In [45, p. 8, and Proposition 1.6 on p. 9] it is shown that a continuous, homogeneous, quasinorm ψ exists on any homogeneous group G.

As such, any given homogeneous group G fits into the framework of Theorem 3.26. Consequently, if G is a homogeneous group and if ψ is a homogeneous quasinorm on G, then the function $\psi_{\mathrm{reg}}: G\to[0,+\infty)$ defined according to the recipe in (3.192), with α (related to the quasitriangle inequality (3.222) considered for $p=1$) given by the expression in (3.223), becomes a continuous, homogeneous quasinorm on G that is equivalent to ψ and is α-subadditive (the homogeneity property is a consequence of (3.204), used with ϕ replaced by the dilation δ_r). In addition, based on [45, Proposition 1.5 on p. 9], it can be shown that the right-topology and left-topology τ^{R}_{ψ} and τ^{L}_{ψ}, respectively, on G coincide with τ_G, the canonical topology on G; hence, (G,τ_G) is a Hausdorff, locally compact topological group whose topology is metrizable by left-invariant (or right-invariant) metrics.

3.3 Metrization Theory in Semigroupoid Setting

As is apparent from the discussion in previous chapters, Theorem 3.26 is constructed from gluing together a number of separate, more specialized results that collectively are stronger than the aforementioned theorem in the sense that they indicate how various individual parts of the hypotheses in Theorem 3.26 imply various parts of the conclusions.

For example, on the algebraic side, a large portion of our results actually holds in the more general setting of semigroupoids, and on the analytical side, our treatment highlights the somewhat incidental role of the quasisymmetry condition (3.187), of the nondegeneracy condition (3.188), and of the fact that the function ψ is (pointwise) finite. The main goal in this section is to further expand the scope of Theorem 3.26 by adopting a similarly more general point of view. In particular, the aim is to develop a quantitative metrization theory in the algebraic setting of semigroupoids that, on the analytical side, illustrates the principle that the quasisubadditivity property alone lends itself to some type of metrization method.

3.3.1 A Sharp Semigroupoid Metrization Theorem

The following result may be regarded as the counterpart of Theorem 3.26 in the more general algebraic context of semigroupoids.

Theorem 3.28 (Semigroupoid Metrization Theorem). *Let $(G, *)$ be a semigroupoid, and assume that $\psi : G \to [0, +\infty]$ is a quasisubadditive function, i.e., there exists a constant $C_1 \in [1, +\infty)$ such that*

$$\psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\}, \quad \text{for all} \ \ (a, b) \in G^{(2)}. \tag{3.314}$$

Introduce

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty], \tag{3.315}$$

and define the regularization $\psi_\# : G \to [0, +\infty]$ of the function ψ by setting, for each $a \in G$,

$$\psi_\#(a) := \inf\left\{\left(\sum_{i=1}^{N} \psi(a_i)^\alpha\right)^{\frac{1}{\alpha}} : \ N \in \mathbb{N}, \ (a_1, \ldots, a_N) \in G^{(N)}, \ a = a_1 * \cdots * a_N\right\} \tag{3.316}$$

if $\alpha < +\infty$ *and, corresponding to the case* $\alpha = +\infty$,

$$\psi_\#(a) := \inf\left\{\max_{1\le i\le N}\psi(a_i) : N\in\mathbb{N},\ (a_1,\dots,a_N)\in G^{(N)},\ a = a_1 * \cdots * a_N\right\}. \tag{3.317}$$

Then, with C_1 *the same constant as in* (3.314)*, one has*

$$C_1^{-2}\,\psi \le \psi_\# \le \psi \ \text{on}\ \ G \tag{3.318}$$

and

$$\psi = \psi_\# \iff \begin{cases} \text{for every } \ (a,b)\in G^{(2)} \ \text{ one has} \\ \psi(a*b) \le (\psi(a)^\alpha + \psi(b)^\alpha)^{1/\alpha} \end{cases} \tag{3.319}$$

with the convention that the last expression in (3.319) *is interpreted as* $\max\{\psi(a),\psi(b)\}$ *in the case when* $\alpha = +\infty$.

Moreover, for every finite number $\beta \in (0,\alpha]$ *the function* $\psi_\#^\beta$ *is genuinely subadditive, i.e.,*

$$\psi_\#(a*b)^\beta \le \psi_\#(a)^\beta + \psi_\#(b)^\beta, \qquad \forall\,(a,b)\in G^{(2)}. \tag{3.320}$$

As a consequence,

$$\psi_\#(a*b) \le C_1 \max\{\psi_\#(a),\psi_\#(b)\} \quad \text{for all}\ \ (a,b)\in G^{(2)}, \tag{3.321}$$

with C_1 *as in* (3.314)*. Also, for each* $N\in\mathbb{N}$ *the original function* ψ *satisfies*

$$\psi(a_1 * \cdots * a_N) \le C_1^2\left\{\sum_{i=1}^{N}\psi(a_i)^\beta\right\}^{\frac{1}{\beta}} \tag{3.322}$$

whenever $a_1,\dots,a_N \in G$ *are such that*

$$(a_i,a_{i+1})\in G^{(2)} \quad \text{for every}\ \ i\in\{1,\dots,N-1\}. \tag{3.323}$$

In particular, if $(a_i)_{i\in\mathbb{N}} \subseteq G$ *is a sequence with the property that* (3.323) *holds for every number* $N\in\mathbb{N}$ *with* $N\ge 2$*, then for each finite number* $\beta\in(0,\alpha]$ *one has*

$$\sup_{N\in\mathbb{N}}\psi(a_1 * \cdots * a_N) \le C_1^2\left\{\sum_{i=1}^{\infty}\psi(a_i)^\beta\right\}^{\frac{1}{\beta}}. \tag{3.324}$$

Finally, all quantitative aspects of the assertions formulated above are sharp in the sense discussed in Propositions 3.30–3.32 stated below.

Proof. The claims in the statement of the theorem up to (and including) (3.320) are consequences of Theorem 3.17 and properties (3), (4) and (5) in Lemma 3.14. Also, estimate (3.322) is implied by (3.318) and iterations of (3.320), whereas (3.324) follows directly from (3.322). □

Remark 3.29. Let $(G, *)$ be a semigroupoid, and assume that $\psi : G \to [0, +\infty)$ is a function with the property that there exists $C_1 \in [1, \infty)$ for which

$$\psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\}, \qquad \forall\, (a, b) \in G^{(2)}. \tag{3.325}$$

Then, via iterations of (3.325), for each $N \in \mathbb{N}$ and for each N-tuple $(a_1, \dots, a_N) \in G^{(N)}$ there holds

$$\begin{aligned}\psi(a_1 * \cdots * a_N) &\leq \max\big\{C_1\psi(a_1), C_1^2\psi(a_2), \dots, \\ &\qquad \dots C_1^{N-2}\psi(a_{N-2}), C_1^{N-1}\psi(a_{N-1}), C_1^{N-1}\psi(a_N)\big\} \\ &\leq \sum_{i=1}^{N-1} C_1^i\psi(a_i) + C_1^{N-1}\psi(a_N) \leq C_1^{N-1}\sum_{i=1}^{N}\psi(a_i).\end{aligned} \tag{3.326}$$

On the other hand, for each $\beta \in (0, (\log_2 C_1)^{-1}]$ finite number and for each $N \in \mathbb{N}$

$$\psi(a_1 * \cdots * a_N) \leq C_1^2 \left\{\sum_{i=1}^{N}\psi(a_i)^\beta\right\}^{1/\beta}, \ \forall\, (a_1, \dots, a_N) \in G^{(N)}. \tag{3.327}$$

Thus, for each $\beta \in \big(0, \min\{1, (\log_2 C_1)^{-1}\}\big]$ Hölder's inequality gives

$$\begin{aligned}\psi(a_1 * \cdots * a_N) &\leq C_1^2 \left\{\left(\sum_{i=1}^{N}(\psi(a_i)^\beta)^{1/\beta}\right)^\beta \cdot \left(N^{\frac{1/\beta-1}{1/\beta}}\right)\right\}^{1/\beta} \\ &= C_1^2 N^{1/\beta-1}\left\{\sum_{i=1}^{N}\psi(a_i)\right\}, \ \forall\, (a_1, \dots, a_N) \in G^{(N)}.\end{aligned} \tag{3.328}$$

Note that while estimates (3.326) and (3.328) have the same format, the multiplicative constant on the right-hand side of (3.326) grows exponentially in N, whereas the multiplicative constant on the right-hand side of (3.328) grows only polynomially in N.

In relation to Theorem 3.28, we wish to note that, remarkably, the upper bound $\beta \leq \alpha := (\log_2 C_1)^{-1}$ guaranteeing that (compare with (3.322))

$$\exists\, C \in [0, +\infty) \quad \text{such that} \quad \psi(a_1 * \cdots * a_N) \leq C \left\{ \sum_{i=1}^{N} \psi(a_i)^{\beta} \right\}^{\frac{1}{\beta}} \tag{3.329}$$

for any $N \in \mathbb{N}$ and any $\{a_i\}_{1 \leq i \leq N} \subseteq G$ satisfying (3.323) is actually sharp, as seen from the following proposition.

Proposition 3.30. *There exists a semigroupoid $(G, *)$ with the property that for any given $C_1 \in (1, +\infty)$ the following statement holds: one can find a function $\psi : G \to [0, +\infty]$ that satisfies (3.314) for the given constant C_1 such that if (3.329) holds for some $\beta \in (0, +\infty)$, then necessarily $\beta \leq (\log_2 C_1)^{-1}$.*

Proof. Denote by G the collection of all subintervals of $(0, +\infty]$ of the form $I = (x, y]$ for some $0 < x < y \leq +\infty$. For each such interval set $I_{\text{left}} := x$, $I_{\text{right}} := y$. Call two intervals $I, J \in G$ composable if $I_{\text{right}} = J_{\text{left}}$, and in such a case, define $I * J$ to be the interval $I \cup J = (I_{\text{left}}, J_{\text{right}}] \in G$. Then $(G, *)$ is a semigroupoid and

$$G^{(2)} = \{(I, J) : \ I = (x, y], \ \ J = (y, z] \ \text{ with } \ 0 \leq x < y < z \leq +\infty\}. \tag{3.330}$$

Consider now an arbitrary, fixed constant $C_1 \in (1, +\infty)$, and introduce the function $\psi : G \to [0, +\infty]$ by setting

$$\psi(I) := [\text{length of } I]^{\log_2 C_1}, \qquad \text{for each interval } I \in G. \tag{3.331}$$

One may then verify without difficulty that the optimal constant for which an estimate of the type (3.314) holds in the present setting is precisely the given number C_1. Having noted this, it remains to prove that the claim in (3.329) (corresponding to the current case) fails for each exponent $\beta \in (\alpha, +\infty)$ where, as in (3.315), we let $\alpha := (\log_2 C_1)^{-1}$. To see this, for each $n \in \mathbb{N}$ take $N := 2^n$, and consider the family of intervals in G given by

$$I_i := ((i-1)2^{-n}, i2^{-n}], \ i \in \{1, \dots, N\}. \tag{3.332}$$

Then condition (3.323) is satisfied and, in fact,

$$I_1 * \cdots * I_N = (0, 1]. \tag{3.333}$$

Hence, $\psi(I_1 * \cdots * I_N) = 1$, and if (3.329) were true, it would be possible to find a constant $C \in [0, +\infty)$ that is independent of n and has the property that

$$1 \leq C \left\{ \sum_{i=1}^{2^n} [\text{length}(I_i)]^{\beta/\alpha} \right\}^{\frac{1}{\beta}}. \tag{3.334}$$

However, given that $\beta > \alpha$, we have $\sum_{i=1}^{2^n} [\text{length}(I_i)]^{\beta/\alpha} = \sum_{i=1}^{2^n} 2^{-n\beta/\alpha} = 2^{n(1-\beta/\alpha)} \to 0$ as $n \to \infty$, contradicting (3.334). $\square$

The construction in the proof of Proposition 3.30 may also be used to show that the value of α in (3.315) is optimal vis-a-vis the validity of (3.318). This is formulated precisely in the result below.

Proposition 3.31. *There exists a semigroupoid $(G, *)$ with the property that for any given $C_1 \in (1, +\infty)$ the following statement holds: one can find a function $\psi : G \to [0, +\infty]$ that satisfies (3.314) for the given constant C_1 such that if the number $\beta \in (0, +\infty)$ and the function $\psi_\beta : G \to [0, +\infty]$, given by*

$$\psi_\beta(a) := \inf\left\{\left(\sum_{i=1}^{N} \psi(a_i)^\beta\right)^{\frac{1}{\beta}} : N\in\mathbb{N},\ (a_1, \dots, a_N) \in G^{(N)},\ a = a_1 * \cdots * a_N\right\} \tag{3.335}$$

for each $a \in G$, satisfy

$$\exists\, C \in [0, +\infty) \quad \text{such that} \quad \psi \le C\psi_\beta \quad \text{on } G, \tag{3.336}$$

then necessarily $\beta \le (\log_2 C_1)^{-1}$. In fact, there are examples for which ψ is a function taking strictly positive values but ψ_β is identically zero whenever $\beta > (\log_2 C_1)^{-1}$.

Proof. Assume that an arbitrary constant $C_1 \in (1, +\infty)$ has been given, and fix a number $\beta \in \big((\log_2 C_1)^{-1}, +\infty\big)$. Let the semigroupoid $(G, *)$ and the function $\psi : G \to [0, +\infty]$ be as in the proof of Proposition 3.30. Then, as already noted, (3.314) is satisfied (for the given constant C_1), and we claim that (3.336) fails in this case. To see this, let ψ_β be as in (3.335). Also, for an arbitrary $n \in \mathbb{N}$, consider $N := 2^n$ and recall the family of intervals in G from (3.332). Then, if (3.336) were true, it would be possible to select some $C \in [0, +\infty)$ with the property that for each $n \in \mathbb{N}$ we had (cf. (3.333) and (3.316))

$$\begin{aligned}
1 = \psi\left((0, 1]\right) &\le C\psi_\beta\left((0, 1]\right) \le C\left(\sum_{i=1}^{N} \psi(I_i)^\beta\right)^{\frac{1}{\beta}} \\
&= C\left\{\sum_{i=1}^{2^n} [\text{length}(I_i)]^{\beta(\log_2 C_1)}\right\}^{\frac{1}{\beta}} = C\left\{\sum_{i=1}^{2^n} 2^{-n\beta(\log_2 C_1)}\right\}^{\frac{1}{\beta}} \\
&= C\, 2^{n(\beta^{-1} - \log_2 C_1)}.
\end{aligned} \tag{3.337}$$

However, this can never happen since the fact that we are assuming $\beta > (\log_2 C_1)^{-1}$ forces $2^{n(\beta^{-1}-\log_2 C_1)} \to 0$ as $n \to \infty$.

In fact, this reasoning shows that if $\beta > (\log_2 C_1)^{-1}$, then $\psi_\beta((0,1]) = 0$, and this type of argument may be naturally adapted to show that actually $\psi_\beta(I) = 0$ for each $I \in G$ if $\beta > (\log_2 C_1)^{-1}$. □

The same circle of ideas may be further employed to show that, in the class of semigroupoids, the largest exponent $\beta > 0$ for which a quasisubadditive function is equivalent (in a pointwise, uniform fashion) with some β-subadditive function is the reciprocal of the base-two logarithm of its quasisubadditivity constant. This is made precise in the following proposition.

Proposition 3.32. *There exists a semigroupoid $(G, *)$ with the property that for any given $C_1 \in (1, +\infty)$ the following statement holds: one can find a function $\psi : G \to [0, +\infty]$ that satisfies (3.314) for the given constant C_1 and with the property that the existence of a function $\widetilde{\psi} : G \to [0, +\infty]$ satisfying*

$$\exists\, C', C'' \in [0, +\infty) \quad \text{such that} \quad C'\psi \le \widetilde{\psi} \le C''\psi \quad \text{on } G \text{ and} \tag{3.338}$$

$$\exists\, \beta \in (0, +\infty) \text{ so that } \widetilde{\psi}(a * b)^\beta \le \widetilde{\psi}(a)^\beta + \widetilde{\psi}(b)^\beta, \quad \forall\, (a, b) \in G^{(2)}, \tag{3.339}$$

forces $\beta \le (\log_2 C_1)^{-1}$.

Proof. Fix an arbitrary constant $C_1 \in (1, +\infty)$, and let the semigroupoid $(G, *)$ and the function $\psi : G \to [0, +\infty]$ be as in Proposition 3.31. In addition, assume that there exists $\widetilde{\psi} : G \to [0, +\infty]$ such that the properties listed in (3.338) and (3.339) hold. Finally, recall the piece of notation introduced in (3.335). Then, on the one hand, $C'\psi_\beta \le \big(\widetilde{\psi}\big)_\beta \le C''\psi_\beta$ by (3.338), while on the other hand, $\big(\widetilde{\psi}\big)_\beta = \widetilde{\psi}$ by (3.339) (cf. Lemma 3.14 for more details). Based on these facts and (3.338), we may therefore conclude that

$$\psi \le (C''/C')\psi_\beta \quad \text{on } G. \tag{3.340}$$

With this in hand, the conclusion in Proposition 3.31 forces $\beta \le (\log_2 C_1)^{-1}$. □

It is instructive to state separately Theorem 3.28 when the quasisubadditivity condition (3.314) is formulated differently, as in the following corollary.

Corollary 3.33. *Let $(G, *)$ be a semigroupoid, and assume that $\psi : G \to [0, +\infty]$ is a function with the property that there exists a finite constant $C \ge 1$ such that*

$$\psi(a * b) \le C(\psi(a) + \psi(b)), \quad \text{for all } (a, b) \in G^{(2)}. \tag{3.341}$$

Introduce

$$\alpha := \frac{1}{1 + \log_2 C} \in (0, 1], \tag{3.342}$$

and define the function $\psi_\# : G \to [0, +\infty]$ *as in* (3.316) *(with* α *as in* (3.342)*).*

Then $\psi_\# \approx \psi$*. More specifically, with* C *the same constant as in* (3.341)*, one has*

$$(2C)^{-2}\psi \leq \psi_\# \leq \psi \text{ on } G. \tag{3.343}$$

In particular, $(\psi_\#)^{-1}(\{0\}) = \psi^{-1}(\{0\})$*. Furthermore, for every* $\beta \in (0, \alpha]$ *one has*

$$\psi_\#(a * b)^\beta \leq \psi_\#(a)^\beta + \psi_\#(b)^\beta, \ \forall\, (a, b) \in G^{(2)}, \tag{3.344}$$

and $\psi = \psi_\#$ *on* G *if and only if* $\psi(a * b)^\alpha \leq \psi(a)^\alpha + \psi(b)^\alpha$ *for all* $(a, b) \in G^{(2)}$*. Finally, estimates* (3.322) *and* (3.324) *hold (under the same algebraic conditions as before) for any* $\beta \in \big(0, (1 + \log_2 C)^{-1}\big]$*, provided the constant* C_1 *is replaced by* $2C$*.*

This is a direct consequence of Theorem 3.28 (cf. also Remark 3.18 with $p = 1$). Significantly, in Proposition 3.35 (stated and proved below), we will actually show that, via an argument based on Fekete's lemma, the value of the exponent α from (3.342) is sharp in the context of Corollary 3.33.

In the last part of this subsection we will show that, in the class of all semigroupoids, Corollary 3.33 is actually sharp. This is going to be a consequence of the result proved in Lemma 3.34 below, dealing with the issue of growth for quasisubadditive sequences. To set the stage, recall first that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real, nonnegative numbers is said to be `subadditive` if $x_{n+m} \leq x_n + x_m$ for all $n, m \in \mathbb{N}$. More generally, call a sequence of real, nonnegative numbers $\{x_n\}_{n\in\mathbb{N}}$ `quasisubadditive` provided there exists a finite constant $C > 0$ with the property that

$$x_{n+m} \leq C(x_n + x_m) \quad \text{for all } n, m \in \mathbb{N}. \tag{3.345}$$

For each given finite constant $C > 0$ let us denote by $\mathscr{Q}(C)$ the class of all sequences with nonnegative terms that satisfy (3.345). We are now ready to state and prove the following lemma.

Lemma 3.34. *Fix an arbitrary, finite constant* $C \geq 1$ *and recall the class of quasisubadditive sequences* $\mathscr{Q}(C)$ *just introduced. Then for any sequence* $\{x_n\}_{n\in\mathbb{N}}$ *belonging to* $\mathscr{Q}(C)$ *one has*

$$x_n = O(n^{1+\log_2 C}) \quad \text{as } n \to +\infty, \tag{3.346}$$

and this rate of growth is optimal in the class $\mathscr{Q}(C)$*.*

In addition, for each real number $N > 0$ *the following three statements are equivalent:*

(I) $N \geq 1 + \log_2 C$*.*
(II) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* $\mathscr{Q}(C)$ *there exists a sequence* $\{y_n\}_{n\in\mathbb{N}}$ *of real, nonnegative numbers satisfying*

$$y_n \approx x_n, \quad \textit{uniformly for } n \in \mathbb{N}, \textit{ and} \tag{3.347}$$

$$\{y_n^{\beta}\}_{n\in\mathbb{N}} \textit{ is subadditive}, \quad \forall\, \beta \in (0, 1/N]. \tag{3.348}$$

(III) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{Q}(C)$ *there exists a sequence* $\{y_n\}_{n\in\mathbb{N}}$ *of real, nonnegative numbers satisfying*

$$y_n \approx x_n, \quad \textit{uniformly for } n \in \mathbb{N}, \textit{ and } \{y_n^{1/N}\}_{n\in\mathbb{N}} \textit{ is subadditive.} \tag{3.349}$$

Proof. Fix a finite constant $C \geq 1$ and assume that $N \in \mathbb{R}$ is such that $N \geq 1 + \log_2 C$. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ belonging to the class $\mathcal{Q}(C)$, consider $\psi : \mathbb{N} \to [0, +\infty)$ defined by $\psi(n) := x_n$ for each $n \in \mathbb{N}$. The incisive observation is that $(\mathbb{N}, +)$ is a semigroup and condition (3.345) implies that, in the framework of this algebraic structure, the function ψ is quasisubadditive. More precisely, ψ satisfies the version of (3.341) corresponding to the current context, with C the constant fixed at the beginning of the proof.

Moving on, let us now define the function $\psi_{\#} : \mathbb{N} \to [0, +\infty)$, according to the recipe described in (3.316) for the exponent $\alpha := (1 + \log_2 C)^{-1}$, and set $y_n := \psi_{\#}(n)$ for each $n \in \mathbb{N}$. Then (3.343) implies that

$$(2C)^{-2} x_n \leq y_n \leq x_n \text{ for every } n \in \mathbb{N}, \tag{3.350}$$

which shows that (3.347) holds. Next, fix $\beta \in (0, 1/N]$, and observe that the assumptions on N and the choice of the exponent α ensure that $\beta \in (0, \alpha]$. Consequently, by (3.344), the sequence $\{y_n^{\beta}\}_{n\in\mathbb{N}}$ is subadditive, so (3.348) is valid as well. This reasoning proves the implication $(I) \Rightarrow (II)$.

That $(II) \Rightarrow (III)$ is obvious. The starting point in the proof of the implication *(III)* $\Rightarrow (I)$ is to recall the elementary inequality

$$(a + b)^{\theta} \leq 2^{\theta-1}(a^{\theta} + b^{\theta}), \ \forall\, \theta \geq 1, \ \forall\, a, b \geq 0. \tag{3.351}$$

With $\alpha = (1 + \log_2 C)^{-1} \in (0, 1]$ as before, this readily implies that the sequence $\{n^{\frac{1}{\alpha}}\}_{n\in\mathbb{N}}$ belongs to the class $\mathcal{Q}(C)$. Hence, given $N > 0$ for which condition (III) holds, it follows from this that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of real, nonnegative numbers with the property that

$$y_n \approx n^{\frac{1}{\alpha}}, \quad \text{uniformly for } n \in \mathbb{N}, \text{ and } \{y_n^{1/N}\}_{n\in\mathbb{N}} \text{ is subadditive.} \tag{3.352}$$

To proceed, we recall a result known as Fekete's lemma, to the effect that if a sequence $\{z_n\}_{n\in\mathbb{N}}$ of nonnegative numbers is subadditive, then $\left\{\frac{z_n}{n}\right\}_{n\in\mathbb{N}}$ is bounded below and converges to $\inf\limits_{n\in\mathbb{N}} \frac{z_n}{n}$. In particular, $z_n = O(n)$ as $n \to +\infty$. In concert with (3.352), this yields $N \geq 1 + \log_2 C$, after some simple algebra. Thus, the proof of the implication *(III)* $\Rightarrow (I)$ is complete.

Consider next the issue of estimating the growth of a sequence $\{x_n\}_{n\in\mathbb{N}}$ from $\mathscr{Q}(C)$ as in (3.346). As a consequence of Fekete's lemma (recalled above) and the fact that, as we have just shown, there exists a nonnegative sequence $\{y_n\}_{n\in\mathbb{N}}$ satisfying (3.352) for $N := 1 + \log_2 C$, we may write $x_n \approx y_n = (y_n^{1/N})^N = O(n^N) = O(n^{1+\log_2 C})$ as $n \to \infty$. This justifies (3.346). Finally, the fact that (3.346) is optimal follows from the observation (made earlier) that the sequence $\{n^{1+\log_2 C}\}_{n\in\mathbb{N}}$ belongs to $\mathscr{Q}(C)$. This completes the proof of the lemma. □

From Lemma 3.34 it is now immediate that, in the context of Corollary 3.33, the value of the exponent α in (3.342) is sharp, in the precise sense described in the next result.

Proposition 3.35. *There exists a semigroup* $(G, *)$ *with the property that if a given function* $\psi : G \to [0, +\infty)$ *satisfies* (3.341) *for some finite constant* $C \geq 1$*, then the existence of a function* $\widetilde{\psi} : G \to [0, +\infty)$ *such that* $\widetilde{\psi} \approx \psi$ *and for which there exists* $\beta > 0$ *with the property that*

$$\widetilde{\psi}(a * b)^\beta \leq \widetilde{\psi}(a)^\beta + \widetilde{\psi}(b)^\beta, \ \forall\, (a, b) \in G^{(2)}, \tag{3.353}$$

necessarily forces $\beta \leq (1 + \log_2 C)^{-1}$.

Proof. Taking $(G, *)$ to be the semigroup $(\mathbb{N}, +)$, the conclusion we seek is a direct consequence of the equivalence $(I) \Leftrightarrow (III)$ in Lemma 3.34. □

3.3.2 An Application to Analytic Capacity

Part of the motivation in considering the more general algebraic framework of semigroupoids (compared with the setting of groupoids in which Theorem 3.26 has been stated) stems from the realization that there are many interesting examples, arising naturally in various applications, of quasisubadditive functions defined on semigroupoids that lack a genuine groupoid structure. Below, we elaborate briefly on one such instance, related to the notion of analytic capacity.

Recall that the analytic capacity of a compact set $E \subseteq \mathbb{C}$ was defined by Ahlfors in the late 1940s as the quantity

$$\gamma(E) := \sup\Big\{ f'(\infty) := \Big|\lim_{z\to\infty} z(f(z) - f(\infty))\Big| : \\ f \text{ analytic on } (\mathbb{C} \cup \{\infty\}) \setminus E \text{ and } |f| \leq 1 \text{ on } \mathbb{C} \setminus E \Big\}, \tag{3.354}$$

where $f(\infty)$ stands for $\lim\limits_{|z|\to\infty} f(z)$. Its relevance is most apparent from a (by now classical) theorem proved by Ahlfors in [1], to the effect that a compact set $E \subseteq \mathbb{C}$ is removable relative to the class of bounded analytic functions if and only if E has vanishing analytic capacity (i.e., $\gamma(E) = 0$). The longstanding, deep issue of deciding whether Ahlfors's analytic capacity (3.354) is quasisemiadditive, i.e., whether

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)) \quad \text{for all compact sets } E, F \subseteq \mathbb{C}, \tag{3.355}$$

for some $C > 0$ was relatively recently settled by Tolsa in [116], who showed that there exists a finite, universal constant $C > 1$ such that (3.355) holds. In [117, p. 1521], Tolsa also asks whether the analytic capacity γ is subadditive, in the sense that (3.355) is valid with $C = 1$. While this question remains open at the moment, here we wish to note that it is possible to construct a version of Ahlfors's analytic capacity that is equivalent to it and that, when raised to an appropriate power, becomes subadditive (cf. (3.359) below). Most significantly, based on our abstract regularization procedure, we are able to obtain capacitary estimates for sequences of compact subsets of the plane (cf. (3.361) and (3.362)).

Corollary 3.36. *Define Tolsa's constant as the number*

$$C_{\mathrm{T}} := \sup_{\substack{E,F\subseteq\mathbb{C}\\ E,F \text{ compact}}} \left(\frac{\gamma(E \cup F)}{\gamma(E) + \gamma(F)} \right) \in [1, +\infty). \tag{3.356}$$

Also, introduce the exponent $\alpha := (1 + \log_2 C_{\mathrm{T}})^{-1} \in [1, +\infty)$*, and define the regularized analytic capacity of a compact set* $E \subseteq \mathbb{C}$ *as the quantity*

$$\gamma_{\#}(E) := \inf \left\{ \left(\sum_{i=1}^{N} \gamma(E_i)^{\alpha} \right)^{\frac{1}{\alpha}} : N \in \mathbb{N}, \right. \\ \left. E_1, \dots, E_N \subseteq \mathbb{C} \text{ compact, } E = \bigcup_{1 \leq i \leq N} E_i \right\}. \tag{3.357}$$

Then the following properties hold.

(1) The regularized analytic capacity considered in (3.357) *is equivalent to Ahlfors's analytic capacity considered in* (3.354)*. More precisely, for every compact set* $E \subseteq \mathbb{C}$

$$(4C_{\mathrm{T}}^2)^{-1}\gamma(E) \leq \gamma_{\#}(E) \leq \gamma(E); \tag{3.358}$$

hence, for every compact set $E \subseteq \mathbb{C}$ *one has* $\gamma_{\#}(E) = 0$ *if and only if* $\gamma(E) = 0$*.*

(2) For each $\beta \in (0, \alpha]$ *the regularized analytic capacity* $\gamma_{\#}$ *is* β*-subadditive in the sense that* $\gamma_{\#}(E \cup F)^{\beta} \leq \gamma_{\#}(E)^{\beta} + \gamma_{\#}(F)^{\beta}$ *for all compact sets* $E, F \subseteq \mathbb{C}$*. In particular,*

$$\gamma_{\#}(E \cup F)^{\alpha} \leq \gamma_{\#}(E)^{\alpha} + \gamma_{\#}(F)^{\alpha} \quad \text{for all compact sets } E, F \subseteq \mathbb{C}. \tag{3.359}$$

(3) One has $\gamma_{\#} = \gamma$ if and only if γ satisfies

$$\gamma(E \cup F)^{\alpha} \leq \gamma(E)^{\alpha} + \gamma(F)^{\alpha} \quad \textit{for all compact sets } E, F \subseteq \mathbb{C}. \tag{3.360}$$

(4) For any exponent $\beta \in (0, \alpha]$ and any finite family of compact subsets $E_1, \dots, E_N$ of $\mathbb{C}$ there holds

$$\gamma\left(\bigcup_{i=1}^{N} E_i\right) \leq 4C_{\mathrm{T}}^2 \left\{\sum_{i=1}^{N} \gamma(E_i)^{\beta}\right\}^{\frac{1}{\beta}}. \tag{3.361}$$

Moreover, for any sequence $(E_i)_{i \in \mathbb{N}}$ of compact subsets of $\mathbb{C}$ and any exponent $\beta \in (0, \alpha]$ one has

$$\lim_{N \to \infty} \gamma\left(\bigcup_{i=1}^{N} E_i\right) \leq 4C_{\mathrm{T}}^2 \left\{\sum_{i=1}^{\infty} \gamma(E_i)^{\beta}\right\}^{\frac{1}{\beta}}. \tag{3.362}$$

Proof. It is clear that the family Comp ($\mathbb{C}$) of all compact subsets of the complex plane becomes a semigroupoid when equipped with the usual union of sets (considered as the semigroupoid binary operation). Since by Tolsa's result mentioned in (3.355) the function γ : Comp ($\mathbb{C}$) $\to [0, +\infty]$ is quasisubadditive, the desired conclusions, with the exception of (3.362), are direct consequences of their abstract counterparts in Corollary 3.33. As regards (3.362), Corollary 3.33 only ensures that for any sequence $(E_i)_{i \in \mathbb{N}}$ of compact subsets of $\mathbb{C}$ and any exponent $\beta \in (0, \alpha]$ there holds

$$\sup_{N \in \mathbb{N}} \gamma\left(\bigcup_{i=1}^{N} E_i\right) \leq 4C_1^2 \left\{\sum_{i=1}^{\infty} \gamma(E_i)^{\beta}\right\}^{\frac{1}{\beta}}. \tag{3.363}$$

However, once this has been established, (3.362) follows since the analytic capacity is monotone (cf., e.g., [42, Proposition 1.8]). □

The remarkable aspect of (3.361) is that the multiplicative constant on the right-hand side of this estimate does not depend on N.

Incidentally, note that if $C_{\mathrm{T}} = 1$ (i.e., if Tolsa's question has a positive answer), then the regularized analytic capacity coincides with Ahlfors's analytic capacity. Also, it is clear that the regularization result described in (3.357) works equally well for other types of capacities used in analysis (such as the electric intensity capacity; cf. [89]).

Remark 3.37. Actually, Tolsa proved that the analytic capacity function γ is countably quasisemiadditive in the sense that there exists a finite constant $C > 0$ such that

$$\gamma\left(\bigcup_{i \in \mathbb{N}} E_i\right) \leq C \sum_{i \in \mathbb{N}} \gamma(E_i) \tag{3.364}$$

for any family $(E_i)_{i\in\mathbb{N}}$ of compact subsets of $\mathbb{C}$. As a result, if for any compact set $E \subseteq \mathbb{C}$ we define

$$\widetilde{\gamma}(E) := \inf\left\{\sum_{i\in\mathbb{N}} \gamma(E_i) : (E_i)_{i\in\mathbb{N}} \text{ compact subsets of } \mathbb{C} \text{ such that } E = \bigcup_{i\in\mathbb{N}} E_i\right\}, \tag{3.365}$$

then $\widetilde{\gamma} \approx \gamma$ in the sense that (with C as in (3.364))

$$C^{-1}\gamma(E) \le \widetilde{\gamma}(E) \le \gamma(E) \quad \text{for each compact set } E \subseteq \mathbb{C} \tag{3.366}$$

and $\widetilde{\gamma}$ is genuinely subadditive, i.e.,

$$\widetilde{\gamma}(E \cup F) \le \widetilde{\gamma}(E) + \widetilde{\gamma}(F) \quad \text{for all compact sets } E, F \subseteq \mathbb{C}. \tag{3.367}$$

3.3.3 Metrization Results with Additional Constraints

The sharp metrization result proved in Theorem 3.28 in the abstract setting of semigroupoids may be further adapted to accommodate additional conditions imposed on the semigroupoid $(G,*)$ as well as the quasisubadditive function ψ and its regularization $\psi_\#$. Schematically, the philosophy behind subsequent results may be described as follows:

$$\begin{aligned} &\text{additional structure on } G \text{ or properties for } \psi \\ &\Longrightarrow \text{ more information about the regularized version } \psi_\# \text{ of } \psi. \end{aligned} \tag{3.368}$$

We have already seen this principle at work in several instances so far. For example, this is the way in which Theorem 3.26, where the assumption on G is strengthened (compared to Theorem 3.28) by demanding that this be a groupoid and where the quasisubadditive function ψ is also assumed to be quasisymmetric and nondegenerate, is related to Theorem 3.28. Also, in the case of Theorem 3.27, where the background semigroupoid is the underlying Abelian group of a vector space X and the function $\psi := \|\cdot\|$ is homogeneous if $\|\cdot\|$ is a quasinorm, the regularized version $\|\cdot\|_\#$ of $\|\cdot\|$ (defined as in (3.284), which is consistent with the recipe described in Theorem 3.28) is also homogeneous and, hence, ultimately, a quasinorm. Finally, as will become apparent later, a similar principle is at work in the statement and proof of Theorem 3.46 (sharpening the Macías–Segovia result presented in Theorem 1.2).

Here we wish to further exemplify the manner in which the generic scheme (3.368) may be implemented by proving several other metrization theorems with additional constraints. The first such result involves dealing with a background partial order structure.

Theorem 3.38. *Let* $(S, *)$ *be a semigroup, and suppose that* $\preceq$ *is a partial order relation on the set* S *that is compatible with the semigroup multiplication operation in the sense that*

$$\text{for any } \ a,b,c,d \in S \ \text{ satisfying } \ a \preceq b \ \text{ and } \ c \preceq d \implies a * c \preceq b * d. \tag{3.369}$$

In this context, assume that $\psi : S \to [0, +\infty]$ *is a function that is quasisubadditive, i.e., there exists a constant* $C_1 \in [1, +\infty)$ *such that*

$$\psi(a * b) \leq C_1 \max\{\psi(a), \psi(b)\} \quad \text{for all } \ a,b \in S, \tag{3.370}$$

and that is quasimonotone, i.e., there exists a constant $C_2 \in (0, +\infty)$ *such that*

$$\psi(a) \leq C_2 \psi(b) \quad \text{whenever } \ a,b \in S \ \text{ are such that } \ a \preceq b. \tag{3.371}$$

Consider

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty], \tag{3.372}$$

and define the regularization $\psi_\star : S \to [0, +\infty]$ *of the function* ψ *by setting, for each* $a \in S$,

$$\psi_\star(a) := \inf\left\{ \left(\sum_{i=1}^{N} \psi(a_i)^\alpha \right)^{\frac{1}{\alpha}} : N \in \mathbb{N},\, a_1, \dots, a_N \in S,\ a \preceq a_1 * \cdots * a_N \right\} \tag{3.373}$$

if $\alpha < +\infty$ *and, corresponding to the case* $\alpha = +\infty$,

$$\psi_\star(a) := \inf\left\{ \max_{1 \leq i \leq N} \psi(a_i) : N \in \mathbb{N},\, a_1, \dots, a_N \in S,\ a \preceq a_1 * \cdots * a_N \right\}. \tag{3.374}$$

Then, with C_1, C_2 *the same constants as in* (3.370) *and* (3.371)*, for every finite number* $\beta \in (0, \alpha]$ *one has*

$$C_2^{-1} C_1^{-2}\, \psi \leq \psi_\star \leq \psi \quad \text{pointwise on } \ S, \tag{3.375}$$

$$\psi_\star(a * b)^\beta \leq \psi_\star(a)^\beta + \psi_\star(b)^\beta,\ \forall\, a,b \in S, \tag{3.376}$$

$$\psi_\star(a) \leq \psi_\star(b) \ \text{ for all } \ a,b \in S \ \text{ such that } \ a \preceq b. \tag{3.377}$$

Proof. Viewing $(S, *)$ as a semigroupoid, define $\psi_\# : S \to [0, +\infty]$ as in (3.316)–(3.317) (with G replaced by S). Then, from the manner in which the functions $\psi_\#$,

$\psi_\star$ are designed, we obviously have (cf. also (3.318))

$$\psi_\star(a) \le \psi_\#(a) \le \psi(a), \ \forall\, a \in S. \tag{3.378}$$

Furthermore, given an arbitrary $a \in S$, if $a_1, \dots, a_N \in S$ are such that $a \preceq a_1 * \cdots * a_N$, then, thanks to the quasimonotonicity of ψ, (3.318), and the definition of $\psi_\star$, we have, in the case $\alpha < +\infty$,

$$\psi(a) \le C_2 \psi(a_1 * \cdots * a_N) \le C_2 C_1^2 \psi_\#(a_1 * \cdots * a_N) \le C_2 C_1^2 \left(\sum_{i=1}^{N} \psi(a_i)^\alpha \right)^{\frac{1}{\alpha}}. \tag{3.379}$$

If we take the infimum over all $N \in \mathbb{N}$ and $a_1, \dots, a_N \in S$ such that $a \preceq a_1 * \cdots * a_N$, then estimate (3.379) yields

$$\psi(a) \le C_2 C_1^2 \psi_\star(a). \tag{3.380}$$

The same type of argument works in the case $\alpha = +\infty$ (this time using (3.374)); hence, ultimately,

$$\psi \le C_2 C_1^2 \psi_\star \qquad \text{on } \ S. \tag{3.381}$$

In concert with (3.378), this proves (3.375). Next, assume that $a, b \in S$ are such that $a \preceq b$. Then, if $N \in \mathbb{N}$ and $b_1, \dots, b_N \in S$ satisfy $b \preceq b_1 * \cdots * b_N$, then it follows that $a \preceq b_1 * \cdots * b_N$; hence, in the case $\alpha < +\infty$,

$$\psi_\star(a) \le \left(\sum_{i=1}^{N} \psi(b_i)^\alpha \right)^{\frac{1}{\alpha}}. \tag{3.382}$$

Taking the infimum over all $N \in \mathbb{N}$ and $b_1, \dots, b_N \in S$ such that $b \preceq b_1 * \cdots * b_N$, we arrive at the conclusion that, if $\alpha < +\infty$, then

$$\psi_\star(a) \le \psi_\star(b), \qquad \forall\, a, b \in S, \ \ \text{satisfying} \ \ a \preceq b. \tag{3.383}$$

The same type of analysis works in the case when $\alpha = +\infty$, and this completes the proof of (3.377). As far as (3.376) is concerned, pick two arbitrary elements $a, b \in S$ and assume that $a_1, \dots, a_{N_1} \in S$ and $b_1, \dots, b_{N_2} \in S$ are such that

$$a \preceq a_1 * \cdots * a_{N_1}, \qquad b \preceq b_1 * \cdots * b_{N_2}. \tag{3.384}$$

Then (3.369) implies that $a * b \preceq a_1 * \cdots * a_{N_1} * b_1 * \cdots * b_{N_2}$. In turn, in light of (3.373), this forces (assuming that $\alpha < +\infty$)

$$\psi_\star(a * b) \leq \left(\sum_{i=1}^{N_1} \psi(a_i)^\alpha + \sum_{i=1}^{N_2} \psi(b_i)^\alpha \right)^{\frac{1}{\alpha}} . \tag{3.385}$$

Taking the infimum over all finite families of elements $a_1, \dots, a_{N_1} \in S$ and $b_1, \dots, b_{N_2} \in S$ satisfying (3.384) then leads to

$$\psi_\star(a * b) \leq \left(\psi_\star(a)^\alpha + \psi_\star(b)^\alpha \right)^{1/\alpha} . \tag{3.386}$$

With this in hand, (3.377) readily follows in the case when $0 < \beta \leq \alpha < +\infty$. The case $\alpha = +\infty$ is treated similarly, and this concludes the proof of the theorem. □

Examples of semigroups $(S, *)$ satisfying (3.369) include:

- Any semigroup $(S, *)$ equipped with the trivial partial order relation

$$a \preceq b \iff a = b, \qquad \forall\, a, b \in S; \tag{3.387}$$

- The underlying Abelian additive group of any partially ordered vector space;
- Any lattice $(S, \preceq, \vee, \wedge)$ (defining $a * b := a \vee b$ for each $a, b \in S$);
- More generally, let $(S, \preceq)$ be a partially ordered set with the property that $\sup\{a, b\}$ exists for each $a, b \in S$; then, if $a * b := \sup\{a, b\}$ for each $a, b \in S$, it follows that $(S, *)$ is a semigroup in which (3.369) is satisfied.

Parenthetically, we note that in the case of a semigroup $(S, *)$ equipped with the trivial partial order relation (3.387), we have $\psi_\star = \psi_\#$; hence Theorems 3.38 and 3.28 coincide in such a situation.

Our next theorem may be regarded as yet another generalization of the Aoki–Rolewicz theorem (stated as Theorem 1.3).

Theorem 3.39. *Let X be a vector space, and assume that $\|\cdot\| : X \to [0, +\infty]$ is a function satisfying the following properties:*

(1) (Quasisubaditivity) there exists a constant $C_0 \in [1, +\infty)$ for which

$$\|x + y\| \leq C_0 \max\{\|x\|, \|y\|\}, \qquad \forall\, x, y \in X; \tag{3.388}$$

(2) (Pseudohomogeneity) there exist $C_1 \in (0, +\infty)$ and $\theta \in \mathbb{R}$ such that

$$\|\lambda x\| \leq C_1 |\lambda|^\theta \|x\|, \qquad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{R} \setminus \{0\}. \tag{3.389}$$

Set

$$\alpha := \frac{1}{\log_2 C_0} \in (0, \infty], \tag{3.390}$$

and for each $x \in X$ *define*

$$\|x\|_\star := \sup_{\lambda \in \mathbb{R}\setminus\{0\}} \inf \Bigg\{ |\lambda|^{-\theta} \Big(\sum_{i=1}^{N} \|\lambda x_i\|^\alpha \Big)^{\frac{1}{\alpha}} : N \in \mathbb{N}, \ \textit{and} \ x_1, \dots, x_N \in X \ \textit{are such that} \ \sum_{i=1}^{N} x_i = x \Bigg\} \tag{3.391}$$

if $\alpha < +\infty$ *and, corresponding to the case* $\alpha = +\infty$*,*

$$\|x\|_\star := \sup_{\lambda \in \mathbb{R}\setminus\{0\}} \inf \Big\{ |\lambda|^{-\theta} \max_{1 \le i \le N} \|\lambda x_i\| : N \in \mathbb{N}, \ \textit{and} \ x_1, \dots, x_N \in X \ \textit{are such that} \ \sum_{i=1}^{N} x_i = x \Big\}. \tag{3.392}$$

Then $\|\cdot\|_\star : X \to [0, +\infty]$ *satisfies*

$$C_0^{-2}\|x\| \le \|x\|_\star \le C_1\|x\| \quad \textit{for all} \ \ x \in X, \tag{3.393}$$

$$\|\eta x\|_\star = |\eta|^\theta \|x\|_\star \quad \textit{for all } x \in X \textit{ and all } \eta \in \mathbb{R} \setminus \{0\}, \tag{3.394}$$

$$\|x + y\|_\star^\beta \le \|x\|_\star^\beta + \|y\|_\star^\beta \quad \textit{for all } x, y \in X \textit{ and each } \beta \in (0, \alpha] \textit{ finite}, \tag{3.395}$$

$$\|x + y\|_\star \le C_0 \max\{\|x\|_\star, \|y\|_\star\}, \quad \forall\, x, y \in X. \tag{3.396}$$

Proof. To get started, denote by $\|\cdot\|_\#$ the regularization of the given $\|\cdot\|$ relative to the group $(X, +)$, in the sense of Theorem 3.28, and observe that (3.391) and (3.392) amount to

$$\|x\|_\star = \sup_{\lambda \in \mathbb{R}\setminus\{0\}} \big[|\lambda|^{-\theta} \|\lambda x\|_\#\big] \quad \text{for all} \ \ x \in X. \tag{3.397}$$

Now Theorem 3.28 gives

$$C_0^{-2}\|x\| \le \|x\|_\# \le \|x\| \quad \text{for all} \ \ x \in X, \tag{3.398}$$

$$\|x + y\|_\#^\beta \le \|x\|_\#^\beta + \|y\|_\#^\beta \text{ for all } x, y \in X \text{ and each } \beta \in (0, \alpha] \text{ finite}. \tag{3.399}$$

Thus, on account of (3.398), taking $\lambda = 1$ in the supremum process in (3.397) yields that, on the one hand,

$$C_0^{-2}\|x\| \leq \|x\|_{\#} \leq \|x\|_{\star} \quad \text{for all} \quad x \in X, \tag{3.400}$$

and, on the other hand, (3.398) and the pseudohomogeneity condition permit us to estimate

$$|\lambda|^{-\theta}\|\lambda x\|_{\#} \leq |\lambda|^{-\theta}\|\lambda x\| \leq C_1\|x\|, \quad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{R} \setminus \{0\}, \tag{3.401}$$

and by taking the supremum over all $\lambda \in \mathbb{R}\setminus\{0\}$ in the most extreme sides of (3.401) we arrive at

$$\|x\|_{\star} \leq C_1\|x\| \quad \text{for all} \quad x \in X. \tag{3.402}$$

Now, (3.393) follows from (3.400) and (3.402). Going further, given $x \in X$, for each fixed $\eta \in \mathbb{R} \setminus \{0\}$ we have

$$\begin{aligned} \|\eta x\|_{\star} &= \sup_{\lambda\in\mathbb{R}\setminus\{0\}} \left[|\lambda|^{-\theta}\|(\lambda\eta)x\|_{\#}\right] \\ &= |\eta|^{\theta} \sup_{\lambda\in\mathbb{R}\setminus\{0\}} \left[|\lambda\eta|^{-\theta}\|(\lambda\eta)x\|_{\#}\right] = |\eta|^{\theta}\|x\|_{\star}, \end{aligned} \tag{3.403}$$

proving (3.394). Moving on, if $x, y \in X$ and $\beta \in (0, \alpha]$ is a finite number, then (3.399) allows us to estimate, for each $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \left[|\lambda|^{-\theta}\|\lambda(x+y)\|_{\#}\right]^{\beta} &= |\lambda|^{-\beta\theta}\|\lambda x + \lambda y\|_{\#}^{\beta} \leq |\lambda|^{-\beta\theta}\|\lambda x\|_{\#}^{\beta} + |\lambda|^{-\beta\theta}\|\lambda y\|_{\#}^{\beta} \\ &\leq \|x\|_{\star}^{\beta} + \|y\|_{\star}^{\beta}. \end{aligned} \tag{3.404}$$

Taking the supremum over all $\lambda \in \mathbb{R} \setminus \{0\}$ in the most extreme sides of (3.404) produces (3.395). Finally, given any $x, y \in X$, then for each number $\beta \in (0, \alpha)$ we have, thanks to (3.395),

$$\|x + y\|_{\star}^{\beta} \leq 2 \max\{\|x\|_{\star}^{\beta}, \|y\|_{\star}^{\beta}\}, \qquad \forall\, x, y \in X. \tag{3.405}$$

Hence, $\|x + y\|_{\star} \leq 2^{1/\beta}\max\{\|x\|_{\star}, \|y\|_{\star}\}$ for each $x, y \in X$, and (3.396) follows, on account of (3.390), by letting $\beta \nearrow \alpha$. This completes the proof of the theorem. □

A natural context in which the pseudohomogeneity condition (3.389) from Theorem 3.39 occurs is as follows. Let $(X, \|\cdot\|)$ be a quasinormed vector space, and assume that $\|\cdot\|' : X \to [0, +\infty)$ is a function with the property that $\|\cdot\|' \approx \|\cdot\|$, i.e., there exist constants $c_0, c_1 \in (0, +\infty)$ such that

$$c_0\|x\| \leq \|x\|' \leq c_1\|x\|, \qquad \forall\, x \in X. \tag{3.406}$$

While in general $\|\cdot\|'$ itself might fail to be a quasinorm (since it may lack homogeneity), we nonetheless have

$$\|\lambda x\|' \leq c_1\|\lambda x\| = c_1|\lambda|\|x\| \leq c_0^{-1}c_1|\lambda|\|x\|', \qquad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{R}. \tag{3.407}$$

Hence, (3.389) holds for $\|\cdot\|'$ with $C_1 := c_0^{-1}c_1$ and $\theta := 1$. Another situation where (3.389) occurs naturally is in considering powers of a given quasinorm.

We also note that our regularization scheme may be naturally adapted to other types of settings. To illustrate this point, recall that an F-`norm` (short for Fréchet-norm) on a vector space X is a function $\|\cdot\| : X \to [0, +\infty)$ satisfying

$$\|x\| = 0 \iff x = 0 \ \text{ for each } \ x \in X, \tag{3.408}$$

$$\|x + y\| \leq \|x\| + \|y\| \ \text{ for all } \ x, y \in X, \tag{3.409}$$

$$\|\lambda x\| \leq \|x\| \ \text{ for all } x \in X \text{ if } |\lambda| \leq 1, \tag{3.410}$$

$$\|\lambda_n x\| \to 0 \ \text{ for each fixed } x \in X \text{ if } \lambda_n \to 0. \tag{3.411}$$

A word of caution is in order here. Specifically, the reader is made aware that frequently (cf., e.g., [74, p. 163], [91, p. 19]) authors impose an extra axiom in the definition of an F-norm, namely,

$$\|\lambda x_n\| \to 0 \ \text{ for each fixed scalar } \lambda \text{ if } \|x_n\| \to 0. \tag{3.412}$$

However, as the next lemma demonstrates, the latter axiom is redundant in the context of (3.409) and (3.410).

Lemma 3.40. *Let X be a vector space over $\mathbb{C}$, and consider a function $\|\cdot\| : X \to [0, +\infty)$ with the property that there exist $c_0, c_1 \in [1, +\infty)$ such that*

$$\|\lambda x\| \leq c_0\|x\|, \quad \forall\, x \in X \ \text{ and } \ \forall\, \lambda \in \mathbb{C} \ \text{ such that } \ |\lambda| \leq 1, \tag{3.413}$$

$$\|x + y\| \leq c_1\left(\|x\| + \|y\|\right), \quad \forall\, x, y \in X. \tag{3.414}$$

Then

$$\|\lambda x\| \leq c_0c_1\left(c_0 + 4c_1^2|\lambda|^{1+\log_2 c_1}\right)\|x\|, \quad \forall\, \lambda \in \mathbb{C} \ \text{ and } \ \forall\, x \in X. \tag{3.415}$$

In particular,

$$\lim_{n\to\infty}\|\lambda x_n\| = 0, \quad \forall\, \{x_n\}_{n\in\mathbb{N}} \subseteq X \ \text{ such that } \ \lim_{n\to\infty}\|x_n\| = 0 \ \text{ and } \ \forall\, \lambda \in \mathbb{C}. \tag{3.416}$$

Proof. En route to (3.415) we will first establish that

$$\forall\,\lambda \in \mathbb{C} \text{ with } |\lambda| \leq 1 \text{ and } \forall\,x \in X \text{ there holds}$$
$$c_0^{-1}\|\lambda x\| \leq \big\| |\lambda| x \big\| \leq c_0 \|\lambda x\|. \tag{3.417}$$

Clearly (3.417) is valid if $\lambda = 0$. Assume next that $\lambda \in \mathbb{C} \setminus \{0\}$, and for any arbitrarily fixed $x \in X$ write

$$\big\||\lambda| x\big\| = \left\| \left(\frac{|\lambda|}{\lambda} \right) \cdot \lambda x \right\| \leq c_0 \|\lambda x\|, \tag{3.418}$$

where the inequality follows from property (3.413) since $\left|\frac{|\lambda|}{\lambda}\right| = 1$. Using again (3.413) we have

$$\|\lambda x\| = \left\| \left(\frac{\lambda}{|\lambda|} \right) (|\lambda| x) \right\| \leq c_0 \big\| |\lambda| x \big\| \tag{3.419}$$

since $\frac{\lambda}{|\lambda|} \in \mathbb{C}$ satisfies $\left|\frac{\lambda}{|\lambda|}\right| = 1$. In concert, (3.418) and (3.419) prove the claim made in (3.417).

Moving on, we will prove that

$$\forall\,\lambda \in \mathbb{R}_+ \text{ and } \forall\,x \in X \text{ there holds } \|\lambda x\| \leq c_1 \left(c_0 + 4c_1^2 \lambda^{1+\log_2 c_1}\right) \|x\|. \tag{3.420}$$

To see this, fix $\lambda \in \mathbb{R}_+$ and denote by N the greatest integer $\leq \lambda$. Then

$$N \in \mathbb{N} \cup \{0\} \text{ and } N \leq \lambda < N + 1. \tag{3.421}$$

Next, for any $x \in X$ write

$$\begin{aligned} \|\lambda x\| &= \|(\lambda - N)x + Nx\| \\ &\leq c_1 \left(\|(\lambda - N)x\| + \|Nx\|\right) \leq c_1 \left(c_0\|x\| + \|Nx\|\right), \end{aligned} \tag{3.422}$$

where the first inequality in (3.422) follows from (3.414), while the second inequality uses (3.413) and (3.421). At this stage, bring in (3.287) to conclude that

$$\|Nx\| = \|x + \cdots + x\| \leq 4c_1^2 N^{1+\log_2 c_1} \|x\|. \tag{3.423}$$

Together, (3.422), (3.423), and the fact that $N \leq \lambda$ prove (3.420).

After this preamble, (3.415) is obtained by writing, based on (3.417) and (3.420),

$$\|\lambda x\| \leq c_0 \big\| \,|\lambda| x \big\| \leq c_0 c_1 \left(c_0 + 4c_1^2 |\lambda|^{1+\log_2 c_1}\right) \|x\| \tag{3.424}$$

for every $\lambda \in \mathbb{C}$ and every $x \in X$. □

Of course, any norm on the vector space X is an F-norm on X. The "standard" example of an F-norm is as follows: if $(\Sigma, \mathfrak{M}, \mu)$ is a finite measure space, and X is the vector space of scalar-valued, $\mathfrak{M}$-measurable functions on Σ that are finite μ-a.e., then an F-norm on X is given by

$$\|f\| := \int_\Sigma \frac{|f(x)|}{1 + |f(x)|} \, d\mu(x), \qquad \forall f \in X. \tag{3.425}$$

It is then natural to consider the related, yet more flexible, notion of `quasi-F-norm`, defined in a similar manner as before but with (3.409) weakened to a quasitriangle inequality, and with (3.410) altered by allowing a fixed multiplicative constant on the right-hand side (thus, in particular, any quasinorm is a quasi-F-norm). Specifically, given a vector space X, assume that $\|\cdot\| : X \to [0, +\infty)$ is a function with the property that, for some constants $c_0, c_1 \in [1, +\infty)$,

$$\|x\| = 0 \iff x = 0 \text{ for each } x \in X, \tag{3.426}$$

$$\|x + y\| \le c_0 \left(\|x\| + \|y\|\right) \text{ for all } x, y \in X, \tag{3.427}$$

$$\|\lambda x\| \le c_1 \|x\| \text{ for all } x \in X \text{ if } |\lambda| \le 1, \tag{3.428}$$

$$\|\lambda_n x\| \to 0 \text{ for each fixed } x \in X \text{ if } \lambda_n \to 0. \tag{3.429}$$

From Lemma 3.40 it follows that property (3.412) continues to hold even under these more relaxed conditions.

To give an example of a quasi-F-norm, assume that $(\Sigma, \mathfrak{M}, \mu)$ is a finite measure space, and, as before, denote by X the vector space of scalar-valued, $\mathfrak{M}$-measurable functions on Σ, which are finite μ-a.e. on Σ. Then a quasi-F-norm on X is given by

$$\|f\| := \int_\Sigma \varphi\left(|f(x)|\right) d\mu(x), \qquad \forall f \in X, \tag{3.430}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a bounded function that vanishes continuously at the origin, is strictly positive on $(0, +\infty)$, is Borel-measurable, and satisfies, for some fixed constant $C \in [1, +\infty)$,

$$\varphi(\lambda t) \le C\varphi(t), \ \forall \lambda \in (0, 1), \quad \forall t \in [0, +\infty), \tag{3.431}$$

$$\varphi(t + s) \le C\left(\varphi(t) + \varphi(s)\right), \ \forall t, s \in [0, +\infty). \tag{3.432}$$

In relation to this, it is relevant to note that for a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq X$ one has $\|f_n\| \to 0$ as $n \to \infty$ if and only if $f_n \to 0$ in measure as $n \to \infty$. Also, a convenient way of producing examples of functions φ satisfying the aforementioned properties is to take

$$\varphi(t) := \left(\frac{t}{1+t}\right)\theta(t), \qquad \forall\, t \geq 0, \tag{3.433}$$

where θ is any Borel-measurable function bounded away from zero and infinity. Observe that (3.425) corresponds to the recipe from (3.430) used with φ as in (3.433) with $\theta(t) \equiv 1$, in which case (3.431) and (3.432) hold with $C = 1$. In general, if $1 \leq \theta(t) \leq C$ for every $t \geq 0$, then this observation may be used to give a short proof of the fact that (3.431) and (3.432) hold as stated whenever φ is as in (3.433).

Our next goal is to show that the properties of a given quasi-F-norm improve once it undergoes a suitable regularization procedure in the spirit of the discussion in this subsection. This is made precise in the theorem below. Before we state it, recall that a p-`norm` $(0 < p \leq 1)$ on a vector space X is a function $\|\cdot\| : X \to [0, +\infty)$ satisfying

$$\|x\| = 0 \iff x = 0 \ \text{ for each } \ x \in X, \tag{3.434}$$

$$\|x + y\| \leq \|x\| + \|y\| \ \text{ for all } \ x, y \in X, \tag{3.435}$$

$$\|\lambda x\| = |\lambda|^p \|x\| \ \text{ for all } x \in X \text{ and all } \lambda \in \mathbb{R}. \tag{3.436}$$

The following result may be regarded as a version of the Aoki–Rolewicz theorem in the more general context of vector spaces equipped with quasi-F-norms.

Theorem 3.41. *Let X be a vector space, and assume that $\|\cdot\| : X \to [0, +\infty)$ is a quasi-F-norm. That is, there exist $c_0, c_1 \in [1, +\infty)$ such that (3.426)–(3.429) hold. Define*

$$p := \frac{1}{1 + \log_2 c_0} \in (0, 1], \tag{3.437}$$

and for each $x \in X$ set

$$\|x\|_* := \sup_{|\lambda| \leq 1} \inf \left\{ \sum_{i=1}^{N} \|\lambda x_i\|^p : \ N \in \mathbb{N}, \ \ x_1, \dots, x_N \in X \ \ \textit{with} \ \ \sum_{i=1}^{N} x_i = x \right\}. \tag{3.438}$$

Then $\|\cdot\|_* : X \to [0, +\infty)$ is a genuine F*-norm, and $\|\cdot\|_* \approx \|\cdot\|^p$ in the precise sense that*

$$4^{-1}\|x\|^p \leq \|x\|_* \leq c_1^p \|x\|^p \ \ \textit{for all} \ \ x \in X. \tag{3.439}$$

In addition,

$$\|\cdot\| \text{ is a quasinorm on } X \implies \|\cdot\|_* \text{ is a } p\text{-norm on } X. \tag{3.440}$$

This result is sharp, in the following sense. For any (nontrivial) normed vector space X and any $c_0 \in [1,+\infty)$ one can find a quasi-F-norm $\|\cdot\|$ on X that satisfies (3.427) for the given constant c_0, satisfies (3.428) with $c_1 := 1$, and has the property that the existence of an F-norm $\|\cdot\|_\star$ on X such that $\|\cdot\|^q \approx \|\cdot\|_\star$ for some $q \in (0,+\infty)$ necessarily forces $q \leq (1+\log_2 c_0)^{-1}$.

Proof. Note that $\|x+y\| \leq 2c_0 \max\{\|x\|,\|y\|\} = 2^{1/p}\max\{\|x\|,\|y\|\}$ for all vectors $x, y \in X$. Denote by $\|\cdot\|_\# : X \to [0,+\infty)$ the function obtained by regularizing the quasisubadditive function $\|\cdot\| : X \to [0,+\infty)$ relative to the group $(X,+)$, as described in Theorem 3.28 with $C_1 := 2^{1/p}$ and $\alpha := p$ (hence, $\|\cdot\|_\#$ is defined as in (3.316) with $\alpha := p$). Then, by design,

$$\|x\|_* = \sup_{\lambda\in\mathbb{R},\,|\lambda|\leq 1} \|\lambda x\|_\#^p, \qquad \forall\, x \in X, \tag{3.441}$$

while Theorem 3.28 ensures that

$$2^{-2/p}\|x\| \leq \|x\|_\# \leq \|x\| \quad \text{for all} \quad x \in X, \tag{3.442}$$

$$\|x+y\|_\#^p \leq \|x\|_\#^p + \|y\|_\#^p \quad \text{for all} \quad x, y \in X. \tag{3.443}$$

As a result, $\|\lambda x\|_\#^p \leq \|\lambda x\|^p \leq c_1^p\|x\|^p$ for every $x \in X$ and every $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. Taking the supremum over all such λ proves the second inequality in (3.439). Also, taking $\lambda = 1$ in the supremum process in (3.441) implies that $\|x\|_* \geq \|x\|_\#^p \geq 4^{-1}\|x\|$ for each $x \in X$. Thus, the first inequality in (3.439) holds as well. Note that axioms (3.408) and (3.411), (3.412), formulated for $\|\cdot\|_*$, subsequently follow from (3.439) with the help of (3.426), (3.429), and Lemma 3.40. As regards the triangle inequality for $\|\cdot\|_*$, observe that if $x, y \in X$, and if $\lambda \in \mathbb{R}$ satisfies $|\lambda| \leq 1$, then

$$\|\lambda(x+y)\|_\#^p = \|\lambda x + \lambda y\|_\#^p \leq \|\lambda x\|_\#^p + \|\lambda y\|_\#^p \leq \|x\|_* + \|y\|_*, \tag{3.444}$$

by (3.443) and (3.441). Taking the supremum over all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ then yields $\|x+y\|_* \leq \|x\|_* + \|y\|_*$, as desired.

At this point, as far as the claim that $\|\cdot\|_*$ is an F-norm is concerned, it remains to check the analog of (3.410) for $\|\cdot\|_*$. To this end, if $x \in X$ and $\lambda, \eta \in \mathbb{R}$ with $|\lambda| \leq 1$, $|\eta| \leq 1$ are arbitrary, then we have $\|(\lambda\eta)x\|_\#^p \leq \|x\|_*$ thanks to (3.441) and the fact that $|\lambda\eta| \leq 1$. Hence,

$$\|\eta x\|_* = \sup_{\lambda\in\mathbb{R},\,|\lambda|\leq 1} \|(\lambda\eta)x\|_\#^p \leq \|x\|_*. \tag{3.445}$$

This shows that $\|\cdot\|_* : X \to [0,+\infty)$ is a genuine F-norm. The implication in (3.440) is then justified by observing that $\|\lambda x\|_* = |\lambda|^p\|x\|_*$ for all $x \in X$ and all $\lambda \in \mathbb{R}$ by (3.441) if $\|\cdot\|$ is homogeneous since the latter condition forces $\|\cdot\|_\#$ to be homogeneous as well.

Finally, consider the claim pertaining to the optimality of the exponent p from (3.437) formulated in the last part of the statement of the theorem. In this regard, assume that $(X, |\cdot|)$ is an arbitrary, nontrivial normed vector space, and, given an arbitrary constant $c_0 \in [1, +\infty)$, define $\|\cdot\| : X \to [0, +\infty)$ by setting $\|x\| := |x|^{1+\log_2 c_0}$ for each $x \in X$. Then, clearly, $\|\cdot\|$ is a quasi-F-norm on X that satisfies

$$\begin{aligned}\|x + y\| &= |x + y|^{1+\log_2 c_0} \leq (|x| + |y|)^{1+\log_2 c_0} \\ &\leq 2^{\log_2 c_0}\left(|x|^{1+\log_2 c_0} + |y|^{1+\log_2 c_0}\right) = c_0\,(\|x\| + \|y\|), \qquad \forall\, x, y \in X,\end{aligned} \tag{3.446}$$

i.e., (3.427) holds for the given constant c_0. Also, for each $x \in X$ and each $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ we have $\|\lambda x\| = |\lambda x|^{1+\log_2 c_0} = |\lambda|^{1+\log_2 c_0}|x|^{1+\log_2 c_0} \leq |x|^{1+\log_2 c_0} = \|x\|$, which shows that (3.428) is satisfied with $c_1 := 1$. Assume next that $\|\cdot\|_\star$ is an F-norm on X such that $C^{-1}\|\cdot\|_\star \leq \|\cdot\|^q \leq C\|\cdot\|_\star$ on X for some $q \in (0, +\infty)$ and some finite constant $C \geq 1$. Fix a unit vector $x \in X$, and for each $n \in \mathbb{N}$ consider the finite family of vectors $x_i := 2^{-n}x$, $i \in \{1, \dots, 2^n\}$. Then

$$\begin{aligned}1 = \|x\|^q &= \Big\|\sum_{i=1}^{2^n} x_i\Big\|^q \leq C\Big\|\sum_{i=1}^{2^n} x_i\Big\|_\star \leq C\sum_{i=1}^{2^n}\|x_i\|_\star \\ &\leq C^2\sum_{i=1}^{2^n}\|x_i\|^q = C^2\sum_{i=1}^{2^n} 2^{-nq(1+\log_2 c_0)} = C^2\, 2^{n-nq(1+\log_2 c_0)},\end{aligned} \tag{3.447}$$

which necessarily forces $q \leq (1 + \log_2 c_0)^{-1}$. This completes the proof of the theorem. □

We wish to note that other variants of the notion of quasi-F-norm are amenable to the same type of regularization procedure as in Theorem 3.41. For example, this is the case for a notion of a quasi-Fréchet norm $\|\cdot\|$ on a vector space X satisfying [for some fixed $c \in [1, +\infty)$]

$$\|x\| \in [0, +\infty) \;\text{ and }\; \|x\| = 0 \Leftrightarrow x = 0 \;\text{ for each }\; x \in X, \tag{3.448}$$

$$\|x + y\| \leq c\,(\|x\| + \|y\|) \quad \text{for all } \; x, y \in X, \tag{3.449}$$

$$\|\lambda x\| \leq c\|x\| \quad \text{for all } x \in X \text{ and scalars with } |\lambda| = 1, \tag{3.450}$$

$$\|\lambda_n x\| \to 0 \;\text{ for each fixed } x \in X \text{ if } \lambda_n \to 0, \tag{3.451}$$

$$\|\lambda_n x_n\| \to 0 \;\text{ if } \{\lambda_n\}_n \text{ bounded and } \|x_n\| \to 0. \tag{3.452}$$

We conclude this section by presenting yet another example of the regularization procedure under additional constraints. To set the stage, we make the following definition.

Definition 3.42. Let $(H, \circ)$ be a group, and consider X an arbitrary set. A `left group action` of H on X is a binary operation $\odot : H \times X \to X$ satisfying the following axioms:

(1) (Associativity)

$$(\lambda_1 \circ \lambda_2) \odot a = \lambda_1 \odot (\lambda_2 \odot a), \qquad \forall\, \lambda_1, \lambda_2 \in H \quad \text{and} \quad \forall\, a \in X. \tag{3.453}$$

(2) (Identity)

$$e_H \odot a = a, \qquad \forall\, a \in H, \tag{3.454}$$

where e_H denotes the identity element in H.

What follows is the statement of the regularization with constraints mentioned earlier.

Theorem 3.43. *Let $(G, *)$ be a semigroupoid, and consider $(H, \circ)$ a group that has a left group action $\odot : H \times G \to H$ on the set G. In addition, suppose that $\odot$ satisfies the following distributivity axiom:*

$$\left.\begin{array}{l} (a,b) \in G^{(2)} \\ \lambda \in H \end{array}\right\} \implies \left\{\begin{array}{l} (\lambda \odot a, \lambda \odot b) \in G^{(2)} \text{ and} \\ (\lambda \odot a) * (\lambda \odot b) = \lambda \odot (a * b). \end{array}\right. \tag{3.455}$$

Let $\varphi : H \to (0, +\infty)$ be a semigroup homomorphism, and assume that the function $\psi : G \to [0, +\infty]$ has the property that there exist two constants $C_0, C_1 \in [1, +\infty)$ such that

$$\psi(\lambda \odot a) \le C_0 \varphi(\lambda) \psi(a), \qquad \forall\, \lambda \in H \quad \text{and} \quad \forall\, a \in G, \tag{3.456}$$

$$\psi(a * b) \le C_1 \max\{\psi(a), \psi(b)\}, \qquad \forall\, (a,b) \in G^{(2)}. \tag{3.457}$$

Define $\psi_\star : G \to [0, +\infty]$ by setting

$$\psi_\star(a) := \sup_{\lambda \in H} \left[\varphi(\lambda^{-1}) \psi(\lambda \odot a)\right], \qquad \forall\, a \in G, \tag{3.458}$$

where λ^{-1} denotes the inverse of $\lambda \in H$ in the group $(H, \circ)$, and introduce

$$\psi_{\star\star} := (\psi_\star)_\#, \tag{3.459}$$

where the subscript # denotes α-regularization as described in (3.316) and (3.317) with

$$\alpha := \frac{1}{\log_2 C_1}. \tag{3.460}$$

Then the following properties hold:

$$C_1^{-2}\psi \le \psi_{\star\star} \le C_0\psi \quad \textit{on} \quad G, \quad \textit{and in particular } \psi_{\star\star} \approx \psi; \tag{3.461}$$

$$\psi_{\star\star}(\lambda \odot a) = \varphi(\lambda)\psi_{\star\star}(a), \quad \forall\, \lambda \in H \quad \textit{and} \quad \forall\, a \in G; \tag{3.462}$$

$$\psi_{\star\star}(a * b) \le C_1 \max\{\psi_{\star\star}(a), \psi_{\star\star}(b)\}, \quad \forall\, (a,b) \in G^{(2)}; \tag{3.463}$$

$$\beta \in \Big(0, (\log_2 C_1)^{-1}\Big] \Rightarrow \psi_{\star\star}(a * b) \le \Big(\psi_{\star\star}(a)^\beta + \psi_{\star\star}(b)^\beta\Big)^{1/\beta}, \quad \forall\, (a,b) \in G^{(2)}. \tag{3.464}$$

Proof. We begin with a couple of useful observations. First, since $\varphi : H \to (0, +\infty)$ is a semigroup homomorphism, $\varphi(e_H) = \varphi(e_H \circ e_H) = \varphi(e_H)\varphi(e_H)$, it follows that

$$\varphi(e_H) = 1. \tag{3.465}$$

Second, using identity (3.465), for each $\lambda \in H$ we may write that $1 = \varphi(e_H) = \varphi(\lambda \circ \lambda^{-1}) = \varphi(\lambda)\varphi(\lambda^{-1})$, from which we obtain that

$$\varphi(\lambda^{-1}) = \frac{1}{\varphi(\lambda)}, \qquad \forall\, \lambda \in H. \tag{3.466}$$

Since $e_H^{-1} = e_H$, based on (3.465) and the identity axiom (3.454), we have

$$\varphi(e_H^{-1})\psi(e_H \odot a) = \psi(a), \qquad \forall\, a \in G, \tag{3.467}$$

and thus, using the definition of $d_\star$ from (3.458),

$$\psi(a) \le \psi_\star(a), \qquad \forall\, a \in G. \tag{3.468}$$

Going further, using (3.456) for each $\lambda \in H$ we may write

$$\varphi(\lambda^{-1})\psi(\lambda \odot a) \le C_0\varphi(\lambda^{-1})\varphi(\lambda)\psi(a) = C_0\psi(a), \qquad \forall\, a \in G, \tag{3.469}$$

and thus, taking the supremum over $\lambda \in H$ in (3.469) allows us to conclude that

$$\psi_\star(a) \le C_0\psi(a), \qquad \forall\, a \in G. \tag{3.470}$$

Next, for each $\lambda \in H$ and each $(a,b) \in G^{(2)}$, employing the distributivity axiom (3.455), (3.457), and (3.458), there holds

$$\begin{aligned} \varphi(\lambda^{-1})\psi(\lambda \odot (a * b)) &= \varphi(\lambda^{-1})\psi\left((\lambda \odot a) * (\lambda \odot b)\right) \\ &\le C_1 \max\left\{\varphi(\lambda^{-1})\psi(\lambda \odot a), \varphi(\lambda^{-1})\psi(\lambda \odot b)\right\} \\ &\le C_1 \max\{\psi_\star(a), \psi_\star(b)\}. \end{aligned} \tag{3.471}$$

Taking the supremum over $\lambda \in H$ in (3.471) and using again the definition of $d_\star$ from (3.458) allows us to conclude that

$$\psi_\star(a * b) \le C_1 \max\{\psi_\star(a), \psi_\star(b)\}, \qquad \forall\,(a,b) \in G^{(2)}. \tag{3.472}$$

From Theorem 3.28 and the fact that # denotes α-regularization as described in (3.316) and (3.317), with α as in (3.460), estimate (3.472) further implies that

$$C_1^{-2}\psi_\star \le (\psi_\star)_\# = \psi_{\star\star} \le \psi_\star. \tag{3.473}$$

Consequently, using (3.473), (3.468), and (3.470) we obtain

$$\psi \le \psi_\star \le C_1^2\,(\psi_\star)_\# \quad \text{and} \quad (\psi_\star)_\# \le \psi_\star \le C_0\psi. \tag{3.474}$$

Now (3.461) immediately follows from the two sequences of estimates in (3.474).

Turning our attention to proving (3.462), we start by claiming that

$$\psi_*(\lambda \odot a) = \varphi(\lambda)\psi_\star(a), \qquad \forall\,\lambda \in H \quad \text{and} \quad \forall\, a \in G. \tag{3.475}$$

To see this, fix $\lambda \in H$ and $a \in G$, and observe that since φ is a semigroup homomorphism and $(\eta\circ\lambda)^{-1} = \lambda^{-1}\circ\eta^{-1}$, there holds $\varphi(\eta^{-1}) = \varphi(\lambda)\varphi\big((\eta\circ\lambda)^{-1}\big)$ for each $\eta \in H$. As such, using this and the associativity axiom (3.453), for each $\eta \in H$ we may write

$$\varphi(\lambda)\varphi\Big((\eta\circ\lambda)^{-1}\Big)\,\psi\left((\eta\circ\lambda)\odot a\right) = \varphi(\eta^{-1})\psi\left(\eta\odot(\lambda\odot a)\right) \le \psi_\star(\lambda\odot a), \tag{3.476}$$

where the inequality follows from the definition of $\psi_\star$ in (3.458). Given that the mapping $H \ni \eta \to \eta\circ\lambda \in H$ is a bijection, if we take the supremum over $\eta \in H$ in (3.476) and use again (3.458), then

$$\varphi(\lambda)\psi_\star(a) \le \psi_\star(\lambda\odot a). \tag{3.477}$$

On the other hand, it follows from the definition of $\psi_\star$ that, for each $\eta \in H$, the first term in (3.476) is $\le \varphi(\lambda)\psi_\star(a)$. Thus, using the identity from (3.476) we obtain that

$$\varphi(\eta^{-1})\psi\left(\eta\odot(\lambda\odot a)\right) \le \varphi(\lambda)\psi_\star(a), \qquad \forall\,\eta \in H. \tag{3.478}$$

Taking the supremum over $\eta \in H$ in (3.478) further implies

$$\psi_\star(\lambda\odot a) \le \varphi(\lambda)\psi_\star(a). \tag{3.479}$$

In concert, (3.477), (3.478), and the fact that $\lambda \in H$ and $a \in G$ were arbitrary give (3.475), as desired.

Going further, fix again $\lambda \in H$ and $a \in G$, and recall α from (3.460). Since by definition $\psi_{\star\star}$ is the α-regularization of $\psi_\star$, it follows that

$$\psi_{\star\star}(\lambda \odot a) := \inf\left\{\left(\sum_{i=1}^{N} \psi_*(a_i)^\alpha\right)^{1/\alpha} : N \in \mathbb{N},\ (a_1, \dots, a_N) \in G^{(N)},\right.$$
$$\left. \text{and } a_1 * \cdots * a_N = \lambda \odot a\right\}. \tag{3.480}$$

Next, note that using repeatedly the distributivity axiom (3.455), for each $N \in \mathbb{N}$ there holds

$$\left.\begin{array}{l}(a_1, \dots, a_N) \in G^{(N)} \\ a_1 * \cdots * a_N = \lambda \odot a\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\big(\lambda^{-1} \odot a_1, \dots, \lambda^{-1} \odot a_N\big) \in G^{(N)}, \\ (\lambda^{-1} \odot a_1) * \cdots * (\lambda^{-1} \odot a_N) = a.\end{array}\right. \tag{3.481}$$

Indeed, an N-tuple $(a_1, \dots, a_N)$ belongs to $G^{(N)}$ provided $(a_i, a_{i+1}) \in G^{(2)}$ for each $i \in \{1, \dots, N-1\}$. Using (3.455) this further implies $(\lambda^{-1} \odot a_i, \lambda^{-1} \odot a_{i+1}) \in G^{(2)}$ for all $i \in \{1, \dots, N-1\}$ and, consequently, $\big(\lambda^{-1} \odot a_1, \dots, \lambda^{-1} \odot a_N\big) \in G^{(N)}$. If in addition to $(a_1, \dots, a_N) \in G^{(N)}$ we also have that $a_1 * \cdots * a_N = \lambda \odot a$, then it follows from (3.455), a straightforward induction argument, and the associativity axiom (3.453) that

$$(\lambda^{-1} \odot a_1) * \cdots * (\lambda^{-1} \odot a_N) = \lambda^{-1} \odot (a_1 * \cdots * a_N) = \lambda^{-1} \odot (\lambda \odot a) = a. \tag{3.482}$$

Hence, for every participant $(a_1, \dots, a_N) \in G^{(N)}$ in the infimum defining $\psi_{\star\star}(\lambda \odot a)$ in (3.480), the N-tuple $\big(\lambda^{-1} \odot a_1, \dots, \lambda^{-1} \odot a_N\big) \in G^{(N)}$ is a participant in the infimum defining $\psi_{\star\star}(a)$. This and (3.475) give that for each $N \in \mathbb{N}$ and $(a_1, \dots, a_N) \in G^{(N)}$ such that $a_1 * \cdots * a_N = \lambda \odot a$ there holds

$$\psi_{\star\star}(a) \leq \left(\sum_{i=1}^{N} \psi_\star(\lambda^{-1} \odot a_i)^\alpha\right)^{1/\alpha} = \varphi(\lambda^{-1}) \left(\sum_{i=1}^{N} \psi_\star(a_i)^\alpha\right)^{1/\alpha}. \tag{3.483}$$

Taking the infimum in (3.483) over N and all N-tuples $(a_1, \dots, a_N) \in G^{(N)}$ such that $a_1 * \cdots * a_N = \lambda \odot a$, and using that $\lambda \in H$ and $a \in G$ are arbitrary, we obtain

$$\psi_{\star\star}(a) \leq \varphi(\lambda^{-1})\psi_{\star\star}(\lambda \odot a), \qquad \forall\, \lambda \in H \quad \text{and} \quad \forall\, a \in G. \tag{3.484}$$

Replacing first a by $\lambda^{-1} \odot a$ in (3.484) and then λ^{-1} by λ allows us to establish that also

$$\psi_{\star\star}(\lambda \odot a) \leq \varphi(\lambda)\psi_{\star\star}(a), \qquad \forall\, \lambda \in H \quad \text{and} \quad \forall\, a \in G. \tag{3.485}$$

Now (3.484) and (3.485) immediately give (3.462).

Next we will prove (3.463). To this end, consider $(a, b) \in G^{(2)}$, and write

$$\psi_{\star\star}(a * b) = (\psi_{\star})_{\#}(a * b) \leq C_1 \max\{(\psi_{\star})_{\#}(a), (\psi_{\star})_{\#}(b)\}$$
$$= C_1 \max\{\psi_{\star\star}(a), \psi_{\star\star}(b)\}, \tag{3.486}$$

where the first equality follows from the definition of $\psi_{\star\star}$ in (3.459), the first inequality follows from (3.472) and (3.321), and the last equality relies again on the definition (3.459). This completes the proof of (3.463). Finally, (3.464) is a direct consequence of the definition of $\psi_{\star\star}$ in (3.459) and (3.472). □

Below we discuss a couple of concrete settings in which the regularization result established in Theorem 3.43 applies.

Example 3.44. Let $(X, +)$ be an arbitrary Abelian group, and take $(G, *)$ to be the pair semigroupoid associated with the set X. That is, $G := X \times X$, with the set of composable pairs $G^{(2)} := \{((x, y), (y, z)) : x, y, z \in X\}$ and the semigroupoid binary operation $(x, y) * (y, z) := (x, z)$ for every $((x, y), (y, z)) \in G^{(2)}$. Consider the group $(H, \circ) := (X, +)$ and its left group action $\odot : H \times G \to G$ on G defined by

$$\lambda \odot (x, y) := (x + \lambda, y + \lambda), \qquad \forall\, \lambda \in H \quad \text{and} \quad \forall\, (x, y) \in G, \tag{3.487}$$

along with the semigroup homomorphism $\varphi : H \to (0, +\infty)$ given by $\varphi(\lambda) = 1$ for each $\lambda \in H$. Then Theorem 3.43 applies and yields a regularization result for any function $\psi : X \times X \to [0, +\infty]$ with the property that there exist two constants $C_0, C_1 \in [1, +\infty)$ such that

$$\psi(x + z, y + z) \leq C_0 \varphi(z) \psi(x, y), \ \forall\, x, y, z \in X, \tag{3.488}$$

$$\psi(x, y) \leq C_1 \max\{\psi(x, z), \psi(z, y)\}, \ \forall\, x, y, z \in X, \tag{3.489}$$

such that the regularization $\psi_{\star\star}$ satisfies (3.488) with equality and $C_0 = 1$, among other things. ■

Example 3.45. Let X be a vector space, and consider the semigroupoid $(G, *)$ as in Example 3.44, along with the group $(H, \circ) := (\mathbb{R} \setminus \{0\}, \cdot)$, where $\cdot$ denotes the multiplication of real numbers. Define the left group action $\odot$ of H on G by setting

$$\odot : H \times G \to G, \ \lambda \odot (x, y) := (\lambda x, \lambda y), \ \forall\, \lambda \in \mathbb{R} \setminus \{0\}, \ \ \forall\, (x, y) \in G. \tag{3.490}$$

Finally, let $\varphi : H \to (0, +\infty)$ be the semigroup homomorphism given by $\varphi(\lambda) := |\lambda|$ for each $\lambda \in H = \mathbb{R} \setminus \{0\}$. Then, once again, Theorem 3.43 applies and yields a regularization result for any function $\psi : X \times X \to [0, +\infty]$ that has the property that there exist two constants $C_0, C_1 \in [1, +\infty)$ such that

$$\psi(\lambda x, \lambda y) \leq C_0 \varphi(\lambda)\psi(x,y), \ \forall\, \lambda \in \mathbb{R}\setminus\{0\} \quad \text{and} \quad \forall\, x,y \in X, \tag{3.491}$$

$$\psi(x,y) \leq C_1 \max\{\psi(x,z), \psi(z,y)\}, \ \forall\, x,y,z \in X, \tag{3.492}$$

such that the regularization $\psi_{\star\star}$ satisfies (3.491) with equality and $C_0 = 1$, among other things. ■

3.4 A Sharp Metrization Result for Quasimetric Spaces

Here we will specialize Theorem 3.26 to the case where the groupoid in question is the pair groupoid associated with the ambient set X of a quasimetric structure (X, ρ). As mentioned in Sect. 1, this yields a sharpened version of the Macías–Segovia result presented in Theorem 1.2. Since the latter result has been hailed as a *cornerstone of the analysis on quasimetric spaces* (cf. [122, p. 25], [123, p. 113], etc.) and is a result of wide, independent interest, we will actually record the most general and nuanced version that the current work yields. In particular, it is now possible to further refine many of the existing results in the literature whose proofs make use of Macías–Segovia's theorem (cf. Theorem 1.2) by using instead Theorem 3.46 below. The reader is referred to Sect. 4, where a large number of such concrete examples are considered in detail.

Compared with the generic statement that the canonical topology induced by a quasimetric is metrizable (which follows directly from the Alexandroff–Urysohn theorem; cf. Theorem 1.1), our Theorem 3.46 gives a concrete formula for the distance yielding the same topology as that induced by the original quasidistance (see (3.519) below), which, in particular, makes it transparent how this distance is related to the original quasidistance.

Theorem 3.46. *Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \to [0, +\infty]$ and an arbitrary exponent $\alpha \in (0, +\infty]$, define the function*

$$\rho_\alpha : X \times X \longrightarrow [0, +\infty] \tag{3.493}$$

by setting for each $x, y \in X$

$$\rho_\alpha(x,y) := \inf\Bigg\{\Bigg(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha\Bigg)^{\frac{1}{\alpha}} : N \in \mathbb{N} \text{ and } \xi_1, \ldots, \xi_{N+1} \in X$$
$$\text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Bigg\}, \tag{3.494}$$

whenever $\alpha < +\infty$ *and its natural counterpart, corresponding to the case when* $\alpha = +\infty$*, i.e.,*

$$\rho_\infty(x,y) := \inf\Big\{ \max_{1\le i\le N} \rho(\xi_i,\xi_{i+1}) : N\in\mathbb{N} \text{ and } \xi_1,\dots,\xi_{N+1}\in X \text{ (not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y \Big\}. \tag{3.495}$$

Then the following conclusions hold for any function $\rho : X\times X \to [0,+\infty]$ *and any exponent* $\alpha\in(0,+\infty]$.

(1) *One has:*

$$\rho_\alpha \text{ is well defined and } \rho_\alpha \le \rho \text{ on } X\times X; \tag{3.496}$$

$$\beta\in(0,\alpha] \text{ finite } \Rightarrow \rho_\alpha(x,y) \le \big([\rho_\alpha(x,z)]^\beta + [\rho_\alpha(z,y)]^\beta\big)^{\frac{1}{\beta}}, \quad \forall\, x,y,z\in X; \tag{3.497}$$

$$\rho_\infty(x,y) \le \max\{\rho_\infty(x,z),\, \rho_\infty(z,y)\}, \quad \forall\, x,y,z\in X; \tag{3.498}$$

$$\rho_\alpha(x,y) \le 2^{1/\alpha}\max\{\rho_\alpha(x,z),\, \rho_\alpha(z,y)\}, \quad \forall\, x,y,z\in X. \tag{3.499}$$

In addition, if $\alpha < +\infty$*, then*

$$\rho = \rho_\alpha \iff \rho \text{ is } \alpha\text{-subadditive}, \tag{3.500}$$

i.e., $\rho(x,y) \le (\rho(x,z)^\alpha + \rho(z,y)^\alpha)^{\frac{1}{\alpha}}$, $\forall\, x,y,z\in X$.

Also,

$$(\rho_\alpha)_\beta = \rho_\alpha \quad \text{whenever } \beta\in(0,\alpha], \tag{3.501}$$

$$(\rho^\beta)_\alpha = (\rho_{\alpha\beta})^\beta \quad \text{whenever } \beta\in(0,+\infty), \tag{3.502}$$

$$(\lambda\rho)_\alpha = \lambda\,\rho_\alpha \text{ for each } \lambda\in[0,+\infty), \tag{3.503}$$

$$\rho_\alpha \le \rho'_\alpha \text{ for each } \rho' : X\times X\to[0,+\infty] \text{ such that } \rho\le\rho' \text{ on } X. \tag{3.504}$$

(2) *Consider the function* $\rho_{\rm sym} : X\times X\to[0,+\infty]$ *defined by*

$$\rho_{\rm sym}(x,y) := \max\{\rho(x,y),\rho(y,x)\}, \quad \forall\, x,y\in X. \tag{3.505}$$

Then

$$\rho_{\rm sym} \text{ is symmetric, i.e., } \rho_{\rm sym}(x,y) = \rho_{\rm sym}(y,x) \text{ for every } x,y\in X; \tag{3.506}$$

$$\rho \text{ is symmetric } \iff \rho = \rho_{\rm sym}; \tag{3.507}$$

$$(\rho_{\rm sym})_{\rm sym} = \rho_{\rm sym}. \tag{3.508}$$

Moreover, $\rho_{\rm sym}$ *may be characterized as the smallest* $[0,+\infty]$*-valued function defined on* $X \times X$ *that is symmetric and pointwise* $\geq \rho$*.*

(3) *One has* $\rho \approx \rho_{\rm sym}$ *if and only if* ρ *is quasisymmetric, i.e., there exists a finite constant* $C_0 \geq 0$ *such that*

$$\rho(y,x) \leq C_0\,\rho(x,y), \qquad \forall\, x,y \in X. \tag{3.509}$$

Furthermore, in the case when (3.509) *holds, one actually has*

$$\rho \leq \rho_{\rm sym} \leq \max\{1, C_0\}\,\rho. \tag{3.510}$$

(4) *The function* ρ *is quasisymmetric if and only if there exists* $\rho' : X \times X \to [0,+\infty]$ *that is symmetric and such that* $\rho \approx \rho'$*.*

(5) *If* $\iota : X \times X \to X \times X$ *is defined by* $\iota(x,y) := (y,x)$ *for every* $x,y \in X$*, then*

$$\rho_\alpha \circ \iota = (\rho \circ \iota)_\alpha \quad \text{on} \;\; X \times X, \tag{3.511}$$

$$\rho \text{ symmetric} \Longrightarrow \rho_\alpha \text{ symmetric}, \tag{3.512}$$

$$\rho \text{ quasisymmetric} \Longrightarrow \rho_\alpha \text{ quasisymmetric}. \tag{3.513}$$

Specifically, if ρ *satisfies* (3.509)*, then so does* ρ_α*, and with the same constant* C_0*.*

Say that $\rho : X \times X \to [0,+\infty]$ *satisfies a quasiultrametric condition if there exists a finite constant* $C_1 \geq 1$ *with the property that*

$$\rho(x,y) \leq C_1\, \max\{\rho(x,z), \rho(z,y)\}, \qquad \forall\, x,y,z \in X. \tag{3.514}$$

Moreover, whenever this is the case, define

$$\alpha := \frac{1}{\log_2 C_1} \in (0,+\infty]. \tag{3.515}$$

Then the following properties hold.

(6) *If* $\rho : X \times X \to [0,+\infty]$ *satisfies the quasiultrametric condition* (3.514)*, then its symmetrized version,* $\rho_{\rm sym}$ *defined as in* (3.505)*, also satisfies a quasiultrametric condition and with the same constant as in the case of* ρ*.*

(7) *If* $\rho : X \times X \to [0,+\infty]$ *satisfies the quasiultrametric condition* (3.514)*, and if* α *is as in* (3.515)*, then* $\rho_\alpha \approx \rho$*. More specifically, with* C_1 *the same constant as in* (3.514)*, one has*

$$C_1^{-2}\rho \leq \rho_\alpha \leq \rho \qquad \text{on} \;\; X \times X. \tag{3.516}$$

In particular, ρ_α is finite precisely when ρ is finite, and they vanish simultaneously, i.e.,

$$(\rho_\alpha)^{-1}(\{0\}) = \rho^{-1}(\{0\}). \tag{3.517}$$

(8) Conversely, if $\rho : X \times X \to [0, +\infty]$ is a function for which there exist some exponent $\alpha' \in (0, +\infty]$ and some finite constant $C \geq 1$ with the property that

$$\rho \leq C\, \rho_{\alpha'} \quad \text{on } X \times X \tag{3.518}$$

(hence $\rho \approx \rho_{\alpha'}$ since the estimate $\rho_{\alpha'} \leq \rho$ is always true), then ρ satisfies the quasiultrametric condition (3.514) for the choice $C_1 := C\, 2^{1/\alpha'}$.

(9) Assume that $\rho : X \times X \to [0, +\infty]$ is a function satisfying the quasiultrametric condition (3.514) for some finite constant $C_1 \geq 1$. Define α as in (3.515) and construct ρ_α as in (3.494) and (3.495). Finally, fix a finite number $\beta \in (0, \alpha]$. Then the function

$$d_{\rho,\beta} : X \times X \to [0, +\infty], \qquad d_{\rho,\beta}(x, y) := [\rho_\alpha(x, y)]^\beta, \qquad \forall\, x, y \in X, \tag{3.519}$$

satisfies the following properties:

$$d_{\rho,\beta}(x, y) \leq d_{\rho,\beta}(x, z) + d_{\rho,\beta}(z, y), \qquad \forall\, x, y, z \in X; \tag{3.520}$$

$$\rho \text{ satisfies } (3.509) \Rightarrow d_{\rho,\beta}(y, x) \leq C_0^\beta\, d_{\rho,\beta}(x, y) \quad \forall\, x, y \in X; \tag{3.521}$$

$$\rho \text{ symmetric} \iff d_{\rho,\beta} \text{ symmetric}; \tag{3.522}$$

$$C_1^{-2}\rho(x, y) \leq \big[d_{\rho,\beta}(x, y)\big]^{1/\beta} \leq \rho(x, y), \quad \forall\, x, y \in X; \tag{3.523}$$

$$\rho^{-1}(\{0\}) = d_{\rho,\beta}^{-1}(\{0\}). \tag{3.524}$$

(10) Assume that $\rho : X \times X \to [0, +\infty)$ is a symmetric function satisfying the quasiultrametric condition (3.514) for some finite constant $C_1 \geq 1$. Define α as in (3.515), construct ρ_α as in (3.494) and (3.495), and, for a fixed, arbitrary, finite number $\beta \in (0, \alpha]$, define the function $d_{\rho,\beta}$ as in (3.519). Then

$$\begin{aligned} &\rho^{-1}(\{0\}) = \operatorname{diag}(X) \iff d_{\rho,\beta} \text{ is a distance on } X, \\ &\operatorname{diag}(X) \subseteq \rho^{-1}(\{0\}) \iff d_{\rho,\beta} \text{ is a pseudodistance on } X. \end{aligned} \tag{3.525}$$

Also, if $\operatorname{diag}(X) \subseteq \rho^{-1}(\{0\})$, then the pseudodistance $d_{\rho,\beta}$ induces the same topology on X as τ_ρ, where the latter is defined as the largest topology on X

with the property that a fundamental system of neighborhoods for each point $x \in X$ is given by the family of ρ-balls

$$B_\rho(x,r) := \{y \in X : \rho(x,y) < r\}, \tag{3.526}$$

indexed by all $r \in (0,+\infty)$. Furthermore, if $\operatorname{diag}(X) \subseteq \rho^{-1}(\{0\})$, *then*

$$\rho_\alpha : \big(X \times X, \tau_\rho \times \tau_\rho\big) \longrightarrow [0,+\infty) \text{ is continuous.} \tag{3.527}$$

(11) *Assume that $\rho : X \times X \to [0,+\infty]$ satisfies both the quasiultrametric condition* (3.514) *and the quasisymmetry condition* (3.509)*. Introduce $\rho_{\rm sym}$ as in* (3.505)*, and, with α as in* (3.515)*, construct the canonical regularization of ρ, namely,*

$$\rho_\# := (\rho_{\rm sym})_\alpha, \tag{3.528}$$

as described in (3.494) *and* (3.495) *but using $\rho_{\rm sym}$ instead of ρ. Finally, let $\beta \in (0,\alpha]$ be arbitrary. Then, with C_0 and C_1 denoting the same constants as in* (3.509) *and* (3.514)*, respectively, the following properties hold:*

$$\rho_\# \text{ is symmetric;} \tag{3.529}$$

$$\rho_\# \approx \rho \text{ in the precise sense that } C_1^{-2}\rho \leq \rho_\# \leq \max\{1, C_0\}\,\rho; \tag{3.530}$$

$$\rho_\#^{-1}(\{0\}) = \rho^{-1}(\{0\}); \tag{3.531}$$

$$\rho_\#(x,y) \leq \big(\rho_\#(x,z)^\beta + \rho_\#(z,y)^\beta\big)^{\frac{1}{\beta}}, \quad \forall\, x,y,z \in X; \tag{3.532}$$

$$\rho_\#(x,y) \leq C_1 \max\{\rho_\#(x,z)\,,\ \rho_\#(z,y)\}, \quad \forall\, x,y,z \in X; \tag{3.533}$$

$$C_1 = 1 \Longrightarrow \rho_\# = (\rho_{\rm sym})_\infty = \rho_{\rm sym}. \tag{3.534}$$

In addition, for any set Y,

$$\text{if } \phi : Y \to X \text{ is a bijection, and if } \widetilde{\rho}(x,y) := \rho(\phi(x),\phi(y)),\ \forall\, x,y \in Y$$

$$\Longrightarrow \widetilde{\rho}(y,x) \leq C_0\widetilde{\rho}(x,y),\ \widetilde{\rho}(x,y) \leq C_1 \max\{\widetilde{\rho}(x,z),\widetilde{\rho}(z,y)\},\ \forall\, x,y,z \in Y, \tag{3.535}$$

$$\text{and for every } x,y \in Y \text{ there holds } (\widetilde{\rho})_\#(x,y) = \rho_\#(\phi(x),\phi(y)).$$

(12) *Let $\rho : X \times X \to [0,+\infty)$ be a function satisfying the quasiultrametric condition* (3.514)*, the quasisymmetry condition* (3.509)*, and the nondegeneracy condition $\rho^{-1}(\{0\}) = \operatorname{diag}(X)$. Define α as in* (3.515)*, and introduce $\rho_\#$, the canonical regularization of ρ, as in* (3.528)*.*

Then the function $\rho_\#$ is a quasidistance on X (which is equivalent to ρ). In addition, the quasidistance $\rho_\#$ is actually a genuine distance on X if $\alpha \in [1,+\infty]$ (i.e., when $C_1 \in [1,2]$) and, in fact,

$$\rho \text{ distance on } X \text{ and } \alpha = 1 \text{ (i.e., } C_1 = 2) \Longrightarrow \rho_\# = \rho. \tag{3.536}$$

Furthermore, if for each finite number $\beta \in (0, \alpha]$ *one considers the function*

$$\widetilde{d}_{\rho,\beta} : X \times X \to [0, +\infty), \quad \widetilde{d}_{\rho,\beta}(x, y) := [\rho_\#(x, y)]^\beta, \quad \forall\, x, y \in X, \tag{3.537}$$

then $\widetilde{d}_{\rho,\beta}$ *is a distance on* X *that induces the same topology on* X *as* ρ *and, in fact, has the property that*

$$C_1^{-2}\rho(x, y) \le \left[\widetilde{d}_{\rho,\beta}(x, y)\right]^{1/\beta} \le \max\{1, C_0\}\, \rho(x, y), \quad \forall\, x, y \in X. \tag{3.538}$$

In particular,

$$\text{the topological space } (X, \tau_\rho) \text{ is metrizable.} \tag{3.539}$$

Moreover,

$$\rho_\# : \big(X \times X, \tau_\rho \times \tau_\rho\big) \longrightarrow [0, +\infty) \text{ is continuous.} \tag{3.540}$$

In fact, for each finite number $\beta \in (0, \alpha]$ *the canonical regularization* $\rho_\#$ *satisfies the following Hölder-type regularity condition of order* β *in both variables (simultaneously):*

$$\begin{aligned} &\big|\rho_\#(x, y) - \rho_\#(w, z)\big| \\ &\le \tfrac{1}{\beta} \max\left\{\rho_\#(x, y)^{1-\beta}, \rho_\#(w, z)^{1-\beta}\right\} \Big([\rho_\#(x, w)]^\beta + [\rho_\#(y, z)]^\beta\Big) \end{aligned} \tag{3.541}$$

whenever $x, y, z, w \in X$ *(with the understanding that when* $\beta \ge 1$*, one also imposes the conditions* $x \ne y$ *and* $z \ne w$*). As a consequence, for each finite number* $\beta \in (0, \alpha]$ *one has*

$$\big|\rho_\#(x, y) - \rho_\#(x, z)\big| \le \tfrac{1}{\beta} \max\left\{\rho_\#(x, y)^{1-\beta}, \rho_\#(x, z)^{1-\beta}\right\} [\rho_\#(y, z)]^\beta \tag{3.542}$$

whenever $x, y, z \in X$ *(with the understanding that when* $\beta \ge 1$*, one also imposes the condition that* $x \notin \{y, z\}$*).*

In addition,

$$B_\rho(x, r) \text{ is a neighborhood of } x \in X \text{ in } \tau_\rho \text{ for every } r > 0; \tag{3.543}$$

$$B_\rho(x, r/C) \subseteq \big(B_\rho(x, r)\big)^\circ \quad \text{if } C > C_\rho, \quad \forall\, x \in X, \ \forall\, r > 0; \tag{3.544}$$

$$\overline{B_\rho(x, r/C)} \subseteq B_\rho(x, r) \quad \text{if } C > C_\rho, \quad \forall\, x \in X, \ \forall\, r > 0, \tag{3.545}$$

where the interior $(\dots)^\circ$ and closure $\overline{(\dots)}$ are taken with respect to the topology τ_ρ.

(13) *The Hölder-type regularity result described in claim* (12) *is sharp, in the following precise sense. Given a finite number $C_1 > 1$, there exist a nonempty set X and a symmetric quasidistance $\rho : X \times X \to [0, +\infty)$ satisfying the quasiultrametric condition for the given C_1 and having the property that if $\rho' : X \times X \to [0, +\infty)$ is such that $\rho' \approx \rho$ and there exist $\beta \in (0, +\infty)$ and $C \in [0, +\infty)$ for which*

$$\left|\rho'(x, y) - \rho'(x, z)\right| \leq C \ \max\left\{\rho'(x, y)^{1-\beta}, \rho'(x, z)^{1-\beta}\right\}\left[\rho'(y, z)\right]^\beta \quad (3.546)$$

whenever $x, y, z \in X$ (and also $x \notin \{y, z\}$ if $\beta \geq 1$), then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \quad (3.547)$$

(14) *If throughout all preceding claims the quasiultrametric condition* (3.514) *is replaced by a condition stating that*

$$\rho(x, y) \leq C_2 \left([\rho(x, z)]^p + [\rho(z, y)]^p\right)^{1/p}, \ \textit{for all} \ \ x, y, z \in X, \quad (3.548)$$

for some exponent $p \in (0, +\infty)$ and some finite constant $C_2 \geq 2^{-1/p}$, then the same conclusions as before hold if in place of (3.515) *one defines α to be*

$$\alpha_p := \frac{p}{1 + p \ \log_2 C_2} \in (0, +\infty]. \quad (3.549)$$

Proof. With a couple of exceptions (treated separately below), this theorem is a summary of the results proved in Sect. 3.2 in the particular case when the groupoid G is the canonical pair groupoid on $X \times X$, as described in Example 2.31, and when the function $\psi : X \times X \to [0, +\infty]$ from Sect. 3.2 has been redenoted here as ρ. One of the exceptions mentioned above is (3.527) [which, in turn, has (3.540) as a consequence]. However, this follows from part *(ii)* in Lemma 2.70 applied to the pseudodistance $d_{\rho,\beta}$ from (3.519), keeping in mind that $\tau_{d_{\rho,\beta}} = \tau_{(\rho_\alpha)^\beta} = \tau_{\rho_\alpha} = \tau_\rho$. The other exception referred to previously is (3.541) (which implies (3.542) by making $w := x$). To prove this, in the case when $\beta < 1$, for arbitrary $x, y, w, z \in X$ we write based on (3.537), (3.183), and (2.185)

$$\begin{aligned}|\rho_\#(x, y) - \rho_\#(w, z)| &= \left|\left[\rho_\#(x, y)^\beta\right]^{\frac{1}{\beta}} - \left[\rho_\#(w, z)^\beta\right]^{\frac{1}{\beta}}\right| = \left|\widetilde{d}_{\rho,\beta}(x, y)^{\frac{1}{\beta}} - \widetilde{d}_{\rho,\beta}(w, z)^{\frac{1}{\beta}}\right| \\ &\leq \tfrac{1}{\beta}\left|\widetilde{d}_{\rho,\beta}(x, y) - \widetilde{d}_{\rho,\beta}(w, z)\right| \left[\max\left\{\rho_\#(x, y)^\beta, \rho_\#(w, z)^\beta\right\}\right]^{\frac{1}{\beta}-1} \\ &\leq \tfrac{1}{\beta}\left[\widetilde{d}_{\rho,\beta}(x, w) + \widetilde{d}_{\rho,\beta}(y, z)\right] \times\end{aligned}$$

$$\times \max\left\{\rho_\#(x,y)^{1-\beta}, \rho_\#(w,z)^{1-\beta}\right\}$$

$$= \frac{1}{\beta}\left[\rho_\#(x,w)^\beta + \rho_\#(y,z)^\beta\right] \times$$

$$\times \max\left\{\rho_\#(x,y)^{1-\beta}, \rho_\#(w,z)^{1-\beta}\right\}, \tag{3.550}$$

as desired. The case when $\beta \geq 1$ is treated similarly, this time making use of (3.185). This completes the proof of Theorem 3.46. □

In closing, we wish to note that a proof of conclusion (12) in Theorem 3.46 appears in [59, Proposition 14.5, p. 110] and in [2], albeit for smaller values of the exponent α (specifically, for $\alpha = (2\log_2 C_1)^{-1}$ in [59], which is half the value of α in (3.515), and for $\alpha_1 = (\log_2(3C_2^2))^{-1}$ in [2], which is strictly smaller than the value $\alpha_1 = (1 + \log_2 C_2)^{-1}$ predicted by (3.549)). Under the stronger condition of genuine symmetry for the function ρ and under the unnecessary restriction that α from (3.515) is ≤ 1, this portion of our result has also been recently dealt with in [93].[2] The monograph [50] also contains an alternative proof of conclusion (11) of Theorem 3.46 (cf. Theorem 1.1.3 on p. 5 in [50]), but for the nonoptimal value of the exponent α given in (1.2).

[2] We became aware of this reference right after we had completed the present work.

Chapter 4
Applications to Analysis on Quasimetric Spaces

The metrization result contained in Theorem 3.46 leads to natural improvements for a large number of basic results in the area of analysis on quasimetric spaces, and in this chapter we single out several topics for which we intend to show that our work so far has a significant impact. The presentation here underscores the thesis that a considerable portion of the well-established body of results in the area of analysis on metric spaces may be quite successfully developed in the more general context of quasimetric spaces. This is relevant since not only is the category of quasimetric spaces more inclusive than the category of metric spaces but, at the same time, the former constitutes a more natural and flexible setting than the latter. For example, the category of quasimetric spaces contains the family of all quasi-Banach spaces, which in turn encompasses a multitude of function spaces (measuring smoothness, on various scales) that are of fundamental importance in analysis. Also, as opposed to the case of metric spaces, the category of quasimetric spaces is stable under any positive "power dilation," $\rho \mapsto \rho^\alpha$, of the quasimetric. Lastly, we wish to note that while our results here are either generalizations of known facts or altogether new, in all cases the emphasis is on the optimality of smoothness measured on the Hölder scale.

4.1 Category of Quasimetric Spaces

We start with some preliminary considerations. For the remainder of our work, if X is a given set of cardinality ≥ 2, then we denote by $\mathfrak{Q}(X)$ the collection of all quasidistances on X. The reader is reminded that, as in (1.1), a quasidistance on X is a function $\rho : X \times X \to [0, +\infty)$ with the property that there exists a finite constant $c \geq 1$ such that for every $x, y, z \in X$ one has

$$\rho(x, y) = 0 \Longleftrightarrow x = y, \quad \rho(x, y) = \rho(y, x), \quad \rho(x, y) \leq c(\rho(x, z) + \rho(z, y)). \tag{4.1}$$

D. Mitrea et al., *Groupoid Metrization Theory*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-8397-9_4,

Next, for each $\rho \in \mathfrak{Q}(X)$ define

$$C_\rho := \sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z),\rho(z,y)\}}, \tag{4.2}$$

and note that

$$\forall\, \rho \in \mathfrak{Q}(X) \implies \rho(x,y) \le C_\rho \max\{\rho(x,z),\rho(z,y)\}, \quad \forall\, x,y,z \in X, \tag{4.3}$$

$$\forall\, \rho \in \mathfrak{Q}(X) \implies C_\rho \in [1,+\infty), \tag{4.4}$$

$$\forall\, \rho \in \mathfrak{Q}(X), \;\; \forall\, \beta \in (0,+\infty) \implies \rho^\beta \in \mathfrak{Q}(X) \;\text{ and }\; C_{\rho^\beta} = (C_\rho)^\beta, \tag{4.5}$$

$$\forall\, \rho \in \mathfrak{Q}(X), \;\; \forall\, \alpha \in (0,+\infty] \implies \rho_\alpha \in \mathfrak{Q}(X) \;\text{ and }\; C_{\rho_\alpha} \le C_\rho, \tag{4.6}$$

where the last inequality is a consequence of (3.107).

Remark 4.1. (i) Recall that a distance d on the set X is called an `ultrametric` provided the stronger version of the triangle inequality

$$d(x,y) \le \max\{d(x,z),d(z,y)\} \quad \text{for all } \; x,y,z \in X \tag{4.7}$$

holds. Hence,

$$\rho \text{ ultrametric on } X \iff \rho \in \mathfrak{Q}(X) \;\text{ and }\; C_\rho = 1. \tag{4.8}$$

In light of this observation, it is natural to refer to an inequality of the type (4.3) as a `quasi-ultrametric condition` for ρ. Thus, C_ρ from (4.2) is the optimal constant appearing in a quasi-ultrametric condition for a given $\rho \in \mathfrak{Q}(X)$.

(ii) If (X,d) is an ultrametric space, then a number of highly specialized properties hold:

(1) Each triangle in X is isosceles, i.e., for all $x,y,z \in X$

$$d(x,y) = \max\{d(x,z),d(z,y)\} \quad \text{whenever } \; d(x,z) \neq d(z,y). \tag{4.9}$$

(2) Every point inside a ball is its center, i.e., $B_d(x,r) = B_d(y,r)$ for each $x,y \in X$ and $r > d(x,y)$. In particular, a ball does not determine its center uniquely.

(3) Intersecting balls are contained in each other. More precisely, if $x,y \in X$ and $0 < r \le R < +\infty$ are such that $B_d(x,r) \cap B_d(y,R) \neq \emptyset$, then $B_d(x,r) \subseteq B(y,R)$. In particular, intersecting balls of equal radii are necessarily identical.

(4) All balls are both open and closed sets in the topology induced by the ultrametric.

Given any nonempty set X, the discrete distance on X, i.e., $d(x,y) := 1$ if $x \neq y$, and $d(x,y) := 0$ if $x = y$, is an ultrametric. Further examples of utrametrics on X may be constructed starting with a function $\phi : X \to [0,+\infty)$ that vanishes at most once (i.e., $\#\phi^{-1}(\{0\}) \leq 1$) and then considering

$$d(x,y) := \begin{cases} \max\{\phi(x), \phi(y)\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \qquad \forall\, x, y \in X. \tag{4.10}$$

In particular, given a quasimetric space (X, ρ) and a fixed point $x_o \in X$, the function $d : X \times X \to [0,+\infty)$ given by

$$d(x,y) := \begin{cases} \max\{\rho(x,x_o), \rho(y,x_o)\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \qquad \forall\, x, y \in X, \tag{4.11}$$

is an ultrametric on X that satisfies $d \geq C_\rho^{-1}\rho$, though in general d and ρ are not equivalent. In fact, in this scenario, the d-balls have the form

$$B_d(x,r) = \{x\} \text{ if } \rho(x,x_o) \geq r \quad \text{and} \quad B_d(x,r) = B_\rho(x_o,r) \text{ if } \rho(x,x_o) < r. \tag{4.12}$$

Convention 4.2. *Given a set X and $\rho \in \mathfrak{Q}(X)$, it is agreed that for the remainder of this work $\rho_\#$ stands for ρ_α, as defined in* (3.494) *and* (3.495) *for the value $\alpha := [\log_2 C_\rho]^{-1}$, with C_ρ as in* (4.2).

Remark 4.3. In view of Convention 4.2, (3.502), and the last formula in (4.5), it may be readily verified that for any set X and any quasidistance $\rho \in \mathfrak{Q}(X)$,

$$(\rho^\gamma)_\# = (\rho_\#)^\gamma, \qquad \forall\, \gamma \in (0,+\infty). \tag{4.13}$$

Moreover, (4.6) entails

$$C_{\rho_\#} \leq C_\rho, \tag{4.14}$$

whereas (3.530) yields

$$C_\rho^{-2}\rho \leq \rho_\# \leq \rho. \tag{4.15}$$

Recall that we have a natural equivalence relation on $\mathfrak{Q}(X)$, as described in Definition 3.15, and we call each equivalence class $\mathbf{q} \in \mathfrak{Q}(X)/_\approx$ a `quasimetric space structure` on X. Finally, for each $\rho \in \mathfrak{Q}(X)$, denote by $[\rho] \in \mathfrak{Q}(X)/_\approx$ the equivalence class of ρ.

By a `quasimetric space` we will understand a pair $(X, \mathbf{q})$, where X is a set of cardinality ≥ 2 and $\mathbf{q} \in \mathfrak{Q}(X)/_\approx$. If X is a set of cardinality ≥ 2 and $\rho \in \mathfrak{Q}(X)$, then we will frequently use the simpler notation (X, ρ) in place of $(X, [\rho])$ and still refer to (X, ρ) as a quasimetric space. Finally, we note that any quasimetric space $(X, \mathbf{q})$ has a canonical topology, denoted $\tau_{\mathbf{q}}$, that is (unequivocally) defined as the topology τ_ρ naturally induced by a choice of a quasidistance ρ in $\mathbf{q}$. As discussed in conclusion (10) of Theorem 3.46, the latter is defined as the largest topology on X with the property that for each point $x \in X$ the family $\{B_\rho(x,r)\}_{r>0}$ is a fundamental system of neighborhoods of x (here it is relevant to note that this family of ρ-balls, defined in (3.526), satisfies axioms *(i)–(iv)* in Proposition 2.65; cf. Proposition 2.66).

Call a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in a quasimetric space $(X,\mathbf{q})$ `Cauchy` provided that for some (hence, any) quasidistance $\rho \in \mathbf{q}$ and any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ with the property that $\rho(x_j,x_k) < \varepsilon$ whenever $j,k \in \mathbb{N}$ are such that $\min\{j,k\} \geq N_\varepsilon$. Also, call a quasimetric space $(X,\mathbf{q})$ `complete` provided any Cauchy sequence in $(X,\mathbf{q})$ converges, in the topology $\tau_{\mathbf{q}}$, to a (unique, a posteriori) point in X.

Given a quasimetric space $(X,\mathbf{q})$, call $E \subseteq X$ `bounded` if E is contained in a ρ-ball for some (hence all) $\rho \in \mathbf{q}$. In other words, a set $E \subseteq X$ is bounded, relative to the quasimetric space structure $\mathbf{q}$ on X, if and only if for some (hence all) $\rho \in \mathbf{q}$ we have $\operatorname{diam}_\rho(E) < +\infty$, where

$$\operatorname{diam}_\rho(E) := \sup\big\{\rho(x,y) : x,y \in E\big\}. \tag{4.16}$$

4.2 Extensions of Hölder Functions

We start with some preparations. Let X be a nonempty set and assume that $\rho \in \mathfrak{Q}(X)$, $\beta \in (0,+\infty)$, and $E \subseteq X$ has cardinality ≥ 2. Given a real-valued function f on E, define its `Hölder seminorm` (of order β, relative to the quasidistance ρ) by setting

$$\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} := \sup_{x,y\in E,\, x\neq y} \frac{|f(x)-f(y)|}{\rho(x,y)^\beta}. \tag{4.17}$$

Next, if $\mathbf{q}$ is a quasimetric space structure on X, then we define the `homogeneous Hölder space` $\dot{\mathscr{C}}^\beta(E,\mathbf{q})$ as

$$\begin{aligned}\dot{\mathscr{C}}^\beta(E,\mathbf{q}) &:= \big\{f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < +\infty \text{ for some } \rho \in \mathbf{q}\big\}\\ &= \big\{f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < +\infty \text{ for every } \rho \in \mathbf{q}\big\}. \end{aligned}\tag{4.18}$$

Given any $\beta > 0$, it follows that $\{\|\cdot\|_{\dot{\mathscr{C}}^\beta(E,\rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent seminorms on $\dot{\mathscr{C}}^\beta(E,\mathbf{q})$. In the same setting, the space of Lipschitz functions and the associated Lipschitz seminorm correspond to the case when $\beta = 1$, i.e., are defined as

$$\operatorname{Lip}(E,\mathbf{q}) := \dot{\mathscr{C}}^1(E,\mathbf{q}), \qquad \|\cdot\|_{\operatorname{Lip}(E,\rho)} := \|\cdot\|_{\dot{\mathscr{C}}^1(E,\rho)}, \quad \forall\, \rho \in \mathbf{q}. \tag{4.19}$$

As far as these spaces are concerned, if $\rho \in \mathfrak{Q}(X)$ is given, then we will sometimes slightly simplify the notation and write $\dot{\mathscr{C}}^\beta(E,\rho)$ and $\mathrm{Lip}\,(E,\rho)$ in place of $\dot{\mathscr{C}}^\beta(E,[\rho])$ and $\mathrm{Lip}\,(E,[\rho])$, respectively. Occasionally, we will refer to functions in $\mathrm{Lip}\,(E,\rho)$ as being ρ-Lipschitz on E. We continue our series of preparations by proving the following result.

Lemma 4.4. *Let (X,ρ) be a quasimetric space, and suppose that the set $E \subseteq X$ and the index $\beta \in (0,+\infty)$ are fixed. Given a family $\{f_i\}_{i\in I}$ of real-valued functions on E with the property that*

$$M := \sup_{i\in I} \|f_i\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < +\infty, \tag{4.20}$$

consider

$$f^*(x) := \sup_{i\in I} f_i(x), \quad f_*(x) := \inf_{i\in I} f_i(x), \qquad \forall\, x \in E. \tag{4.21}$$

Then the following conclusions hold.

(a) Either $f^*(x) = +\infty$ for every $x \in E$ or $f^* : E \to \mathbb{R}$ is a well-defined function satisfying $\|f^*\|_{\dot{\mathscr{C}}^\beta(E,\rho)} \leq M$.
(b) Either $f_*(x) = -\infty$ for every $x \in E$ or $f_* : E \to \mathbb{R}$ is a well-defined function satisfying $\|f_*\|_{\dot{\mathscr{C}}^\beta(E,\rho)} \leq M$.

Proof. Consider the conclusion in (a). If f^* is not identically $+\infty$ on E, then there exists $x_o \in E$ such that $\sup_{i\in I} f_i(x_o) < +\infty$. On the other hand, condition (4.20) entails that, for each $i \in I$,

$$f_i(x) \leq f_i(y) + M\rho(x,y)^\beta, \qquad \forall\, x,y \in E. \tag{4.22}$$

Using (4.22) with $y := x_o$ gives $\sup_{i\in I} f_i(x) \leq \sup_{i\in I} f_i(x_o) + M\rho(x,x_o)^\beta < +\infty$ for every $x \in E$. Thus, $f^* : E \to \mathbb{R}$ with $f^*(x) := \sup_{i\in I} f_i(x)$ for each $x \in E$ is a well-defined function. Moreover, (4.22) readily implies that $f^*(x) \leq f^*(y) + M\rho(x,y)^\beta$ for all $x,y \in E$; hence, ultimately, $\|f^*\|_{\dot{\mathscr{C}}^\beta(E,\rho)} \leq M$. This completes the proof of (a).

Finally, (b) follows from (a) used for the family $\{-f_i\}_{i\in I}$. □

Recall that a topological space (X,τ) is said to be `normal` if for each pair of disjoint closed subsets E_0, E_1 of X there exist two disjoint open subsets U_0, U_1 of X such that $E_0 \subseteq U_0$ and $E_1 \subseteq U_1$. A classical extension result is the Tietze–Urysohn theorem, which states that a topological space (X,τ) is normal if and only if every real-valued continuous function defined on a closed subset of X may be extended to the entire space with preservation of continuity (cf., e.g., [95, Proposition 3.7, p. 19], [44, Theorem 2.1.8]).

Assume that (X,τ) is a metrizable topological space, i.e., there exists a distance d on X such that $\tau = \tau_d$, the topology canonically induced on X by the distance d. In this setting, for $E \subseteq X$ and $x \in X$ define $\mathrm{dist}_d(x,E) := \inf\{d(x,y) :\ y \in E\}$.

Then, given two disjoint closed subsets E_0, E_1 of X, the function $f : (X, \tau) \to \mathbb{R}$ defined by

$$f(x) := \frac{\mathrm{dist}_d(x, E_1)}{\mathrm{dist}_d(x, E_0) + \mathrm{dist}_d(x, E_1)}, \qquad \forall\, x \in X, \tag{4.23}$$

is continuous and has the property that $f \equiv 0$ on E_0 and $f \equiv 1$ on E_1. Hence, if we define $U_0 := f^{-1}((-\infty, \frac{1}{2}))$ and $U_1 := f^{-1}((\frac{1}{2}, +\infty))$, then U_0, U_1 are disjoint open subsets of X satisfying $E_0 \subseteq U_0$ and $E_1 \subseteq U_1$. Since the Alexandroff–Urysohn metrization theorem (cf. also Theorem 3.46) gives that the topology induced by the quasimetric space structure on any given quasimetric space is metrizable, this shows that

$$\text{if } (X, \mathbf{q}) \text{ is a quasimetric space, then } (X, \tau_{\mathbf{q}}) \text{ is a normal topological space.} \tag{4.24}$$

Of course, these considerations apply to metric spaces, and hence the Tietze–Urysohn theorem yields an abstract extension procedure in such a setting. Interestingly, however, in the case where E is a closed subset of a metric space (X, d) and $f : E \to \mathbb{R}$ is a continuous function, a concrete formula for a continuous extension of f to X is given by Hausdorff's formula

$$F(x) := \begin{cases} \inf\Big\{f(y) + \frac{d(x,y)}{\mathrm{dist}_d(x,E)} - 1 : \, y \in E\Big\} & \text{if } \; x \in X \setminus E, \\ f(x) & \text{if } \; x \in E \end{cases} \tag{4.25}$$

(see e.g., [44, Exercise 4.1.F]). This result can be extended to the setting of quasimetric spaces as follows.

Proposition 4.5. *Assume that $(X, \mathbf{q})$ is a quasimetric space and that the quasidistance $\rho \in \mathbf{q}$ and the real number β satisfy $0 < \beta \leq [\log_2 C_\rho]^{-1}$. Also, recall the regularized version $\rho_\#$ of ρ defined as in Theorem 3.46. Finally, given a closed subset E of $(X, \tau_{\mathbf{q}})$ and a function $f : E \to \mathbb{R}$ that is continuous relative to the topology induced by $\tau_{\mathbf{q}}$ on E, define*

$$F(x) := \begin{cases} \inf\Big\{f(y) + \Big[\frac{\rho_\#(x,y)}{\mathrm{dist}_{\rho_\#}(x,E)}\Big]^\beta - 1 : \, y \in E\Big\} & \textit{if } \; x \in X \setminus E, \\ f(x) & \textit{if } \; x \in E. \end{cases} \tag{4.26}$$

Then $F : X \to \mathbb{R}$ is a continuous extension of f.

Proof. This follows (after some minor adjustments) from Hausdorff's formula (4.25) used for $(X, (\rho_\#)^\beta)$, which, by Theorem 3.46, is known to be a metric space under the hypotheses of the current proposition, and after observing that $\tau_{(\rho_\#)^\beta} = \tau_{\mathbf{q}}$. □

Note that, with the set E fixed, the extension operator $f \mapsto F$ described in Proposition 4.5 is nonlinear. A linear extension operator in the class of continuous functions will be constructed later, in Theorem 4.11 below (based on ideas originally introduced by H. Whitney), but for the more specialized setting of geometrically doubling quasimetric spaces.

Passing from continuous to smoother functions, one has the basic result due to McShane [81] and Whitney [129] to the effect that if (X, d) is a metric space and if $E \subseteq X$ is an arbitrary nonempty set, then for any function $g \in \mathrm{Lip}\,(E, d)$ there exists $f \in \mathrm{Lip}\,(X, d)$ such that $f|_E = g$ and $\|g\|_{\mathrm{Lip}\,(E,d)} = \|f\|_{\mathrm{Lip}\,(X,d)}$. Indeed, one may take

$$f(x) := \inf\big\{g(y) + \|g\|_{\mathrm{Lip}\,(E,d)}\, d(x, y) : \ y \in E\big\}, \qquad \forall\, x \in X. \tag{4.27}$$

However, as opposed to the setting of metric spaces (considered in [72, 81, 129]) where Lipschitz functions are abundant,[1] the space of Lipschitz functions is often trivial (i.e., it reduces to just constants) in the framework of quasimetric spaces. For example, elementary calculus shows that this is the case for (X_o, ρ_o) if, for some fixed $\gamma > 1$, we take

$$\begin{aligned} &X_o \text{ a nonempty, connected, open subset of } \mathbb{R}^n \\ &\text{and } \rho_o(x, y) := |x - y|^\gamma \text{ for every } x, y \in X_o. \end{aligned} \tag{4.28}$$

In turn, the fact that the only functions that are globally Lipschitz on such a quasimetric space (X_o, ρ_o) are constant functions shows that the extension problem of Lipschitz functions fails to have a solution in this setting. Indeed, if the cardinality of $E \subseteq X_o$ is finite and ≥ 2, then any nonconstant real-valued function f defined on E is Lipschitz but does not extend to a Lipschitz function F on (X_o, ρ_o) since the latter would have to be constant. The moral of this discussion is that once the focus shifts from metric spaces to the more general category of quasimetric spaces, one must necessarily formulate the extension problem for classes of functions other than Lipschitz.

Below, the goal is to extend the aforementioned extension result of McShane and Whitney to the more general context of Hölder scales of quasimetric spaces.

Theorem 4.6. *Let $(X, \mathbf{q})$ be a quasimetric space, and assume that β is a finite number with the property that there exists a quasidistance $\rho \in \mathbf{q}$ such that*

$$0 < \beta \leq \big[\log_2 C_\rho\big]^{-1}. \tag{4.29}$$

Finally, fix a nonempty set $E \subseteq X$. Then any function in $\dot{\mathscr{C}}^\beta(E, \mathbf{q})$ may be extended with preservation of the Hölder property to the entire set X, i.e., one has

$$\dot{\mathscr{C}}^\beta(E, \mathbf{q}) = \{f|_E : \ f \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})\}. \tag{4.30}$$

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[1]Trivially, if (X, d) is a metric space, then for each fixed $x_o \in X$ the function $d(\cdot, x_o) : X \to \mathbb{R}$ is Lipschitz.

Furthermore, the Hölder seminorm may be controlled in the process of extending Hölder functions in the sense that there exists a finite constant $C \geq 0$ *having the property that*

$$\begin{aligned}&\forall\, g \in \dot{\mathscr{C}}^{\beta}(E,\mathbf{q}) \quad \exists\, f \in \dot{\mathscr{C}}^{\beta}(X,\mathbf{q}) \text{ such that}\\ &g = f|_E \;\text{ and }\; \|f\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \leq C\|g\|_{\dot{\mathscr{C}}^{\beta}(E,\rho)}.\end{aligned} \tag{4.31}$$

As a corollary, the space $\dot{\mathscr{C}}^{\beta}(X,\mathbf{q})$ *separates the points in* X. *In particular, the space* $\dot{\mathscr{C}}^{\beta}(X,\mathbf{q})$ *contains nonconstant functions.*

Proof. Let $X, \mathbf{q}, \beta, \rho$ be as in the statement of the theorem. Then it follows from (12) in Theorem 3.46 that there exists $\rho_\# \in \mathbf{q}$ with the property that $(\rho_\#)^\beta$ is a distance on X. Consequently, given an arbitrary nonempty set $E \subseteq X$, by virtue of the discussion preceding the statement of the current theorem, we have

$$\begin{aligned}\dot{\mathscr{C}}^{\beta}(E,\rho) &= \dot{\mathscr{C}}^{\beta}(E,\rho_\#) = \mathrm{Lip}\big(E,(\rho_\#)^\beta\big)\\ &= \mathrm{Lip}\big(X,(\rho_\#)^\beta\big)\Big|_E = \dot{\mathscr{C}}^{\beta}(X,\rho_\#)\Big|_E = \dot{\mathscr{C}}^{\beta}(X,\rho)\Big|_E\end{aligned} \tag{4.32}$$

with control of seminorms. This proves (4.30) and (4.31). In fact, if $\rho \in \mathbf{q}$ is such that $\beta \leq [\log_2 C_\rho]^{-1}$, then two explicit extensions of a function $f \in \dot{\mathscr{C}}^{\beta}(E,\rho)$ to X with preservation of Hölder regularity are f^*, f_* defined below. Concretely, upon recalling that $\rho_\#$ is the regularized version of ρ as defined in Theorem 3.46, we set

$$\begin{aligned}&f^*(x) := \inf\nolimits_{z\in E} f_z^+(x), \quad f_*(x) := \sup\nolimits_{z\in E} f_z^-(x), \quad \forall\, x \in X, \quad \text{where,}\\ &\text{for each } z \in \text{E, we define} \quad f_z^\pm(x) := f(z) \pm C\rho_\#(x,z)^\beta, \quad \forall\, x \in X,\end{aligned} \tag{4.33}$$

for some sufficiently large $C > 0$, to be specified shortly. Note that, by Theorem 3.46, each $f_z^\pm$ is $(\rho_\#)^\beta$-Lipschitz on X with constant $\leq C$. Hence, choosing $C \geq C_\rho^{2\beta}\|f\|_{\dot{\mathscr{C}}^{\beta}(E,\rho)}$ ensures that $f_z^- \leq f \leq f_z^+$ on E for every $z \in E$. Since $f_z^\pm(z) = f(z)$ for each $z \in E$, (4.30) and (4.31) follow from this analysis and Lemma 4.4.

To justify the last claim in the statement of the theorem, given two distinct points $x_0, x_1 \in X$, apply (4.31) in the case where $E := \{x_0, x_1\}$ and $f : E \to \mathbb{R}$ is defined as $f(x_0) := 0$, $f(x_1) := 1$. The desired conclusions follow. □

Remark 4.7. Theorem 4.6 is sharp, in the sense that the upper bound in (4.29) is optimal in the class of all quasimetric spaces. To see this, for some fixed $\gamma > 0$ consider (X_o, ρ_o) as in (4.28) and note that, in this scenario, (4.29) becomes $0 < \beta \leq \gamma^{-1}$. On the other hand, if $\beta > \gamma^{-1}$, then, since Hölder functions of order β with respect to ρ_o are Hölder functions of order $\beta\gamma > 1$ with respect to the standard Euclidean distance in the nonempty, open, connected subset X_o of $\mathbb{R}^n$, it follows that this Hölder class contains only constant functions. Hence, much as in the discussion following (4.28), the extension problem does not have a solution in this setting.

According to the last part in Theorem 4.6, if $(X, \mathbf{q})$ is a quasimetric space, $\rho \in \mathbf{q}$, and $\beta \in \mathbb{R}$ is such that $0 < \beta \leq [\log_2 C_\rho]^{-1}$, then the Hölder space $\dot{\mathscr{C}}^\beta(X, \rho)$ contains plenty of nonconstant functions. Note that the case $[\log_2 C_\rho]^{-1} = +\infty$ corresponds precisely to the situation when $C_\rho = 1$, i.e., when ρ is an ultrametric. In this scenario, there are nonconstant Hölder functions that are Hölder of any finite order on X. This is made precise in the remark below.

Remark 4.8. Let X be a set, and assume that ρ is an ultrametric on X. Consider a function $\eta \in \mathscr{C}^\infty(\mathbb{R})$ satisfying $\|\eta'\|_\infty \leq 1$, as well as

$$0 \leq \eta \leq 1, \quad \eta(t) = 1 \text{ for all } t \in [4, 8], \quad \eta(t) = 0 \text{ for all } t \in \mathbb{R} \setminus [2, 16]. \tag{4.34}$$

Finally, fix $z \in X$ and, for each $r > 0$, define

$$\phi_{z,r} : X \to \mathbb{R}, \quad \phi_{z,r}(x) := \eta\big(\rho(x, z)/r\big), \quad \forall\, x \in X. \tag{4.35}$$

Then, making use of (4.34), (4.35), and (3.542), it can be readily verified that

$$0 \leq \phi_{z,r} \leq 1, \quad \phi_{z,r} = 1 \text{ on } B_\rho(z, 8r) \setminus B_\rho(z, 4r), \tag{4.36}$$

$$\phi_{z,r} = 0 \text{ on } X \setminus \big(B_\rho(z, 16r) \setminus B_\rho(z, 2r)\big), \tag{4.37}$$

and that for each real number $\beta > 1$ there exists a finite constant $C = C(\beta) > 0$ with the property that

$$\|\phi_{z,r}\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C r^{-\beta}. \tag{4.38}$$

We proceed by discussing a version of Theorem 4.6 in which the issue is the extension (with control) of functions satisfying a Hölder condition at a single point.

Theorem 4.9. *Suppose that (X, ρ) is a quasimetric space, E is a subset of X, and x_o is a fixed point in E. Let $f : E \to \mathbb{R}$ be a function with the property that there exist an exponent $\alpha \in (0, +\infty)$ and a constant $C \in [0, +\infty)$ such that*

$$|f(x) - f(x_o)| \leq C\rho(x, x_o)^\alpha, \qquad \forall\, x \in E. \tag{4.39}$$

Then there exists $F : X \to \mathbb{R}$ such that $F\big|_E = f$ and

$$|F(x) - F(x_o)| \leq C_o\rho(x, x_o)^\alpha, \qquad \forall\, x \in X, \tag{4.40}$$

where $C_o \in [0, +\infty)$ depends only on α, C, and ρ.

Proof. Let d be the ultrametric on X defined in relation to the quasidistance ρ and the point x_o as in (4.11). Then for each $x, y \in E$ with $x \neq y$ we may estimate

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_o)| + |f(y) - f(x_o)| \\ &\leq 2\max\big\{|f(x) - f(x_o)|, |f(y) - f(x_o)|\big\} \\ &\leq 2C \max\big\{\rho(x, x_o)^\alpha, \rho(y, x_o)^\alpha\big\} = 2C\, d(x, y)^\alpha, \end{aligned} \tag{4.41}$$

which proves that $f : (E, d) \to \mathbb{R}$ is Hölder of order α relative to the ultrametric d. In fact, since $d(x, x_o) = \rho(x, x_o)$ for any $x \in X$, it follows that

$$\text{the class of functions } f \text{ satisfying an estimate of the type (4.39) coincides with } \dot{\mathscr{C}}^{\alpha}(E, d). \tag{4.42}$$

Next, since α is a finite number with the property that $0 < \alpha \le [\log_2 C_d]^{-1} = +\infty$ (given that $C_d = 1$), Theorem 4.6 applies and gives a function $F : X \to \mathbb{R}$ such that $F\big|_E = f$ and (cf. (4.31))

$$\|F\|_{\dot{\mathscr{C}}^{\alpha}(X,d)} \le C_o \|f\|_{\dot{\mathscr{C}}^{\alpha}(E,d)}, \tag{4.43}$$

where $C_o \in [0, +\infty)$ depends only on α, C, and ρ. In turn, from (4.43) we deduce that $|F(x) - F(x_o)| \le C_o d(x, x_o)^{\alpha} = C_o \rho(x, x_o)^{\alpha}$ for every $x \in X$; hence, (4.40) follows. ☐

In the last part of this section we wish to comment on the relationship between our Theorem 4.6 and the technology developed by Whitney for extending functions from rough subsets of $\mathbb{R}^n$ with preservation of smoothness. Specifically, recall that if $n \in \mathbb{N}$ and $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$, then for any closed subset E of $\mathbb{R}^n$ there exists a linear operator $\mathscr{E}$ with the property that if $0 < \beta \le 1$, then $\mathscr{E}$ maps functions from $\dot{\mathscr{C}}^{\beta}(E, |\cdot - \cdot|)$ into $\dot{\mathscr{C}}^{\beta}(\mathbb{R}^n, |\cdot - \cdot|)$, with control of the Hölder seminorm, and which satisfies $\mathscr{E} f\big|_E = f$ for each $f \in \dot{\mathscr{C}}^{\beta}(E, |\cdot - \cdot|)$. See [113, Theorem 3, p. 174] where it is also shown that the norm of the operator $\mathscr{E}$ may be bounded independently of $E \subseteq \mathbb{R}^n$. The proof presented in [113, pp. 174–175] makes use of three basic ingredients, namely: (1) the existence of a Whitney decomposition of the open set $\mathbb{R}^n \setminus E$ (into Whitney balls of bounded overlap), (2) the existence of a smooth partition of unity subordinate (in an appropriate, quantitative manner) to such an open cover of $\mathbb{R}^n \setminus E$, and (3) differential calculus in open subsets of $\mathbb{R}^n$ (the mean-value theorem, to be precise).

The ingredients listed in items (1) and (2) have been subsequently extended to the setting of spaces of homogeneous type, which raises the natural question as to whether the approach in Stein's book [113] may be adapted to this more general framework. Of course, item (3) is utterly ill-suited in the named setting, so the strategy outlined above must be suitably altered. We also wish to mentioned that, as far as spaces of homogeneous type are concerned, the existence of a doubling measure is an unnecessarily strong hypothesis, given the nature of the problem at hand (e.g., the setup of Hölder spaces is free of any measure-theoretical considerations and only requires a quasimetric space structure).

This point is underscored by the extension result discussed in Theorem 4.11 below, whose formulation is akin to the original result of Whitney (as presented in [113, Theorem 3, p. 174]) and which is valid in the general framework of geometrically doubling quasimetric spaces. The formal definition of the latter concept is as follows.

Definition 4.10. A quasimetric space $(X, \mathbf{q})$ is called `geometrically doubling` if there exists $\rho \in \mathbf{q}$ for which one can find a number $N \in \mathbb{N}$, called the `geometrically doubling constant of` $(X, \mathbf{q})$, with the property that any ρ-ball of radius r in X may be covered by at most N ρ-balls in X of radii $r/2$. Finally, if X is an arbitrary, nonempty set and $\rho \in \mathfrak{Q}(X)$, call (X, ρ) geometrically doubling if $(X, [\rho])$ is geometrically doubling.

Note that if $(X, \mathbf{q})$ is a geometrically doubling quasimetric space, then

$$\begin{gathered}\forall\, \rho \in \mathbf{q} \;\; \forall\, \theta \in (0,1) \;\; \exists\, N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r \\ \text{in } X \text{ may be covered by at most } N \;\rho\text{-balls in } X \text{ of radii } \theta r.\end{gathered} \tag{4.44}$$

In particular, this ensures that the last part in Definition 4.10 is meaningful. Another useful consequence of the geometrically doubling property for a quasimetric space $(X, \mathbf{q})$ is as follows. Given $\varepsilon > 0$ and $\rho \in \mathbf{q}$, a set $E \subseteq X$ is said to be (ε, ρ)-`disperse` provided

$$\rho(x, y) \geq \varepsilon, \qquad \forall\, x, y \in E \;\text{ with }\; x \neq y. \tag{4.45}$$

Then, if $(X, \mathbf{q})$ is a geometrically doubling quasimetric space, $\rho \in \mathbf{q}$ and $\varepsilon > 0$,

$$\begin{gathered}\text{any family of } \rho\text{-balls in } X \text{ of bounded radii, and whose centers} \\ \text{make up an } (\varepsilon, \rho)\text{-disperse subset of } X\text{, has bounded overlap}\end{gathered} \tag{4.46}$$

for some bound depending only on ε, the bound on the radii of the ρ-balls in question, and the geometrically doubling constant of the quasimetric space $(X, \mathbf{q})$.

Another point of view on this matter is as follows. Suppose that (X, ρ) is a given quasimetric space. Recall that the diameter of any nonempty set E of X (relative to the quasidistance ρ) is defined as $\mathrm{diam}_\rho(E) := \sup\{\rho(x, y) : \; x, y \in E\}$ and that a subset E of X is called bounded if it has finite diameter. Finally, recall that a subset of E of X is said to be `totally bounded` provided for every $\varepsilon > 0$ there exist some $N = N(\varepsilon) \in \mathbb{N}$ and $x_1, \dots, x_N \in X$ such that $E \subseteq \bigcup_{1 \leq i \leq N} B_\rho(x_i, \varepsilon)$. Then

$$\begin{gathered}\text{a subset of a geometrically doubling quasimetric space} \\ \text{is bounded if and only if it is totally bounded.}\end{gathered} \tag{4.47}$$

Also, in the category of quasimetric spaces,

$$\begin{gathered}\text{the geometric doubling property is a quantitative,} \\ \text{scale-invariant version of total boundedness.}\end{gathered} \tag{4.48}$$

Lastly, let us point out here that

$$\text{if } (X, \mathbf{q}) \text{ is a geometrically doubling quasimetric space,} \\ \text{then the topological space } (X, \tau_{\mathbf{q}}) \text{ is separable.} \tag{4.49}$$

What follows is the extension theorem mentioned previously.

Theorem 4.11. *Let $(X, \mathbf{q})$ be a geometrically doubling quasimetric space and assume that E is a nonempty, closed subset of the topological space $(X, \tau_{\mathbf{q}})$, where $\tau_{\mathbf{q}}$ is the topology canonically induced on X by the quasimetric space structure $\mathbf{q}$.*

Then there exists a linear operator $\mathscr{E}$, extending real-valued continuous functions defined on $\big(E, \tau_{\mathbf{q}}|_E\big)$ into continuous real-valued functions defined on $(X, \tau_{\mathbf{q}})$, that has the following property. Whenever $\rho \in \mathbf{q}$ and $\beta \in \mathbb{R}$ are such that $0 < \beta \leq [\log_2 C_\rho]^{-1}$, then

$$\mathscr{E} : \dot{\mathscr{C}}^\beta(E, \rho) \longrightarrow \dot{\mathscr{C}}^\beta(X, \rho) \tag{4.50}$$

is well defined and bounded. In addition, $\mathscr{E}$ also maps bounded functions on E into bounded functions on X, without increasing the bound, i.e., for any $f : E \to \mathbb{R}$

$$\sup_{x \in X} |\mathscr{E} f(x)| \leq \sup_{x \in E} |f(x)|. \tag{4.51}$$

In particular, $\mathscr{E} : \mathscr{C}^\beta(E, \rho) \to \mathscr{C}^\beta(X, \rho)$ linearly and boundedly for the same range of β as before.

The proof of Theorem 4.11 will be presented later, in the second part of Sect. 4.6, after a number of necessary tools have been properly developed. At this point we only wish to remark that, while Theorem 4.11 yields a stronger conclusion than Theorem 4.6 (in the sense that in the former theorem we manufacture a linear extension operator, compared to the nonlinear extension procedure in the latter theorem), the setting in Theorem 4.11 is more restrictive than that in Theorem 4.6 as it presupposes that the quasimetric space in question is geometrically doubling.

4.3 Separation Properties of Hölder Functions

Here we will improve upon the separation property described in the last part of Theorem 4.6 by establishing the separation result for Hölder functions on quasimetric spaces described in Theorem 4.12. In particular, this result sheds light on the structural richness of the space of Hölder functions (of a given order) on a quasimetric space by specifying an order up to which the named spaces do not reduce to just constant functions. Even though the issue of the triviality of the Hölder spaces for large exponents has come up on earlier occasions (cf., e.g., the comment on the footnote on p. 591 in [35]), to the best of our knowledge no attempt has been made to quantify this statement in a satisfactory manner.

Before stating the aforementioned result, let us recall that, given a quasimetric space (X, ρ), the ρ-distance between two arbitrary, nonempty sets $E, F \subseteq X$ is defined as

$$\operatorname{dist}_\rho(E, F) := \inf\{\rho(x, y) : x \in E, \ y \in F\}. \tag{4.52}$$

Corresponding to the case $E = \{x\}$ for some $x \in X$ and $F \subseteq X$, we therefore have $\operatorname{dist}_\rho(x, F) = \operatorname{dist}_\rho(\{x\}, F)$. Let us also note here that, as a corollary of Lemma 4.4, if (X, ρ) is a quasimetric space and if $\beta > 0$ is a finite number, then for any $f, g \in \dot{\mathscr{C}}^\beta(X, \rho)$ it follows that

$$\max\{f, g\} \in \dot{\mathscr{C}}^\beta(X, \rho), \quad \min\{f, g\} \in \dot{\mathscr{C}}^\beta(X, \rho), \tag{4.53}$$

and

$$\max\Big\{\|\max\{f, g\}\|_{\dot{\mathscr{C}}^\beta(X,\rho)}, \|\min\{f, g\}\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\Big\} \le \max\big\{\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)}, \|g\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big\}. \tag{4.54}$$

What follows is the theorem mentioned previously, which should be thought of as a quantitative version of the classical Urysohn's lemma (cf., e.g., [87, Theorem 33.1, p. 207]).

Theorem 4.12. *Let* $(X, \mathbf{q})$ *be a quasimetric space, and assume that* β *is a finite number with the property that there exists a quasidistance* $\rho \in \mathbf{q}$ *such that* $0 < \beta \le \big[\log_2 C_\rho\big]^{-1}$. *Assume that* $F_0, F_1 \subseteq X$ *are two nonempty sets with the property that* $\operatorname{dist}_\rho(F_0, F_1) > 0$. *Then there exists* $\psi \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ *such that*

$$0 \le \psi \le 1 \ \ on \ \ X, \quad \psi \equiv 0 \ \ on \ \ F_0, \quad \psi \equiv 1 \ \ on \ \ F_1, \tag{4.55}$$

and for which there exists a finite constant $C > 0$*, depending only on* ρ*, so that*

$$\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \le C\big(\operatorname{dist}_\rho(F_0, F_1)\big)^{-\beta}. \tag{4.56}$$

Proof. Let $F_0, F_1 \subseteq X$ be two sets such that $\operatorname{dist}_\rho(F_0, F_1) > 0$, and consider the function $\varphi : F_0 \cup F_1 \to \mathbb{R}$ given by

$$\varphi(x) := \begin{cases} 0 \ \text{ if } \ x \in F_0, \\ 1 \ \text{ if } \ x \in F_1, \end{cases} \qquad x \in F_0 \cup F_1. \tag{4.57}$$

Notice that if either $x, y \in F_0$ or $x, y \in F_1$, then $|\varphi(x) - \varphi(y)| = 0 \le \rho(x, y)^\beta$. Also, if either $x \in F_0$ and $y \in F_1$ or $x \in F_1$ and $y \in F_0$, then

$$|\varphi(x) - \varphi(y)| = 1 \le (\operatorname{dist}_\rho(F_0, F_1))^{-\beta} \rho(x, y)^\beta \tag{4.58}$$

since $\beta > 0$ and, in the current case, $\rho(x, y) \ge \operatorname{dist}_\rho(F_0, F_1) > 0$. All together these imply

$$\varphi \in \dot{\mathscr{C}}^\beta(F_0 \cup F_1, \rho) \ \text{ and } \ \|\varphi\|_{\dot{\mathscr{C}}^\beta(F_0 \cup F_1,\rho)} \le (\operatorname{dist}_\rho(F_0, F_1))^{-\beta}. \tag{4.59}$$

With this in hand, Theorem 4.6 then ensures the existence of a function $\tilde{\varphi} \in \dot{\mathscr{C}}^{\beta}(X,\rho)$ that extends the function φ and has the property that there exists a finite constant $C = C(\rho) > 0$ such that

$$\|\tilde{\varphi}\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \le C(\mathrm{dist}_{\rho}(F_0,F_1))^{-\beta}. \tag{4.60}$$

At this stage, consider $\psi : X \to \mathbb{R}$ given by

$$\psi := \min\big\{\max\{\tilde{\varphi},0\},1\big\}. \tag{4.61}$$

By design, the function ψ satisfies (4.55). Moreover, (4.53) and (4.54) yield $\psi \in \dot{\mathscr{C}}^{\beta}(X,\rho)$ and $\|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \le C\|\tilde{\varphi}\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)}$. This and (4.60) then prove (4.56), completing the proof of the theorem. □

4.4 Density and Embedding Properties of Hölder Functions

Given a quasimetric space $(X,\mathbf{q})$, for each finite number $\beta > 0$ define

$$\dot{\mathscr{C}}^{\beta}_{c}(X,\mathbf{q}) := \big\{f \in \dot{\mathscr{C}}^{\beta}(X,\mathbf{q}) : \ f \text{ vanishes outside of a bounded subset of } X\big\}. \tag{4.62}$$

Much as in the past, if $\rho \in \mathfrak{Q}(X)$, then we will sometimes write $\dot{\mathscr{C}}^{\beta}_{c}(X,\rho)$ instead of $\dot{\mathscr{C}}^{\beta}_{c}(X,[\rho])$.

The first main result in this section is the following density theorem.

Theorem 4.13. *Let $(X,\mathbf{q})$ be a quasimetric space, and let β be a finite number with the property that there exists a quasidistance $\rho \in \mathbf{q}$ such that $0 < \beta \le \big[\log_2 C_{\rho}\big]^{-1}$. Then, for every locally finite Borel regular measure μ on $(X,\tau_{\mathbf{q}})$, one has*

$$\dot{\mathscr{C}}^{\beta}_{c}(X,\mathbf{q}) \hookrightarrow L^{p}(X,\mu) \text{ densely for each } p \in (0,+\infty). \tag{4.63}$$

The proof of this theorem makes use of the following auxiliary result, which is of interest in its own right.

Lemma 4.14. *Let $(X,\mathbf{q})$ be a quasimetric space, and assume that $\rho \in \mathbf{q}$ and $\beta \in \mathbb{R}$ are such that $0 < \beta \le \big[\log_2 C_{\rho}\big]^{-1}$. Then for every set $E \subseteq X$ that is closed in the topology $\tau_{\mathbf{q}}$ there exists a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\beta}(X,\mathbf{q})$ such that*

$$0 \le f_j \le 1 \text{ on } X \text{ for each } j\in\mathbb{N} \quad\text{and}\quad f_j \searrow \mathbf{1}_E \quad \text{pointwise as } j\to\infty. \tag{4.64}$$

Furthermore, if the set E is bounded, then matters can also be arranged so that all functions f_j vanish outside a common bounded subset of X.

As a corollary, for every set $\mathcal{O} \subseteq X$ that is open in the topology $\tau_{\mathbf{q}}$ there exists a sequence of functions $\{h_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\beta}(X,\mathbf{q})$ such that

$$0 \le h_j \le 1 \text{ on } X \text{ for each } j\in\mathbb{N}, \text{ and } h_j \nearrow \mathbf{1}_{\mathcal{O}} \text{ pointwise as } j\to\infty. \tag{4.65}$$

Proof. With $\rho \in \mathbf{q}$ and $\beta \in \mathbb{R}$ as in the statement of the lemma, let d be a distance on X such that $d \approx \rho^\beta$ (that such a distance exists is a consequence of Theorem 3.46), and for each $j \in \mathbb{N}$ consider the set

$$\mathcal{O}_j := \{x \in X : \operatorname{dist}_d(x, E) < 1/j\}. \tag{4.66}$$

Clearly $\mathcal{O}_j$ is an open subset of X in the topology τ_d for every $j \in \mathbb{N}$. Since $\tau_d = \tau_\rho$, this implies that for every $j \in \mathbb{N}$ the set $\mathcal{O}_j$ is open in (X, τ_ρ). Also, based on (4.66) we conclude that

$$\mathcal{O}_{j+1} \subseteq \mathcal{O}_j \quad \forall\, j \in \mathbb{N} \quad \text{and} \quad E \subseteq \bigcap_{j\in\mathbb{N}} \mathcal{O}_j = \{x \in X : \operatorname{dist}_d(x, E) = 0\}. \tag{4.67}$$

Using the last part of (4.67), the fact that $d \approx \rho^\beta$, and the definition of $\overline{E}$ (the closure of the set E in the topology τ_ρ), we obtain that

$$E \subseteq \bigcap_{j\in\mathbb{N}} \mathcal{O}_j = \{x \in X : \operatorname{dist}_\rho(x, E) = 0\} \subseteq \overline{E} = E \tag{4.68}$$

since we are assuming that E is closed in (X, τ_ρ). Consequently, we have $\bigcap_{j\in\mathbb{N}} \mathcal{O}_j = E$ and, given that the sets $\mathcal{O}_j$ in (4.67) are a nested family, we may ultimately conclude

$$\mathcal{O}_j \searrow E \quad \text{as} \quad j \to \infty. \tag{4.69}$$

This and the fact that $d \approx \rho^\beta$ yield $\operatorname{dist}_\rho(X \setminus \mathcal{O}_j, E) \approx \operatorname{dist}_d(X \setminus \mathcal{O}_j, E) \geq 1/j$ for each $j \in \mathbb{N}$, hence $\operatorname{dist}_\rho(X \setminus \mathcal{O}_j, E) > 0$ for every $j \in \mathbb{N}$. We may therefore invoke Theorem 4.12 to guarantee the existence of a sequence of functions $g_j : X \to \mathbb{R}$, $j \in \mathbb{N}$, satisfying

$$\begin{aligned} &g_j \in \dot{\mathscr{C}}^\beta(X, \rho), \quad 0 \leq g_j \leq 1 \text{ on } X, \\ &g_j \equiv 0 \text{ on } X \setminus \mathcal{O}_j, \quad g_j \equiv 1 \text{ on } E. \end{aligned} \tag{4.70}$$

To proceed, define inductively

$$f_1 := g_1 \quad \text{and} \quad f_j := \min\{f_{j-1}, g_j\}, \quad \forall\, j \in \mathbb{N}, \quad j \geq 2. \tag{4.71}$$

Then, based on (4.70), (4.71), and (4.53), we conclude that, for every $j \in \mathbb{N}$,

$$f_j \in \dot{\mathscr{C}}^\beta(X, \rho), \quad 1 \geq f_j \geq f_{j+1} \geq 0 \text{ pointwise on } X. \tag{4.72}$$

Furthermore,

$$f_j \equiv 1 \text{ on } E \quad \text{and} \quad f_j \equiv 0 \text{ on } X \setminus \mathcal{O}_j, \qquad \forall\, j \in \mathbb{N}, \tag{4.73}$$

where the fact that f_j vanishes on $X \setminus \mathcal{O}_j$ is a consequence of the inequality $0 \leq f_j \leq g_j$ and (4.70). In addition, (4.69) and the monotonicity of the sequence $\{f_j\}_{j\in\mathbb{N}}$ (cf. (4.72)) ensure that the pointwise convergence in (4.64) holds. Moreover, if the set E is bounded in (X, ρ), then the set $\mathcal{O}_1$ is ρ-bounded and all functions f_j vanish outside $\mathcal{O}_1$.

Finally, given an open set $\mathcal{O} \subseteq (X, \tau_{\mathbf{q}})$, if $\{f_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ is the sequence as in (4.64) for the closed set $E := X \setminus \mathcal{O}$, then the functions $h_j := 1 - f_j \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$, for $j \in \mathbb{N}$, satisfy (4.65). The proof of the lemma is therefore complete. □

In general, we will denote by $\mathbf{1}_E$ the characteristic function of a set E. We are now prepared to present the following proof.

Proof of Theorem 4.13. Let $\rho \in \mathbf{q}$ and $\beta \in \mathbb{R}$ be such that $0 < \beta \leq \left[\log_2 C_\rho\right]^{-1}$, and fix an integrability exponent $p \in (0, +\infty)$. The goal is to approximate arbitrarily well in $L^p(X, \mu)$ a given function $f \in L^p(X, \mu)$ with functions from $\dot{\mathscr{C}}_c^\beta(X, \rho)$. Since simple functions are dense in $L^p(X, \mu)$, there is no loss of generality in assuming that $f = \mathbf{1}_E$, where $E \subseteq X$ is μ-measurable and $\mu(E) < +\infty$. Because μ is a Borel regular measure, we may further assume that, in fact, E is a Borel set (relative to the topology τ_ρ) of finite measure. In this case, we have

$$\mu(E) = \sup_{\substack{K \subseteq E,\, K \text{bounded} \\ K = \overline{K}}} \mu(K), \tag{4.74}$$

so ultimately it suffices to approximate $\mathbf{1}_E$ in $L^p(X, \mu)$ with functions from $\dot{\mathscr{C}}_c^\beta(X, \rho)$ in the case where E is a closed and bounded subset of X. At this point, Lemma 4.14 applies and yields the desired conclusion. □

The second main result in this section is the following density theorem.

Theorem 4.15. *Assume that $(X, \mathbf{q})$ is a quasimetric space with the property that $\tau_{\mathbf{q}}$ (the topology canonically associated with the quasimetric space structure $\mathbf{q}$ on X) is locally compact. Let $\mathscr{C}_0(X)$ be the Banach space of real-valued, continuous functions on $(X, \tau_{\mathbf{q}})$ that vanish at infinity, equipped with the supremum norm. Finally, assume that $\beta \in \mathbb{R}$ is such that there exists a quasimetric $\rho \in \mathbf{q}$ for which $0 < \beta \leq \left[\log_2 C_\rho\right]^{-1}$. Then*

$$\dot{\mathscr{C}}_c^\beta(X, \mathbf{q}) \hookrightarrow \mathscr{C}_0(X) \quad \textit{densely}. \tag{4.75}$$

Proof. This follows from the Stone–Weierstrass theorem given that, as a consequence of Theorem 4.12, $\dot{\mathscr{C}}_c^\beta(X, \mathbf{q})$ is a subalgebra of $\mathscr{C}_0(X)$ that separates points and vanishes nowhere. □

We conclude this subsection by discussing embedding properties for Hölder spaces of functions on quasimetric spaces. To set the stage, consider a quasimetric space (X, ρ), and for each $\alpha \in (0, +\infty)$ define the `inhomogeneous Hölder space` $\mathscr{C}^\alpha(X, \rho)$ as

$$\mathscr{C}^{\alpha}(X,\rho) := \Big\{ f : X \to \mathbb{R} : \, \|f\|_{\mathscr{C}^{\alpha}(X,\rho)} := \sup_{x\in X} |f(x)| + \|f\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)} < +\infty \Big\}. \tag{4.76}$$

Our last main result in this subsection then reads as follows.

Theorem 4.16. *Let (X, ρ) be a quasimetric space. Then the inclusion operator*

$$\begin{aligned} &\iota : \mathscr{C}^{\beta}(X,\rho) \hookrightarrow \mathscr{C}^{\alpha}(X,\rho) \ \ \textit{is well defined} \\ &\textit{and compact whenever} \ \ 0 < \alpha < \beta < +\infty \end{aligned} \tag{4.77}$$

if and only if (X, ρ) is totally bounded.

Proof. The proof of the fact that (4.77) holds provided (X, ρ) is totally bounded is organized into two steps.

Step 1. The claim in (4.77) *holds when (X, τ_ρ) is a compact topological space*. In this setting, observe first that, $x_0 \in X$ having been fixed, the family of $\rho_\#$-balls $\big\{B_{\rho_\#}(x_0, j)\big\}_{j\in\mathbb{N}}$ constitutes an open cover of (X, τ_ρ). Since (X, τ_ρ) is compact, we deduce from this and (4.15) that $\operatorname{diam}_\rho(X) < +\infty$. Assume now that $0 < \alpha < \beta < +\infty$. It is then clear from definitions that

$$\|f\|_{\mathscr{C}^{\alpha}(X,\rho)} \le \max\Big\{1, \operatorname{diam}_\rho(X)^{\beta-\alpha}\Big\} \|f\|_{\mathscr{C}^{\beta}(X,\rho)}, \qquad \forall\, f \in \mathscr{C}^{\beta}(X,\rho), \tag{4.78}$$

which shows that the inclusion operator in (4.77) is well defined, linear, and bounded. To prove that this operator is also compact, fix a bounded sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}^{\beta}(X,\rho)$. Hence, there exists $M \in [0, +\infty)$ with the property that

$$|f_j(x)| \le M, \qquad \forall\, x \in X \ \text{ and } \ \forall\, j \in \mathbb{N}, \tag{4.79}$$

$$|f_j(x) - f_j(y)| \le M\rho(x,y)^{\beta}, \qquad \forall\, x, y \in X \ \text{ and } \ \forall\, j \in \mathbb{N}. \tag{4.80}$$

Collectively, (4.79) and (4.80) show that the family of functions $\{f_j\}_{j\in\mathbb{N}}$ is pointwise uniformly bounded and equicontinuous on (X, τ_ρ). Given that (X, τ_ρ) is a Hausdorff, compact topological space, Arzela's theorem (cf., e.g., [87, Exercise 4, p. 293]) may be used to conclude that there exists a subsequence of $\{f_j\}_{j\in\mathbb{N}}$ that converges uniformly on X to a continuous function $f : (X, \tau_\rho) \to \mathbb{R}$. Without loss of generality we will assume in what follows that this is the case for the entire original sequence $\{f_j\}_{j\in\mathbb{N}}$.

Passing to limit $j \to \infty$ in (4.79) and (4.80) then gives that $\|f\|_{\mathscr{C}^{\beta}(X,\rho)} \le 2M$. In particular, $f \in \mathscr{C}^{\beta}(X,\rho)$ and, hence, $f \in \mathscr{C}^{\alpha}(X,\rho)$ by (4.78). As such, the proof is completed the moment we show that $\|f_j - f\|_{\mathscr{C}^{\alpha}(X,\rho)} \to 0$ as $j \to \infty$. In this regard, fix an arbitrary number $\varepsilon > 0$, and introduce $r := \big(\varepsilon/(2M)\big)^{1/(\beta-\alpha)} \in (0, +\infty)$. This choice of r ensures that, on the one hand,

$$\sup_{\substack{x,y\in X,\,x\neq y\\ \rho(x,y)<r}} \frac{|(f_j-f)(x)-(f_j-f)(y)|}{\rho(x,y)^\alpha} \le \sup_{\substack{x,y\in X,\,x\neq y\\ \rho(x,y)<r}} \frac{|f(x)-f(y)|}{\rho(x,y)^\beta}\cdot\rho(x,y)^{\beta-\alpha}$$
$$+\sup_{\substack{x,y\in X,\,x\neq y\\ \rho(x,y)<r}} \frac{|f_j(x)-f_j(y)|}{\rho(x,y)^\beta}\cdot\rho(x,y)^{\beta-\alpha}$$
$$<\varepsilon, \qquad \forall\, j\in\mathbb{N}, \tag{4.81}$$

and, on the other hand,

$$\sup_{\substack{x,y\in X\\ \rho(x,y)\geq r}} \frac{|(f_j-f)(x)-(f_j-f)(y)|}{\rho(x,y)^\alpha} \le 2r^{-\alpha}\cdot\sup_{x\in X}|f_j(x)-f(x)|, \qquad \forall\, j\in\mathbb{N}. \tag{4.82}$$

Choose now $j_o\in\mathbb{N}$ such that the right-hand side of (4.82) is $<\varepsilon$ whenever $j\in\mathbb{N}$ satisfies $j\geq j_o$. That this is possible is guaranteed by the fact that the sequence $\{f_j\}_{j\in\mathbb{N}}$ converges uniformly on X to f. Then from (4.81) and (4.82) we deduce that $\|f_j-f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}<2\varepsilon$ if $j\in\mathbb{N}$ satisfies $j\geq j_o$. Thus, ultimately we have $\|f_j-f\|_{\mathscr{C}^\alpha(X,\rho)}\to 0$ as $j\to\infty$, as desired.

Step 2. The claim in (4.77) *holds whenever* (X,ρ) *is totally bounded*. Recall that a quasimetric space $(\tilde{X},\tilde{\rho})$ is said to be a `completion` of (X,ρ) provided the following conditions are satisfied:

$$X\subseteq\tilde{X} \text{ and } \tilde{\rho}|_X\approx\rho, \text{ say } c^{-1}\rho\le\tilde{\rho}|_X\le c\rho, \text{ for some } c\in[1,+\infty), \tag{4.83}$$

$$X \text{ is dense in } (\tilde{X},\tau_{\tilde{\rho}}), \tag{4.84}$$

$$(\tilde{X},\tilde{\rho}) \text{ is complete.} \tag{4.85}$$

Then any quasimetric space (X,ρ) has a completion $(\tilde{X},\tilde{\rho})$. To see that this is the case, given a quasimetric space (X,ρ), associate to it the metric space $(X,\rho_\#^\beta)$, where β is a finite number belonging to the interval $\big(0,(\log_2 C_\rho)^{-1}\big]$. Then, as is well known (cf., e.g., [27, Theorem 1.5.10, p. 12], [74, p. 25]), there exists a metric space $(\tilde{X},d)$ that is the completion of $(X,\rho_\#^\beta)$ in the sense described in (4.83)–(4.85). Then $(\tilde{X},\tilde{\rho})$ with $\tilde{\rho}:=d^{1/\beta}$ becomes a completion of the original quasimetric space (X,ρ).

In fact, it can be shown (either directly or by reducing matters as was done previously to known results for metric spaces) that the completeness of a quasimetric space (X,ρ) is unique up to bi-Lipschitz homeomorphisms, in the sense that for any two completions $(\tilde{X}_1,\tilde{\rho}_1)$ and $(\tilde{X}_2,\tilde{\rho}_2)$ of the given quasimetric space (X,ρ) there exists a bi-Lipschitz homeomorphism $\Phi:(\tilde{X}_1,\tilde{\rho}_1)\to(\tilde{X}_2,\tilde{\rho}_2)$ such that $\Phi|_X$ is the identity mapping of X.

We next claim that

$$\begin{aligned}&\text{if a quasimetric space } (X,\rho) \text{ is totally bounded,}\\&\text{then its completion } (\tilde{X},\tilde{\rho}) \text{ is also totally bounded.}\end{aligned} \tag{4.86}$$

To see that this is the case, pick an arbitrary $\varepsilon > 0$ and select $C \in \big(0, (C_{\tilde{\rho}})^2 \cdot c\big)$, where c is as in (4.83). Since (X,ρ) is totally bounded, there exist $x_1,\dots,x_N \in X$ such that

$$X \subseteq \bigcup_{1\leq i\leq N} B_\rho(x_i, \varepsilon/C). \tag{4.87}$$

Select an arbitrary point $x_* \in \tilde{X}$. By (4.84), it is possible to find a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$y_n \longrightarrow x_* \quad \text{in} \quad \tau_{\tilde{\rho}} \quad \text{as} \quad n\to\infty. \tag{4.88}$$

Note that since by (4.87) the set X is covered by the finite family of ρ-balls $\big\{B_\rho(x_i,\varepsilon/C)\big\}_{1\leq i\leq N}$, there exist $i_o \in \{1,\dots,N\}$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ with the property that $y_{n_k} \in B_\rho(x_{i_o},\varepsilon/C)$ for every $k\in\mathbb{N}$. Then

$$\begin{aligned}\tilde{\rho}(x_*,x_{i_o}) &\leq (C_{\tilde{\rho}})^2 \tilde{\rho}_\#(x_*,x_{i_o}) = (C_{\tilde{\rho}})^2 \lim_{k\to\infty} \tilde{\rho}_\#(y_{n_k},x_{i_o})\\&\leq (C_{\tilde{\rho}})^2 \limsup_{k\to\infty} \tilde{\rho}(y_{n_k},x_{i_o}) \leq (C_{\tilde{\rho}})^2 c \limsup_{k\to\infty} \rho(y_{n_k},x_{i_o}) < \varepsilon,\end{aligned} \tag{4.89}$$

where the first inequality uses (4.15), the subsequent equality is based on (4.88) and the fact that $\tilde{\rho}_\#$ is continuous, the second inequality once again follows from (4.15), the third inequality is a consequence of (4.83), and the last inequality is implied by the choice of the subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ and of the constant C. Estimate (4.89) proves that $x_* \in B_{\tilde{\rho}}(x_{i_o},\varepsilon)$; hence, ultimately,

$$\tilde{X} \subseteq \bigcup_{1\leq i\leq N} B_{\tilde{\rho}}(x_i,\varepsilon), \tag{4.90}$$

completing the proof of (4.86).

Going further, fix an exponent $\beta \in (0,+\infty)$, and note that, thanks to (4.83) and (4.84), any function $f \in \mathscr{C}^\beta(X,\rho)$ extends uniquely to a function $\tilde{f} \in \mathscr{C}^\beta(\tilde{X},\tilde{\rho})$ in such a way that

$$\begin{aligned}&\text{the operator } \mathscr{C}^\beta(X,\rho) \ni f \mapsto \tilde{f} \in \mathscr{C}^\beta(\tilde{X},\tilde{\rho})\\&\text{is well defined, linear, and bounded.}\end{aligned} \tag{4.91}$$

To see this, given $f \in \mathscr{C}^\beta(X,\rho)$, define $\tilde{f} : \tilde{X} \to \mathbb{R}$ by setting

$$\tilde{f}(x_*) := \lim_{n\to\infty} f(x_n) \text{ whenever } x_* \in \tilde{X} \text{ and } \{x_n\}_{n\in\mathbb{N}} \subseteq X \\ \text{are such that } x_n \to x_* \text{ in } \tau_{\tilde{\rho}} \text{ as } n \to \infty. \tag{4.92}$$

Given that any Hölder function maps Cauchy sequences to Cauchy sequences we deduce that the limit in (4.92) exists. Furthermore, by interlacing sequences it follows that this limit is unambiguously defined under the conditions stipulated in (4.92). As a byproduct, we obtain that $\tilde{f}|_X = f$. Finally, to see that $\tilde{f} \in \mathscr{C}^\beta(\tilde{X}, \tilde{\rho})$ with control of the norm, pick two arbitrary points $\tilde{x}, \tilde{y} \in \tilde{X}$ along with two sequences

$$\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq X \text{ such that } x_n \to \tilde{x} \text{ and } y_n \to \tilde{y} \text{ in } \tau_{\tilde{\rho}} \text{ as } n \to \infty. \tag{4.93}$$

Then we may estimate (based on the continuity of $\tilde{\rho}_\#$)

$$\begin{aligned} |\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y})| &= \big|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)\big| = \lim_{n\to\infty} |f(x_n) - f(y_n)| \\ &\leq \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \limsup_{n\to\infty} \rho(x_n, y_n)^\beta \leq c^\beta \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \limsup_{n\to\infty} \tilde{\rho}(x_n, y_n)^\beta \\ &\leq c^\beta (C_{\tilde{\rho}})^2 \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \limsup_{n\to\infty} \tilde{\rho}_\#(x_n, y_n)^\beta = c^\beta (C_{\tilde{\rho}})^2 \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \tilde{\rho}_\#(\tilde{x}, \tilde{y})^\beta \\ &\leq c^\beta (C_{\tilde{\rho}})^2 \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \tilde{\rho}(\tilde{x}, \tilde{y})^\beta, \end{aligned} \tag{4.94}$$

which goes to show that

$$\|\tilde{f}\|_{\dot{\mathscr{C}}^\beta(\tilde{X},\tilde{\rho})} \leq c^\beta (C_{\tilde{\rho}})^2 \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)}. \tag{4.95}$$

Moreover,

$$|\tilde{f}(\tilde{x})| = \big|\lim_{n\to\infty} f(x_n)\big| = \lim_{n\to\infty} |f(x_n)| \leq \sup_{x\in X} |f(x)|, \tag{4.96}$$

so that

$$\sup_{\tilde{z}\in\tilde{X}} |\tilde{f}(\tilde{z})| \leq \sup_{x\in X} |f(x)|. \tag{4.97}$$

Hence, the boundedness of the operator in (4.91) follows from (4.95) and (4.97).

We are now in a position to justify the claim made at the beginning of Step 2. To this end, assume that $0 < \alpha < \beta < +\infty$, and consider a bounded sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathscr{C}^\beta(X, \rho)$. Then, thanks to (4.91), the sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{C}^\beta(\tilde{X}, \tilde{\rho})$. Since from (4.86) and (4.85) we know that $(\tilde{X}, \tilde{\rho})$ is both totally bounded and complete, it follows that $(\tilde{X}, \tau_{\tilde{\rho}})$ is a compact topological space. As such, the result proved in Step 1 applies and yields the existence of some function $\tilde{f} \in \mathscr{C}^\beta(\tilde{X}, \tilde{\rho})$ with the property that, for some subsequence $\{\tilde{f}_{n_k}\}_{k\in\mathbb{N}}$ of $\{\tilde{f}_n\}_{n\in\mathbb{N}}$,

$$\lim_{k\to\infty} \|\tilde{f}_{n_k} - \tilde{f}\|_{\mathscr{C}^\alpha(\tilde{X},\tilde{\rho})} = 0. \tag{4.98}$$

Consequently, if $f := \tilde{f}|_X \in \mathscr{C}^\beta(X,\rho)$, then we deduce from (4.98) that

$$\begin{aligned}\lim_{k\to\infty} \|f_{n_k} - f\|_{\mathscr{C}^\alpha(X,\rho)} &= \lim_{k\to\infty} \big\|\tilde{f}_{n_k}|_X - \tilde{f}|_X\big\|_{\mathscr{C}^\alpha(X,\rho)} \\ &\leq c^\alpha \lim_{k\to\infty} \|\tilde{f}_{n_k} - \tilde{f}\|_{\mathscr{C}^\alpha(\tilde{X},\tilde{\rho})} = 0. \end{aligned}\tag{4.99}$$

Thus, $f_{n_k} \to f$ in $\mathscr{C}^\alpha(X,\rho)$ as $k\to\infty$, completing the proof of the claim made in Step 2.

Let us now consider the converse implication, i.e., that if (4.77) holds, then (X,ρ) is necessarily totally bounded. Fix an arbitrary $\varepsilon > 0$ and use Zorn's lemma to construct a family of points $\{x_j\}_{j\in J} \subseteq X$, where J is a set of indexes, such that

$$\rho(x_i, x_j) > \varepsilon/2, \qquad \forall\, i,j \in J \text{ with } i \neq j, \tag{4.100}$$

which is maximal, with respect to the partial order induced by the inclusion of subsets of X, with this property. In particular, by the maximality property, this family of points satisfies

$$X \subseteq \bigcup_{j\in J} B_\rho(x_j, \varepsilon). \tag{4.101}$$

Also, since the topological space (X,τ_ρ) is separable (cf. (4.49)), we may actually assume that J is at most countable; hence, in the case when J is infinite there is no loss of generality in assuming that $J = \mathbb{N}$. In this scenario, select a finite number β such that $0 < \beta \leq \big[\log_2 C_\rho\big]^{-1}$, and invoke Theorem 4.12 to construct a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}^\beta(X,\rho)$ with the property that, for each $j \in \mathbb{N}$,

$$\begin{gathered} 0 \leq f_j \leq 1 \text{ on } X, \quad f_j \equiv 0 \text{ on } X \setminus B_\rho(x_j, \varepsilon/2), \\ \text{and } \ f_j \equiv 1 \text{ on } B_\rho\big(x_j, \varepsilon/(2C_\rho)\big), \end{gathered}\tag{4.102}$$

and such that, for some finite constant $C > 0$, depending only on ρ,

$$\|f_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C\Big(\mathrm{dist}_\rho\big(X\setminus B_\rho(x_j,\varepsilon/2), B_\rho(x_j, \varepsilon/(2C_\rho))\big)\Big)^{-\beta}. \tag{4.103}$$

That this is possible is ensured by observing that each ball $B_\rho\big(x_j, \varepsilon/(2C_\rho)\big)$ is nonempty (since it contains its center), each set of the form $X \setminus B_\rho(x_j,\varepsilon/2)$ is nonempty (since $x_k \in X \setminus B_\rho(x_j,\varepsilon/2)$ for every $k \in \mathbb{N}\setminus\{j\}$), and

$$\mathrm{dist}_\rho\big(X\setminus B_\rho(x_j,\varepsilon/2), B_\rho\big(x_j,\varepsilon/(2C_\rho)\big)\big) \geq \varepsilon/(2C_\rho). \tag{4.104}$$

The last inequality is justified by noting that for any points $y \in B_\rho\big(x_j, \varepsilon/(2C_\rho)\big)$ and $z \in X \setminus B_\rho(x_j, \varepsilon/2)$ we have

$$\rho(x_j, y) < \varepsilon/(2C_\rho) \quad \text{and} \quad \varepsilon/2 \le \rho(x_j, z) \le C_\rho \max\big\{\rho(x_j, y), \rho(y, z)\big\}. \tag{4.105}$$

Based on (4.105), we obtain $\rho(y, z) \ge \varepsilon/(2C_\rho)$, and (4.104) follows. Going further, from (4.104), (4.103), and (4.102) we deduce that

$$\|f_j\|_{\mathscr{C}^\beta(X,\rho)} \le 1 + C(2C_\rho)^\beta \varepsilon^{-\beta}, \qquad \forall\, j \in \mathbb{N}. \tag{4.106}$$

Thus, $\{f_j\}_{j\in\mathbb{N}}$ is bounded in $\mathscr{C}^\beta(X, \rho)$. If we now pick some $\alpha \in (0, \beta)$, then it follows from this and (4.77) that there exist a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ and a function $f \in \mathscr{C}^\alpha(X, \rho)$ with the property that $\lim\limits_{k\to\infty} \|f_{j_k} - f\|_{\mathscr{C}^\alpha(X,\rho)} = 0$. In particular,

$$\sup_{x\in X}\big|f_{j_k}(x) - f(x)\big| \longrightarrow 0 \ \text{ as } \ k \to \infty. \tag{4.107}$$

Then for each $k, k' \in \mathbb{N}$ with $k \ne k'$ we have $j_k \ne j_{k'}$ and, hence, $\rho(x_{j_k}, x_{j_{k'}}) > \varepsilon/2$. In particular, $x_{j_k} \in X \setminus B_\rho(x_{j_{k'}}, \varepsilon/2)$, hence $f_{j_{k'}}(x_{j_k}) = 0$. Keeping $k \in \mathbb{N}$ fixed and letting $k' \to \infty$ then yields

$$|f(x_{j_k})| = |f(x_{j_k}) - f_{j_{k'}}(x_{j_k})| \le \sup_{x\in X} |f(x) - f_{j_{k'}}(x)| \to 0 \ \text{ as } \ k' \to \infty \tag{4.108}$$

on account of (4.107). Consequently, $f(x_{j_k}) = 0$ for every $k \in \mathbb{N}$. In turn, this implies that, on the one hand,

$$1 = |f_{j_k}(x_{j_k})| = |f(x_{j_k}) - f_{j_k}(x_{j_k})| \le \sup_{x\in X} |f(x) - f_{j_k}(x)| \ \text{ for every } \ k \in \mathbb{N}, \tag{4.109}$$

and, on the other hand, by (4.107), the supremum in the rightmost side of (4.109) goes to zero as $k \to \infty$, a contradiction. This proves that the sequence of points $\{x_j\}_{j\in J} \subseteq X$ satisfying (4.100) is necessarily finite. Together with (4.101), this implies that (X, ρ) is totally bounded, as desired. □

4.5 Regularized Distance Function to a Set

Given a set X, a quasidistance $\rho \in \mathfrak{Q}(X)$, and a nonempty set $E \subseteq X$, define

$$\operatorname{dist}_\rho(x, E) := \inf\{\rho(x, y) :\ y \in E\}, \qquad \forall\, x \in X. \tag{4.110}$$

If ρ is actually a metric, then $\operatorname{dist}_\rho(\cdot, E) : X \to \mathbb{R}$ is a Lipschitz function (with Lipschitz constant ≤ 1). In the general case corresponding to ρ being merely a quasidistance on X, the function $\operatorname{dist}_\rho(\cdot, E)$ may even fail to be continuous (this is the case even in such simple instances as when the set E is a singleton). The issue that arises is whether there exists a nonnegative function on X that is pointwise equivalent to $\mathrm{dist}_\rho(\cdot, E)$ and exhibits better regularity properties. When the quasimetric space in question is $\mathbb{R}^n$ equipped with the standard Euclidean distance and when "regularity" is understood in the differential sense, this is a classical problem, and the reader is referred to, e.g., [113, Theorem 2, p. 171] for an excellent exposition. Here the goal is to explore the extent to which a result of this flavor is valid in the setting of general quasimetric spaces. Specifically, we have the following theorem.

Theorem 4.17. *Suppose that* $(X, \mathbf{q})$ *is a quasimetric space,* E *is a nonempty subset of* X*, and* $\rho \in \mathbf{q}$. *Then there exist a function* $\delta_E : X \to [0, +\infty)$ *and two constants* $c_0, c_1 \in (0, +\infty)$*, which depend only on* C_ρ*, such that*

$$c_0 \,\mathrm{dist}_\rho(x, E) \le \delta_E(x) \le c_1 \,\mathrm{dist}_\rho(x, E), \qquad \forall\, x \in X. \tag{4.111}$$

Furthermore, if $\beta \in \mathbb{R}$ *is such that* $0 < \beta \le [\log_2 C_\rho]^{-1}$*, then* δ_E *satisfies the following properties:*

(1) If E *is a closed, proper subset of* $(X, \tau_{\mathbf{q}})$*, then* $\delta_E \in \dot{\mathscr{C}}^{\beta}_{loc}(X \setminus E, \rho)$ *in the quantitative sense that for every* $\varepsilon \in (0, C_\rho^{-1})$ *there exists* $C \in (0, +\infty)$*, depending only on* C_ρ*,* β*, and* ε*, such that*

$$\sup \left\{ \frac{|\delta_E(x) - \delta_E(y)|}{\rho(x, y)^\beta} : \, x, y \in B_\rho\big(z, \varepsilon\, \mathrm{dist}_\rho(z, E)\big), \;\; x \neq y \right\}$$
$$\le C\big[\mathrm{dist}_\rho(z, E)\big]^{1-\beta} \;\; \text{for all} \;\; z \in X \setminus E. \tag{4.112}$$

(2) If $0 < \beta \le 1$*, then there exists* $C \in (0, +\infty)$*, which depends only on* C_ρ *and* β *such that*

$$\frac{|\delta_E(x) - \delta_E(y)|}{\rho(x, y)^\beta} \le C\, \Big(\rho(x, y) + \max\{\mathrm{dist}_\rho(x, E)\,,\, \mathrm{dist}_\rho(y, E)\}\Big)^{1-\beta} \tag{4.113}$$

for all $x, y \in X$ *with* $x \neq y$.

Proof. Suppose that $\rho \in \mathbf{q}$, and for now assume that E is a fixed, arbitrary, nonempty subset of X. If $\rho_\#$ denotes the regularized version of ρ as in Theorem 3.46 and we define

$$\delta_E(x) := \inf\{\rho_\#(x, y) : \; y \in E\}, \qquad \forall\, x \in X, \tag{4.114}$$

then (4.111) follows by virtue of the fact that $\rho_\# \in [\rho]$.

Let $\beta \in \mathbb{R}$ be such that $0 < \beta \le [\log_2 C_\rho]^{-1}$. Fix $x, y \in X$ and select a sequence of points $\{z_n\}_{n \in \mathbb{N}}$ in E with the property that

$$\rho_\#(y, z_n) \to \delta_E(y) \;\; \text{as} \;\; n \to \infty. \tag{4.115}$$

Making use of (3.542), for each $n \in \mathbb{N}$ we then write

$$\begin{aligned}\delta_E(x) &\leq \rho_\#(x, z_n) \leq \rho_\#(y, z_n) + |\rho_\#(x, z_n) - \rho_\#(y, z_n)| \\ &\leq \rho_\#(y, z_n) + \tfrac{1}{\beta} \max\big\{\rho_\#(x, z_n)^{1-\beta}, \rho_\#(y, z_n)^{1-\beta}\big\}\big[\rho_\#(x, y)\big]^\beta, \qquad (4.116)\end{aligned}$$

with the understanding that when $\beta \geq 1$, we also assume that $z_n \notin \{x, y\}$.

To treat the claim made in (1), consider now the case when E is a nonempty, proper, closed subset of $(X, \tau_{\mathbf{q}})$. Choose $\varepsilon \in (0, C_\rho^{-1})$, and fix some point $z \in X \setminus E$. Then

$$r := \mathrm{dist}_\rho(z, E) > 0 \qquad (4.117)$$

since $X \setminus E$ is open in $(X, \tau_{\mathbf{q}})$. Finally, assume that $x, y \in B_\rho(z, \varepsilon r)$. We then claim that

$$C_\rho^{-3} r \leq \delta_E(x) \leq C_\rho r \quad \text{and} \quad C_\rho^{-3} r \leq \delta_E(y) \leq C_\rho r. \qquad (4.118)$$

To justify (4.118), fix an arbitrary $\theta > 1$ and select $w \in E$ with the property that $\rho(z, w) < \theta r$ (here, the fact that $r > 0$ is used). Then, by (3.530),

$$\begin{aligned}\delta_E(x) &\leq \rho_\#(x, w) \leq C_\rho \max\{\rho_\#(x, z), \rho_\#(z, w)\} \\ &\leq C_\rho \max\{\rho(x, z), \rho(z, w)\} \leq C_\rho \max\{\varepsilon r, \theta r\} = C_\rho \theta r. \qquad (4.119)\end{aligned}$$

Passing to the limit as $\theta \searrow 1$ then yields $\delta_E(x) \leq C_\rho r$, as desired. In the opposite direction, for every $w \in E$ we may write

$$r = \mathrm{dist}_\rho(z, E) \leq \rho(z, w) \leq C_\rho \max\{\rho(z, x), \rho(x, w)\} \leq C_\rho \max\{\varepsilon r, \rho(x, w)\}, \qquad (4.120)$$

which, given that $C_\rho \varepsilon < 1$, yields $r \leq C_\rho \rho(x, w)$. Hence, $C_\rho^{-1} r \leq \rho(x, w) \leq C_\rho^2 \rho_\#(x, w)$ for every $w \in E$; thus, ultimately, $C_\rho^{-1} r \leq C_\rho^2 \delta_E(x)$. This completes the proof of the first double inequality in (4.118). The second double inequality in (4.118) is proved similarly, completing the proof of (4.118).

Let us also note that if $x, y \in B_\rho(z, \varepsilon r)$, then $\rho(x, z), \rho(y, z) < \varepsilon r$ and, as a result,

$$\rho_\#(x, y) \leq \rho(x, y) \leq C_\rho \max\{\rho(x, z), \rho(z, y)\} \leq \varepsilon C_\rho r. \qquad (4.121)$$

Returning now to the context in which (4.116) was derived, observe that, by (4.118) and (4.121), whenever $x, y \in B_\rho(z, \varepsilon r)$, we may write

$$\begin{aligned}C_\rho^{-3} r &\leq \delta_E(x) \leq \rho_\#(x, z_n) \leq C_\rho \max\{\rho_\#(x, y), \rho_\#(y, z_n)\} \\ &\leq C_\rho \max\{\varepsilon C_\rho r, \rho_\#(y, z_n)\}, \qquad \forall\, n \in \mathbb{N}. \qquad (4.122)\end{aligned}$$

In light of (4.115) and (4.118), this further implies

$$
\begin{aligned}
C_\rho^{-3} r &\le \liminf_{n\to+\infty} \rho_\#(x, z_n) \le \limsup_{n\to+\infty} \rho_\#(x, z_n) \\
&\le C_\rho \max\{\varepsilon C_\rho r, \delta_E(y)\} \le C_\rho^2 r
\end{aligned} \tag{4.123}
$$

whenever $x, y \in B_\rho(z, \varepsilon r)$. In a similar fashion,

$$
C_\rho^{-3} r \le \liminf_{n\to+\infty} \rho_\#(y, z_n) \le \limsup_{n\to+\infty} \rho_\#(y, z_n) \le C_\rho^2 r \tag{4.124}
$$

if $y \in B_\rho(z, \varepsilon r)$. From (4.123) and (4.124) we deduce that if $x, y \in B_\rho(z, \varepsilon r)$, then, for n large,

$$
\max\left\{\rho_\#(x, z_n)^{1-\beta}, \rho_\#(y, z_n)^{1-\beta}\right\}\left[\rho_\#(x, y)\right]^\beta \approx r^{1-\beta}\rho(x, y)^\beta, \tag{4.125}
$$

with proportionality constants depending only on C_ρ and β. Collectively, (4.115), (4.116), and (4.125) allow us to deduce that

$$
\delta_E(x) \le \delta_E(y) + C r^{1-\beta}\rho(x, y)^\beta, \qquad \forall\, x, y \in B_\rho(z, \varepsilon r), \tag{4.126}
$$

for some finite $C \ge 0$ that depends only on C_ρ and β. From (4.126) and its version with x and y interchanged we finally conclude that

$$
|\delta_E(x) - \delta_E(y)| \le C r^{1-\beta}\rho(x, y)^\beta, \qquad \forall\, x, y \in B_\rho(z, \varepsilon r), \tag{4.127}
$$

with C as before. Given that $r = \mathrm{dist}_\rho(z, E)$, this completes the proof of (4.112).

To treat the claim made in (2), suppose now that E is a nonempty subset of X. Also, assume this time that $0 < \beta \le 1$ and that $x, y \in X$ are arbitrary. Of course, the considerations pertaining to (4.114)–(4.116) remain valid, and we augment these by noting that $\rho_\#(x, z_n) \le C_\rho \max\{\rho_\#(x, y), \rho_\#(y, z_n)\}$ for all $n \in \mathbb{N}$. Hence,

$$
\limsup_{n\to+\infty} \rho_\#(x, z_n) \le C_\rho \max\{\rho_\#(x, y), \delta_E(y)\}, \tag{4.128}
$$

which, when used back in (4.116) and after passing to limit $n \to +\infty$, yields, on account of (4.115) and the fact that $\beta \in (0, 1]$,

$$
\begin{aligned}
\delta_E(x) &\le \delta_E(y) + \tfrac{1}{\beta} C_\rho^{1-\beta} \max\{\rho_\#(x, y)^{1-\beta}, \delta_E(y)^{1-\beta}\}\left[\rho_\#(x, y)\right]^\beta \\
&\le \delta_E(y) + C \max\{\rho(x, y)^{1-\beta}, \delta_E(y)^{1-\beta}\}\rho(x, y)^\beta.
\end{aligned} \tag{4.129}
$$

As a consequence, for each $x, y \in X$ we see that

$$
\delta_E(x) \le \delta_E(y) + C\Big(\rho(x, y) + \max\{\delta_E(x), \delta_E(y)\}\Big)^{1-\beta}\rho(x, y)^\beta. \tag{4.130}
$$

Now, (4.113) follows from this, its version written with x and y interchanged, as well as (4.111). This completes the proof of the theorem. □

A conceptually different proof of a slight variant of Theorem 4.17, more akin to that of [113, Theorem 2, p. 171], but in the more specialized setting of geometrically doubling quasimetric spaces, will be presented in the next section, following a discussion of Whitney-like decompositions of open sets and partitions of unity in the said setting.

4.6 Whitney-Like Partitions of Unity via Hölder Functions

An important tool in harmonic analysis is the Whitney decomposition of a nonempty, proper, open subset $\mathcal{O}$ of a quasimetric space (X, ρ) into balls whose distance to the complement of $\mathcal{O}$ in X is proportional to the radius of the ball in question. Frequently, given such a Whitney decomposition, it is useful to have a partition of unity subordinate to it, which is quantitative in the sense that the size of the functions involved is controlled in terms of the size of their respective supports. Details in the standard setting of $\mathbb{R}^n$ may be found in [113, p. 170].

More recently, such quantitative Whitney partitions of unity have been constructed on general metric spaces (see [73, Lemma 2.4, p.339], [51]) and on quasimetric spaces, as in [80, Lemma 2.16, p. 278]. Here we wish to improve upon the latter result both by allowing a more general set-theoretic framework and by providing a transparent description of the order of smoothness of the functions involved in such a Whitney-like partition of unity for an arbitrary quasimetric space.

Theorem 4.18. *Let $(X, \mathbf{q})$ be a quasimetric space, and assume that β is a finite number with the property that there exists a quasidistance $\rho \in \mathbf{q}$ such that $0 < \beta \leq \big[\log_2 C_\rho\big]^{-1}$. In this setting, assume that $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$ and $\{\widehat{E}_j\}_{j\in I}$ are three families of nonempty proper subsets of X satisfying the following properties:*

(a) For each $j \in I$ one has $E_j \subseteq \widetilde{E}_j \subseteq \widehat{E}_j$, $r_j := \operatorname{dist}_\rho(E_j, X \setminus \widetilde{E}_j) > 0$ and

$$\operatorname{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \approx r_j \quad \text{uniformly for } j \in I. \tag{4.131}$$

(b) One has $r_i \approx r_j$ uniformly for $i, j \in I$ such that $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$.

(c) There exists $N \in \mathbb{N}$ such that $\sum_{j\in I} \mathbf{1}_{\widehat{E}_j} \leq N$.

(d) One has $\bigcup_{j\in I} E_j = \bigcup_{j\in I} \widehat{E}_j$.

Then there exist a finite constant $C \geq 1$, depending only on ρ, β, N, and the proportionality constants in (a) and (b) above, along with a family of real-valued functions $\{\varphi_j\}_{j\in I}$ on X such that the following conditions are valid:

(1) For each $j \in I$ one has

$$\varphi_j \in \dot{\mathscr{C}}^\beta(X, \mathbf{q}) \quad \textit{and} \quad \|\varphi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C r_j^{-\beta}. \tag{4.132}$$

(2) For every $j \in I$ one has

$$0 \leq \varphi_j \leq 1 \;\; \textit{on} \;\; X, \quad \varphi_j \equiv 0 \;\; \textit{on} \;\; X \setminus \widetilde{E}_j, \quad \textit{and} \quad \varphi_j \geq 1/C \;\; \textit{on} \;\; E_j. \tag{4.133}$$

(3) One has $\sum_{j\in I} \varphi_j = \mathbf{1}_{\bigcup_{j\in I} E_j} = \mathbf{1}_{\bigcup_{j\in I} \widetilde{E}_j} = \mathbf{1}_{\bigcup_{j\in I} \widehat{E}_j}$.

Proof. Fix ρ and β as in the statement of the theorem. Based on Theorem 4.12 and property (a), for each $j \in I$ there exists a function $\psi_j \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ such that

$$(i)\; \psi_j \equiv 1 \;\text{ on }\; E_j, \quad (ii)\; \psi_j \equiv 0 \;\text{ on }\; X \setminus \widetilde{E}_j, \quad (iii)\; 0 \leq \psi_j \leq 1 \;\text{ on }\; X, \tag{4.134}$$

and

$$\|\psi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C r_j^{-\beta}, \tag{4.135}$$

where $C > 0$ is a finite constant depending only on ρ and β. Consider next the function

$$\Psi : \bigcup_{j\in I} E_j \longrightarrow \mathbb{R}, \quad \Psi := \sum_{j\in I} \psi_j, \tag{4.136}$$

and note that Ψ is well defined and satisfies

$$1 \leq \Psi \leq N \quad \text{on} \quad \bigcup_{j\in I} E_j. \tag{4.137}$$

Indeed, the fact that Ψ is well defined follows from (c) and (iii) in (4.134), the first inequality is due to (i) and (iii) in (4.134), and the second inequality is a consequence of (ii) in (4.134), the fact that $E_j \subseteq \widehat{E}_j$ for each $j \in I$, and statement (c) in the hypotheses. Going further, for each $j \in I$ introduce the function

$$\varphi_j : X \longrightarrow \mathbb{R}, \quad \varphi_j := \begin{cases} \psi_j/\Psi & \text{on } \bigcup_{i\in I} E_i, \\ 0 & \text{on } X \setminus \big(\bigcup_{i\in I} E_i\big). \end{cases} \tag{4.138}$$

By the discussion pertaining to the nature of Ψ, for each $j \in I$ the function φ_j is well defined and, thanks to (4.138), the first inequality in (4.137), and (ii) in (4.134), satisfies

$$0 \leq \varphi_j \leq 1 \;\text{ on }\; X, \quad \varphi_j \equiv 0 \;\text{ on }\; X \setminus \widetilde{E}_j. \tag{4.139}$$

This proves the first two assertions in (2) in the conclusion of the theorem. Also, employing (4.138), (i) in (4.134), and the second inequality in (4.137), we may conclude that

$$\varphi_j = \psi_j/\Psi = 1/\Psi \geq 1/N \quad \text{on} \quad E_j. \tag{4.140}$$

This completes the proof of (2) provided one chooses $C \geq N$. Going further, by (c), the sum $\sum_{j\in I}\varphi_j$ is meaningfully defined and, using (4.138) and (4.136), is identically equal to one on $\bigcup_{j\in I} E_j$, thus proving conclusion (3) from the statement of the theorem.

It remains to prove (1). To this end, as a preliminary step we will show that there exists a finite constant $C > 0$, depending only on ρ, β, and the proportionality constants in (a) and (b), such that, for each $j \in I$, there holds

$$|\psi_j(x) - \psi_j(y)| \leq C r_j^{-\beta}\rho(x,y)^\beta\big[\mathbf{1}_{\widehat{E}_j}(x) + \mathbf{1}_{\widehat{E}_j}(y)\big], \qquad \forall\, x,y \in X. \quad (4.141)$$

To prove (4.141), fix $j \in I$ and, based on $\psi_j \in \dot{\mathscr{C}}^\beta(X,\rho)$ and (4.135), estimate

$$|\psi_j(x) - \psi_j(y)| \leq \|\psi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\rho(x,y)^\beta \leq C r_j^{-\beta}\rho(x,y)^\beta, \quad \forall\, x,y \in X. \quad (4.142)$$

By construction, $\psi_j \equiv 0$ on $X \setminus \widetilde{E}_j$ so that if $x, y \in X \setminus \widetilde{E}_j$, then (4.141) is obviously true. In the case when either $x \in \widetilde{E}_j$ or $y \in \widetilde{E}_j$, using the fact that $\widetilde{E}_j \subseteq \widehat{E}_j$ we may write

$$\mathbf{1}_{\widehat{E}_j}(x) + \mathbf{1}_{\widehat{E}_j}(y) \geq 1, \quad (4.143)$$

and thus (4.141) follows from (4.143) and (4.142).

Having disposed of (4.141) we focus on proving (1), i.e., show that for each fixed index $j \in I$ we have

$$|\varphi_j(x) - \varphi_j(y)| \leq C r_j^{-\beta}\rho(x,y)^\beta, \qquad \forall\, x,y \in X, \quad (4.144)$$

for some finite constant $C > 0$, depending only on ρ, β, N, as well as the proportionality constants implicit in conditions (a) and (b). Fix $j \in I$ and note that (4.144) is obviously true whenever $x, y \in X \setminus \big(\bigcup_{i\in I} E_i\big)$ as the left-hand side in (4.144) vanishes in this case [cf. (4.138)]. Consider next the case when $x, y \in \bigcup_{i\in I} E_i$, in which scenario we write

$$\begin{aligned}
|\varphi_j(x) - \varphi_j(y)| &= \left|\frac{\psi_j(x)}{\Psi(x)} - -\frac{\psi_j(y)}{\Psi(y)}\right| = \left|\frac{\psi_j(x)\Psi(y) - \psi_j(y)\Psi(x)}{\Psi(x)\Psi(y)}\right| \\
&\leq |\psi_j(x)\Psi(y) - \psi_j(y)\Psi(x)| \\
&\leq |\psi_j(x) - \psi_j(y)|\Psi(y) + |\Psi(x) - \Psi(y)|\psi_j(y) \\
&\leq N|\psi_j(x) - \psi_j(y)| + |\Psi(x) - \Psi(y)|\mathbf{1}_{\widetilde{E}_j}(y) =: I_1 + I_2. \quad (4.145)
\end{aligned}$$

Here, the first inequality follows from the first inequality in (4.137), the second estimate is a consequence of the triangle inequality, and the third one follows from (4.137) and (ii)–(iii) in (4.134). Moving on, (4.142) immediately gives

$$I_1 \leq CNr_j^{-\beta}\rho(x,y)^\beta, \qquad \forall\, x,y \in X. \quad (4.146)$$

As for I_2, we make the claim that there exists a finite constant $C > 0$, depending only on ρ, β, N, and the proportionality constants in conditions (a) and (b) from the hypotheses, such that

$$I_2 \leq C r_j^{-\beta} \rho(x,y)^\beta, \qquad \forall\, x, y \in \bigcup_{i\in I} E_i. \tag{4.147}$$

To justify this claim, observe that if $y \in \big(\bigcup_{i\in I} E_i\big) \setminus \widetilde{E}_j$, then $I_2 = 0$, so estimate (4.147) is trivially true. Consider next the case when $y \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j$, and denote by $c > 0$ the lower proportionality constant in (4.131). If $\rho(x,y) \geq c\, r_j$, then, on the one hand, $r_j^{-\beta}\rho(x,y)^\beta \geq c^\beta$, while on the other hand $I_2 \leq 2N$ by the second inequality in (4.137). Hence, (4.147) holds in this case as well. Suppose now that

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in \Big(\bigcup_{i\in I} E_i\Big) \cap \widetilde{E}_j \quad \text{are such that} \quad \rho(x,y) < c\, r_j. \tag{4.148}$$

Given that $y \in \widetilde{E}_j$, and since by (4.131) and (4.148) we have

$$\operatorname{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \geq c r_j > \rho(y,x), \tag{4.149}$$

we deduce that $x \in \widehat{E}_j$. Based on this, the triangle inequality, and (4.141), it follows that

$$\begin{aligned} I_2 &= |\Psi(x) - \Psi(y)| \mathbf{1}_{\widehat{E}_j}(x) \mathbf{1}_{\widehat{E}_j}(y) \leq \sum_{i\in I} |\psi_i(x) - \psi_i(y)| \mathbf{1}_{\widehat{E}_j}(x) \mathbf{1}_{\widehat{E}_j}(y) \\ &\leq C\rho(x,y)^\beta \sum_{i\in I} r_i^{-\beta} \big[\mathbf{1}_{\widehat{E}_i}(x) + \mathbf{1}_{\widehat{E}_i}(y)\big] \mathbf{1}_{\widehat{E}_j}(x) \mathbf{1}_{\widehat{E}_j}(y) \\ &\leq C\rho(x,y)^\beta \big\{I_2' + I_2''\big\} \quad \text{whenever } x, y \text{ are as in (4.148)}, \end{aligned} \tag{4.150}$$

where

$$I_2' := \sum_{i\in I} r_i^{-\beta} \mathbf{1}_{\widehat{E}_i}(x) \mathbf{1}_{\widehat{E}_j}(x) \quad \text{and} \quad I_2'' := \sum_{i\in I} r_i^{-\beta} \mathbf{1}_{\widehat{E}_i}(y) \mathbf{1}_{\widehat{E}_j}(y). \tag{4.151}$$

For each nonzero term in I_2' we necessarily have $x \in \widehat{E}_i \cap \widehat{E}_j$; hence, $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$, which further forces $r_i \approx r_j$, by condition (b) in the hypotheses. Thus, using this and property (c) from the hypotheses,

$$I_2' \leq C r_j^{-\beta} \sum_{i\in I} \mathbf{1}_{\widehat{E}_i}(x) \leq C N r_j^{-\beta}, \tag{4.152}$$

where $C > 0$ is a finite constant that depends only on the proportionality constant in (b). Similarly, $I_2'' \le C r_j^{-\beta}$ for some finite constant $C > 0$ depending only on N and the proportionality constant in (b). Granted the discussion in the paragraph preceding (4.148), it follows from this and (4.150) that (4.147) holds as stated.

In summary, this analysis shows that the estimate in (4.144) holds whenever either $x, y \in X \setminus \big(\bigcup_{i\in I} E_i\big)$ or $x, y \in \bigcup_{i\in I} E_i$. Therefore, to complete the proof of (4.144), it remains to consider the case when

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in X \setminus \Big(\bigcup_{i\in I} E_i\Big), \tag{4.153}$$

or vice versa. Concretely, assume that (4.153) holds (the other case is treated similarly). Then (4.144) is clear when $x \notin \widetilde{E}_j$ since in such a scenario we have $\varphi_j(x) = \varphi_j(y) = 0$ by the second property in (4.139) and the second condition in (4.153). Thus matters have been reduced to considering the case when

$$x \in \Big(\bigcup_{i\in I} E_i\Big) \cap \widetilde{E}_j \quad \text{and} \quad y \in X \setminus \Big(\bigcup_{i\in I} E_i\Big) = X \setminus \Big(\bigcup_{i\in I} \widehat{E}_i\Big), \tag{4.154}$$

where the aforementioned equality is a consequence of condition (d) in the hypotheses. In particular, $x \in \widetilde{E}_j$ and $y \in X \setminus \widehat{E}_j$, and, hence, based on (a), we have

$$\rho(x, y) \ge \mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \ge c r_j, \tag{4.155}$$

where, as before, $c > 0$ is the lower proportionality constant in (4.131). In this situation, using the definition of φ_j from (4.138) and (4.155) we may estimate

$$|\varphi_j(x) - \varphi_j(y)| = \varphi_j(x) = \frac{\psi_j(x)}{\Psi(x)} \le \psi_j(x) \le 1 \le c^{-\beta} r_j^{-\beta} \rho(x, y)^\beta. \tag{4.156}$$

This proves the last case in the analysis of (4.144), completing the proof of (1) in the conclusion of the theorem. The proof of Theorem 4.18 is now complete. □

There are several important instances where the hypotheses of Theorem 4.18 are satisfied; in Comments 4.19, 4.20, and 4.22 we discuss three such scenarios.

Comment 4.19. Let (X, ρ) be a quasimetric space, and fix two finite real constants C_0, C_1 such that $C_1 \ge C_\rho$ and $C_0 \ge C_\rho C_1$. Also, assume that $\{B_\rho(x_j, r_j)\}_{j\in I}$ is a family of ρ-balls in X satisfying

$$0 < r_j < C_0 \,\mathrm{diam}_\rho(X), \quad \forall\, j \in I, \tag{4.157}$$

as well as

$$\begin{gathered} \{B_\rho(x_j, r_j)\}_{i\in I} \text{ have finite overlap, } \textstyle\bigcup_{j\in I} B_\rho(x_j, r_j/C_0) = X, \text{ and} \\ r_i \approx r_j \text{ uniformly for } i, j \in I \text{ such that } B_\rho(x_i, r_i) \cap B_\rho(x_j, r_j) \neq \emptyset. \end{gathered} \tag{4.158}$$

Then taking

$$E_j := B_\rho(x_j, r_j/C_0), \quad \widetilde{E}_j := B_\rho(x_j, r_j/C_1), \quad \widehat{E}_j := B_\rho(x_j, r_j), \quad \forall\, j \in I, \tag{4.159}$$

yields a triplet of families $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$, and $\{\widehat{E}_j\}_{j\in I}$ satisfying the conditions stipulated in Theorem 4.18 (with the radii r_j playing the role of the parameters r_j in the statement of this theorem).

A concrete situation when a family $\{B_\rho(x_j, r_j)\}_{j\in I}$ satisfying (4.157) and (4.158) arises naturally in practice is as follows. Assume that $(X, \mathbf{q})$ is a geometrically doubling quasimetric space, and fix some quasidistance $\rho \in \mathbf{q}$. Also, let C_0, C_1 be as before, and pick a real number r such that $0 < r < C_0 \operatorname{diam}_\rho(X)$. In this setting, suppose that $\{x_j\}_{j\in\mathbb{N}}$ is an at most countable family of points in X, which is $(r/C_0, \rho)$-disperse and such that this family is maximal among all collections of points in X with this property (maximality is understood with respect to the partial order induced by inclusion on the collection of subsets of X). That such a family always exists is a consequence of Zorn's lemma (note that, here, (4.49) is also used in order to ensure that such a family is at most countable). If we then take $B_\rho(x_j, r_j) := B_\rho(x_j, r)$ for each $j \in I$, then properties (4.157) and (4.158) are readily verified with the help of (4.46). ■

Another situation where the hypotheses of Theorem 4.18 are verified is described next. Note that, in contrast to the case presented in the last part in Comment 4.19, this time we do not assume that the quasimetric space in question is geometrically doubling.

Comment 4.20. Suppose that (X, ρ) is a quasimetric space and that C_0, C_1 are two fixed, finite real numbers with the property that $C_1 \geq C_\rho$ and $C_0 \geq C_\rho C_1$. Pick an arbitrary point $x_o \in X$ along with a real number r such that $0 < r < C_0 \operatorname{diam}_\rho(X)$; then define

$$E_1 := B_\rho(x_o, r), \quad \widetilde{E}_1 := B_\rho(x_o, C_1 r), \quad \widehat{E}_1 := B_\rho(x_o, C_0 r), \tag{4.160}$$

and, for each $j \in \mathbb{N}$, $j \geq 2$,

$$\begin{aligned} E_j &:= B_\rho(x_o, 2^j r) \setminus B_\rho(x_o, 2^{j-1} r), \\ \widetilde{E}_j &:= B_\rho(x_o, 2^j C_1 r) \setminus B_\rho(x_o, 2^{j-1} C_1^{-1} r), \\ \widehat{E}_j &:= B_\rho(x_o, 2^j C_0 r) \setminus B_\rho(x_o, 2^{j-1} C_0^{-1} r). \end{aligned} \tag{4.161}$$

Finally, consider $I := \{j \in \mathbb{N} : E_j \neq \emptyset\}$. Then $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$, $\{\widehat{E}_j\}_{j\in I}$ is a triplet of families satisfying the conditions formulated in the statement of Theorem 4.18 (with the parameters r_j in the statement of this theorem given by $r_j := 2^{j-1} r$, $j \in \mathbb{N}$). ■

Yet, perhaps the most basic setting in which families of sets $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$, and $\{\widehat{E}_j\}_{j\in I}$ satisfying the conditions hypothesized in Theorem 4.18 arise in a natural fashion is in relation to the Whitney decomposition of an open subset of a geometrically doubling quasimetric space (for more details see Comment 4.22 below).

A statement of the Whitney decomposition theorem, which extends work in [34, Theorem 3.1, p. 71] and [35, Theorem 3.2, p. 623] done in the context of bounded open sets in spaces of homogeneous type, is recorded next. Note that we only assume that (X, ρ) is a geometrically doubling quasimetric space and, perhaps most importantly, our open set $\mathcal{O}$ is not assumed to be bounded. From the point of view of the strategy of the proof, our approach is entirely self-contained and, as opposed to [35], makes no use of Vitali's covering lemma. This is relevant since the demand of the boundedness of the open sets for which a Whitney-type decomposition is shown to exist in [34, Theorem 3.1, p. 71] and [35, Theorem 3.2, p. 623] is an artifact of the use of Vitali's covering lemma (which applies to families of balls of bounded radii). In this regard, our proof is more akin to that in the classical setting of Euclidean spaces from [113, Theorem 1.1, p. 167].

Theorem 4.21 (Whitney's decomposition). *Let (X, ρ) be a geometrically doubling quasimetric space. Then for each $\lambda \in (1, +\infty)$ there exist $\Lambda \in (\lambda, +\infty)$ and $M \in \mathbb{N}$, both depending only on C_ρ, λ and the geometrically doubling constant of (X, ρ) and which have the following significance.*

For each open, nonempty, proper subset $\mathcal{O}$ of the topological space (X, τ_ρ) there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ in $\mathcal{O}$ along with a family of real numbers $r_j > 0$, $j \in \mathbb{N}$, for which the following properties are valid:

(1) $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$.

(2) $\sum_{j\in\mathbb{N}} \mathbf{1}_{B_\rho(x_j,\lambda r_j)} \leq M$ *on $\mathcal{O}$. In fact, there exists $\varepsilon \in (0, 1)$, which depends only on C_ρ, λ and the geometrically doubling constant of (X, ρ), with the property that for any $x_o \in \mathcal{O}$ one has*

$$\#\Big\{j \in \mathbb{N} :\ B_\rho\big(x_o, \varepsilon\, \mathrm{dist}_\rho(x_o, X \setminus \mathcal{O})\big) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset\Big\} \leq M. \qquad (4.162)$$

(3) $B_\rho(x_j, \lambda r_j) \subseteq \mathcal{O}$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[X \setminus \mathcal{O}\big] \neq \emptyset$ *for every* $j \in \mathbb{N}$.

(4) $r_i \approx r_j$ *uniformly for* $i, j \in \mathbb{N}$ *such that* $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$.

Proof. Set $F := X \setminus \mathcal{O}$, so that F is a nonempty, proper, closed subset of (X, τ_ρ). In a first stage, we will decompose $\mathcal{O}$ into a family of mutually disjoint dyadic layers $(L_n)_{n\in\mathbb{Z}}$ defined as

$$L_n := \{x \in \mathcal{O} :\ 2^{-n-1} \leq \mathrm{dist}_\rho(x, F) < 2^{-n}\}, \quad \forall\, n \in \mathbb{Z}. \qquad (4.163)$$

Clearly, $L_n \cap L_m = \emptyset$ for any two distinct integers n, m, and we claim that

$$\mathcal{O} = \bigcup_{n\in\mathbb{Z}} L_n. \qquad (4.164)$$

Indeed, the right-to-left inclusion is a direct consequence of (4.163). To justify the left-to-right inclusion, pick an arbitrary point $x \in \mathcal{O}$. Since $\mathcal{O}$ is open in (X, τ_ρ), it follows that there exists $r > 0$ with the property that $B_\rho(x, r) \subseteq \mathcal{O}$. In particular, we have $\mathrm{dist}_\rho(x, F) \geq r > 0$. Since $F \neq \emptyset$, this further forces

$$\mathrm{dist}_\rho(x, F) \in (0, +\infty) = \bigcup_{n \in \mathbb{Z}} [2^{-n-1}, 2^{-n}), \tag{4.165}$$

from which the desired conclusion readily follows. This completes the proof of (4.164).

Moving on, assume next that a parameter $\lambda \in (1, +\infty)$ has been given, and fix some number ε that satisfies

$$0 < \varepsilon < 2^{-1}(\lambda C_\rho^3)^{-1}. \tag{4.166}$$

For each $n \in \mathbb{Z}$ use Zorn's lemma to construct a family of points

$$\{x_j^n\}_{j \in I_n} \subseteq L_n, \tag{4.167}$$

where I_n is a set of indexes such that

$$\rho(x_i^n, x_j^n) > \varepsilon C_\rho 2^{-n-1}, \quad \forall\, i, j \in I_n \text{ with } i \neq j, \tag{4.168}$$

which is maximal, with respect to the partial order induced by the inclusion of subsets of L_n, with this property. Observe that since the topological space (X, τ_ρ) is separable (cf. (4.49)), we may assume that I_n is at most countable. We then claim that for each $n \in \mathbb{Z}$ we have

$$\big\{B_\rho(x_j^n, \varepsilon 2^{-n-1})\big\}_{j \in I_n} \text{ are mutually disjoint}, \tag{4.169}$$

$$L_n \subseteq \textstyle\bigcup_{j \in I_n} B_\rho(x_j^n, \varepsilon C_\rho 2^{-n}). \tag{4.170}$$

To justify these properties, note that if $n \in \mathbb{Z}$ is such that one can find two indexes $i, j \in I_n$ with the property that there exists $x \in B_\rho(x_i^n, \varepsilon 2^{-n-1}) \cap B_\rho(x_j^n, \varepsilon 2^{-n-1})$, then $\rho(x_i^n, x) < \varepsilon 2^{-n-1}$ and $\rho(x_j^n, x) < \varepsilon 2^{-n-1}$, which further imply

$$\rho(x_i^n, x_j^n) \leq C_\rho \max\{\rho(x_i^n, x), \rho(x_j^n, x)\} < \varepsilon C_\rho 2^{-n-1}, \tag{4.171}$$

in contradiction with (4.168). This proves (4.169). As far as (4.170) is concerned, from the maximality of the family $\{x_j^n\}_{j \in I_n}$ in the sense described above it follows that for each $x \in L_n$ there exists $j \in I_n$ with $\rho(x, x_j^n) \leq \varepsilon C_\rho 2^{-n-1}$. Hence $x \in B_\rho(x_j^n, \varepsilon C_\rho 2^{-n})$, proving (4.170).

To proceed, we introduce

$$\widehat{L}_n := \{x \in X : \operatorname{dist}_\rho(x, L_n) < \varepsilon\lambda C_\rho^2 2^{-n}\}, \qquad \forall\, n \in \mathbb{Z}, \tag{4.172}$$

then note that, by (4.167) and (4.172), we have

$$\bigcup_{j\in I_n} B_\rho(x_j^n, \varepsilon\lambda C_\rho^2 2^{-n}) \subseteq \widehat{L}_n, \qquad \forall\, n \in \mathbb{Z}. \tag{4.173}$$

The claim we make at this stage is that

$$\widehat{L}_n \subseteq \{x \in X : C_\rho^{-1} 2^{-n-1} \le \operatorname{dist}_\rho(x, F) \le C_\rho 2^{-n}\}, \qquad \forall\, n \in \mathbb{Z}. \tag{4.174}$$

To prove this claim, fix $n \in \mathbb{Z}$, pick an arbitrary point $x_o \in \widehat{L}_n$, and note that this entails $\operatorname{dist}_\rho(x_o, L_n) < \varepsilon\lambda C_\rho^2 2^{-n}$. From this and (4.163) it follows that there exist $x \in L_n$ and $z \in F$ satisfying

$$2^{-n-1} \le \rho(x, z) < 2^{-n} \quad \text{and} \quad \rho(x, x_o) < \varepsilon\lambda C_\rho^2 2^{-n}. \tag{4.175}$$

Thanks to (4.166), on the one hand we then have

$$\begin{aligned} \operatorname{dist}_\rho(x_o, F) &\le \rho(x_o, z) \le C_\rho \max\{\rho(x_o, x), \rho(x, z)\} \\ &\le C_\rho \max\{\varepsilon\lambda C_\rho^2 2^{-n}, 2^{-n}\} = C_\rho 2^{-n}, \end{aligned} \tag{4.176}$$

which suits our purposes. On the other hand, for every $w \in F$ we may write

$$\begin{aligned} 2^{-n-1} &\le \operatorname{dist}_\rho(x, F) \le \rho(x, w) \le C_\rho \max\{\rho(x, x_o), \rho(x_o, w)\} \\ &\le \max\{\varepsilon\lambda C_\rho^3 2^{-n}, C_\rho \rho(x_o, w)\}. \end{aligned} \tag{4.177}$$

In turn, given that $\varepsilon\lambda C_\rho^3 2^{-n} < 2^{-n-1}$, this implies $\rho(x_o, w) \ge C_\rho^{-1} 2^{-n-1}$ for all $w \in F$. Thus, ultimately,

$$\operatorname{dist}_\rho(x_o, F) \ge C_\rho^{-1} 2^{-n-1}, \tag{4.178}$$

and (4.174) follows from (4.176) and (4.178).

Let us now consider the family of intervals $(J_n)_{n\in\mathbb{Z}}$, where

$$J_n := [C_\rho^{-1} 2^{-n-1}, C_\rho 2^{-n}], \qquad \forall\, n \in \mathbb{Z}, \tag{4.179}$$

and note that

$$n, m \in \mathbb{Z} \ \text{ and } \ J_n \cap J_m \neq \emptyset \implies |n - m| \le 1 + 2\log_2 C_\rho. \tag{4.180}$$

As a consequence,

$$\begin{array}{c}\text{the largest number of intervals in the family } \{J_n : n \in \mathbb{Z}\} \\ \text{that have a nonempty intersection is } \leq 4(1 + \log_2 C_\rho).\end{array} \tag{4.181}$$

Going further, for every $x \in \mathcal{O}$ define

$$N(x) := \big\{n \in \mathbb{Z} : C_\rho^{-1} 2^{-n-1} \leq \operatorname{dist}_\rho(x, F) \leq C_\rho 2^{-n}\big\} \quad \text{and} \quad n(x) := \inf N(x). \tag{4.182}$$

Thus,

$$N(x) = \{n \in \mathbb{Z} : \operatorname{dist}_\rho(x, F) \in J_n\}, \qquad \forall\, x \in \mathcal{O}, \tag{4.183}$$

which, when used in concert with (4.181), gives the following estimate for the cardinality of $N(x)$:

$$\#\big(N(x)\big) \leq 4(1 + \log_2 C_\rho), \qquad \forall\, x \in \mathcal{O}. \tag{4.184}$$

Together, (4.180), (4.182), and (4.184) imply that for any $x \in \mathcal{O}$ we have

$$0 \leq n - n(x) \leq 1 + 2\log_2 C_\rho \text{ for any } n \in N(x). \tag{4.185}$$

Suppose next that an arbitrary point $x_o \in \mathcal{O}$ has been fixed. We then claim that

$$\begin{array}{c}\text{whenever } n \in \mathbb{Z} \text{ and } j \in I_n \text{ are such that} \\ B_\rho(x_o, \varepsilon\lambda C_\rho 2^{-n}) \cap B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \neq \emptyset, \\ \text{then } \; B_\rho(x_j^n, \varepsilon 2^{-n-1}) \subseteq B_\rho(x_o, \varepsilon\lambda C_\rho^3 2^{-n(x_o)}).\end{array} \tag{4.186}$$

To prove this claim, assume that $n \in \mathbb{Z}$ has the property that there exists some $j \in I_n$ for which one can find $y \in X$ such that $y \in B_\rho(x_o, \varepsilon\lambda C_\rho 2^{-n})$ and $y \in B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n})$. Then

$$\rho(x_o, x_j^n) \leq C_\rho \max\{\rho(x_o, y), \rho(y, x_j^n)\} < \varepsilon\lambda C_\rho^2 2^{-n}, \tag{4.187}$$

which permits us to conclude that

$$x_o \in B_\rho(x_j^n, \varepsilon\lambda C_\rho^2 2^{-n}). \tag{4.188}$$

By virtue of (4.173), (4.174), (4.182), and (4.188), we may, in a first stage, deduce

$$n \in N(x_o). \tag{4.189}$$

In a second stage, we note that if $z \in B_\rho(x_j^n, \varepsilon 2^{-n-1})$, then $\rho(z, x_j^n) < \varepsilon 2^{-n-1}$ and, hence,

$$\begin{aligned}\rho(x_o, z) &\leq C_\rho \max\{\rho(x_o, x_j^n), \rho(x_j^n, z)\} \leq C_\rho \max\{\varepsilon\lambda C_\rho^2 2^{-n}, \varepsilon 2^{-n-1}\} \\ &= \varepsilon\lambda C_\rho^3 2^{-n} \leq \varepsilon\lambda C_\rho^3 2^{-n(x_o)},\end{aligned} \tag{4.190}$$

where the last inequality is a consequence of (4.189) and (4.182). This completes the proof of the claim made in (4.186). Let us augment this result by observing that, as seen with the help of (4.189) and (4.185), the ratio of the radii of the two balls in the third line of (4.186) satisfies

$$\frac{\varepsilon 2^{-n-1}}{\varepsilon\lambda C_\rho^3 2^{-n(x_o)}} = 2^{n(x_o)-n-1}(\lambda C_\rho^3)^{-1} \in \big[(4\lambda C_\rho^5)^{-1}\,,\, 2^{-1}(\lambda C_\rho^3)^{-1}\big]. \tag{4.191}$$

At this stage, a combination of (4.169), (4.184), (4.189), (4.191), and (4.44) shows that there exists a constant $M \in \mathbb{N}$, depending only on C_ρ, λ and the geometrically doubling constant of (X, ρ), which has the property that for every $x_o \in \mathcal{O}$ we have

$$\#\Big\{(n, j) : n \in \mathbb{Z},\ \ j \in I_n \text{ and } B_\rho(x_o, \varepsilon\lambda C_\rho 2^{-n}) \cap B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \neq \emptyset\Big\} \leq M. \tag{4.192}$$

Hence, in particular,

$$\sum_{n\in\mathbb{Z} \text{ and } j\in I_n} \mathbf{1}_{B_\rho(x_j^n,\, \varepsilon\lambda C_\rho 2^{-n})} \leq M \quad \text{on} \quad \mathcal{O}. \tag{4.193}$$

Furthermore, from (4.173) and (4.174) we may also deduce that

$$B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \subseteq \mathcal{O} \quad \text{ whenever } n \in \mathbb{Z} \text{ and } j \in I_n. \tag{4.194}$$

Also, (4.164), (4.170), and (4.194) entail

$$\mathcal{O} = \bigcup_{n\in\mathbb{Z} \text{ and } j\in I_n} B_\rho(x_j^n, \varepsilon C_\rho 2^{-n}). \tag{4.195}$$

Finally, from (4.167) and (4.163) we conclude that

$$2^{-n-1} \leq \operatorname{dist}_\rho(x_j^n, X \setminus \mathcal{O}) < 2^{-n}, \qquad \forall\, n \in \mathbb{Z} \ \text{ and } \ \forall\, j \in I_n, \tag{4.196}$$

which further implies the existence of a number $\Lambda \in (\lambda, +\infty)$, depending only on C_ρ, λ and the geometrically doubling constant of (X, ρ), with the property that

$$B_\rho(x_j^n, \varepsilon\Lambda C_\rho 2^{-n}) \cap \big[X \setminus \mathcal{O}\big] \neq \emptyset, \qquad \forall\, n \in \mathbb{Z} \ \text{ and } \ \forall\, j \in I_n. \tag{4.197}$$

Thus, properties (1)–(3) in the statement of the theorem are going to be verified if we take $\{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}}$ to be a relabeling of the countable family $\big\{B_\rho(x_j^n, \varepsilon C_\rho 2^{-n})\big\}_{n\in\mathbb{Z},\, j\in I_n}$. Property (4) is also implicit in the preceding construction. This completes the proof of Theorem 4.21. □

Comment 4.22. In the setting of Theorem 4.21, if for some fixed numbers $\lambda, \lambda' > 1$ with the property that $C_\rho < \lambda'$ and $\lambda' C_\rho < \lambda$ we take $E_j := B_\rho(x_j, r_j)$, $\widetilde{E}_j := B_\rho(x_j, \lambda' r_j)$ and $\widehat{E}_j := B_\rho(x_j, \lambda r_j)$ for each $j \in \mathbb{N}$, then conditions (a)–(d) in Theorem 4.18 are valid for the families $\{E_j\}_{j\in\mathbb{N}}$, $\{\widetilde{E}_j\}_{j\in\mathbb{N}}$, $\{\widehat{E}_j\}_{j\in\mathbb{N}}$ (with the radii r_j playing the role of the parameters r_j from the statement of Theorem 4.18). ■

All ingredients are now in place for presenting the proof of Theorem 4.11.

Proof of Theorem 4.11. Suppose $(X, \mathbf{q})$ is a geometrically doubling quasimetric space, and fix an arbitrary nonempty, closed subset E of $(X, \tau_{\mathbf{q}})$. Also, suppose that $\rho \in \mathbf{q}$ and $\beta \in \mathbb{R}$ are such that $0 < \beta \le [\log_2 C_\rho]^{-1}$. If $E = X$, then we take $\mathscr{E}$ to be the identity operator, so we assume in what follows that $E \ne X$. Pick a constant $\lambda > C_\rho^2$, and consider the Whitney decomposition $X \setminus E = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$ as in Theorem 4.21. Next, select $\lambda' \in (C_\rho, \lambda/C_\rho)$ and define the families $\{\widetilde{E}_j\}_{j\in\mathbb{N}}$, $\{\widehat{E}_j\}_{j\in\mathbb{N}}$ as in Comment 4.22, corresponding to this choice of constants. Then, as already noted, the hypotheses of Theorem 4.18 are satisfied, and we consider a partition of unity $\{\varphi_j\}_{j\in\mathbb{N}}$ satisfying the properties listed in the conclusion of Theorem 4.18. Finally, for each $j \in \mathbb{N}$ choose a point $p_j \in E$ with the property that

$$\frac{1}{2}\,\mathrm{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big) \le \mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big) \le \mathrm{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big). \tag{4.198}$$

Hence, since by (3) in Theorem 4.21

$$\mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big) \approx r_j, \qquad \text{uniformly in } j \in \mathbb{N}, \tag{4.199}$$

it follows from this and (4.198) that

$$\mathrm{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big) \approx r_j \qquad \text{uniformly in } j \in \mathbb{N}. \tag{4.200}$$

Given an arbitrary continuous function $f : E \to \mathbb{R}$, we then proceed to define

$$(\mathscr{E} f)(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \sum\limits_{j\in\mathbb{N}} f(p_j)\varphi_j(x) & \text{if } x \in X \setminus E, \end{cases} \qquad \forall\, x \in X, \tag{4.201}$$

and note that, in light of (4.133) and (2) in the conclusion of Theorem 4.21, we have that $\mathscr{E} f : X \to \mathbb{R}$ is a well-defined function. Also, (4.51) is clear from the preceding definition.

We next propose to show that the operator

$$\mathscr{E} : \dot{\mathscr{C}}^\beta(E, \rho) \longrightarrow \dot{\mathscr{C}}^\beta(X, \rho) \quad \text{is well defined, linear, and bounded.} \tag{4.202}$$

Hence, the goal is to prove that there exists a finite constant $C \geq 0$ with the property that for any $f \in \dot{\mathscr{C}}^\beta(E,\rho)$ there holds

$$\big|(\mathscr{E} f)(x) - (\mathscr{E} f)(y)\big| \leq C \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\, \rho(x,y)^\beta, \qquad \forall\, x,y \in X. \tag{4.203}$$

Obviously, the estimate in (4.203) holds if $C \geq 1$ whenever $x,y \in E$. Consider next the case when $x \in X \setminus E$ and $y \in E$. As a preliminary matter, we claim that

$$j \in \mathbb{N} \;\text{ and }\; x \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(x,p_j) \approx r_j, \tag{4.204}$$

with the proportionality constant depending only on ρ. To justify the claim in (4.204), note that if $x \in B_\rho(x_j, \lambda r_j)$ for some $j \in \mathbb{N}$, then

$$\rho(x,z) \leq C_\rho \max\{\rho(x,x_j), \rho(x_j,z)\} < \lambda C_\rho\, r_j, \qquad \forall\, z \in B_\rho(x_j,\lambda r_j); \tag{4.205}$$

hence, further, for every $z \in B_\rho(x_j,\lambda r_j)$,

$$\rho(x,p_j) \leq C_\rho \max\{\rho(x,z), \rho(z,p_j)\} < C_\rho \max\{\lambda C_\rho\, r_j\,,\, \rho(z,p_j)\}. \tag{4.206}$$

Taking the infimum over all $z \in B_\rho(x_j,\lambda r_j)$ and keeping in mind (4.200) we therefore arrive at the conclusion that

$$\begin{aligned}\rho(x,p_j) &\leq C_\rho \max\Big\{\lambda C_\rho r_j\,,\, \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda r_j)\big)\Big\}\\ &\leq C_\rho \max\Big\{\lambda C_\rho r_j\,,\, \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big)\Big\}\\ &\leq C r_j.\end{aligned} \tag{4.207}$$

In summary, this analysis shows that there exists $C = C(\rho) \in (0,+\infty)$ for which

$$j \in \mathbb{N} \;\text{ and }\; x \in B_\rho(x_j, \lambda r_j) \Longrightarrow \rho(x,p_j) \leq C r_j, \tag{4.208}$$

which is a slightly stronger version than what is really needed in (4.204) (however, this will be useful later on). In the opposite direction, if $x \in B_\rho(x_j,\lambda' r_j)$ for some $j \in \mathbb{N}$, then

$$\rho(x,p_j) \geq \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big) \geq c\, r_j, \tag{4.209}$$

by appealing once more to (4.200). Since, as before, $c = c(\rho) \in (0,+\infty)$, this concludes the proof of (4.204). As a consequence of (4.204) and (4.199) we then obtain

$$\rho(x,p_j) \approx \mathrm{dist}_\rho\big(E, B_\rho(x_j,\lambda' r_j)\big) \quad \text{uniformly in } j \in \mathbb{N} \text{ and } x \in B_\rho(x_j,\lambda' r_j). \tag{4.210}$$

Going further, whenever $y \in E$ and $x \in B_\rho(x_j, \lambda' r_j)$ for some $j \in \mathbb{N}$, (4.210) allows us to estimate

$$\rho(y, p_j) \leq C_\rho \max\{\rho(y,x), \rho(x,p_j)\}$$
$$\leq C \max\Big\{\rho(y,x)\,,\, \mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big)\Big\} \leq C\rho(y,x). \tag{4.211}$$

Hence, for some finite $C = C(\rho) > 0$, independent of x, y, j, we have

$$y \in E,\;\; j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(y, p_j) \leq C\rho(x,y). \tag{4.212}$$

Based on (4.212), the fact that $f \in \dot{\mathscr{C}}^\beta(E,\rho)$, and the properties of the functions $\{\varphi_j\}_{j\in\mathbb{N}}$, whenever $x \in X \setminus E$ and $y \in E$, we may therefore write

$$|(\mathscr{E}f)(y) - (\mathscr{E}f)(x)| = \Big|f(y) - \sum_{j\in\mathbb{N}} f(p_j)\varphi_j(x)\Big| = \Big|\sum_{j\in\mathbb{N}} (f(y) - f(p_j))\varphi_j(x)\Big|$$
$$\leq \sum_{j\in\mathbb{N}} |f(y) - f(p_j)|\,\varphi_j(x) = \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x\in B_\rho(x_j,\lambda' r_j)}} |f(y) - f(p_j)|\,\varphi_j(x)$$
$$\leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x\in B_\rho(x_j,\lambda' r_j)}} \rho(y, p_j)^\beta \varphi_j(x)$$
$$\leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta \tag{4.213}$$

since $0 \leq \varphi_j \leq 1$ for every $j \in \mathbb{N}$ and since

$$\text{the cardinality of } \{j \in \mathbb{N} : x \in B_\rho(x_j, \lambda' r_j)\} \text{ is } \leq M. \tag{4.214}$$

Of course, estimate (4.213) suits our purposes. The situation when $y \in X \setminus E$ and $x \in E$ is handled similarly, so it remains to treat the case when $x, y \in X \setminus E$, which we now consider. We will investigate two separate subcases.

`Subcase I:` *Assume that the points $x, y \in X \setminus E$ are such that*

$$\rho(x,y) < \varepsilon\, \mathrm{dist}_\rho(x, E), \quad \textit{where} \quad 0 < \varepsilon < \frac{\lambda}{C_\rho(\Lambda C_\rho^2 + \lambda)}. \tag{4.215}$$

The relevance of the choice made for ε will become more apparent later. For now, we wish to mention that such a choice forces $\varepsilon \in (0, 1/C_\rho)$. To get started in earnest, we make the claim that in the current scenario we have

$$\mathrm{dist}_\rho(x, E) \leq \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big)\,\mathrm{dist}_\rho(y, E). \tag{4.216}$$

Indeed, for every $z \in E$ we may write

$$\mathrm{dist}_\rho(x, E) \le \rho(x,z) \le C_\rho\big(\rho(x,y) + \rho(y,z)\big) \le C_\rho\big(\varepsilon\,\mathrm{dist}_\rho(x,E) + \rho(y,z)\big), \tag{4.217}$$

hence $(1 - \varepsilon C_\rho)\,\mathrm{dist}_\rho(x,E) \le C_\rho\,\rho(y,z)$. Taking the infimum over all $z \in E$, (4.216) follows. Moving on, observe that

$$(\mathscr{E} f)(x) - (\mathscr{E} f)(y) = \sum_{j\in\mathbb{N}} (f(p_j) - f(z))(\varphi_j(x) - \varphi_j(y)), \quad \forall\, z \in E. \tag{4.218}$$

Choose now $z \in E$ such that

$$\frac{1}{2}\,\rho(x,z) \le \mathrm{dist}_\rho(x,E) \le \rho(x,z), \tag{4.219}$$

and note that this forces $\rho(x,z) \approx \mathrm{dist}_\rho(x,E) \le \rho(x,p_j)$. In concert with (4.208), this implies

$$j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda r_j) \Longrightarrow \rho(p_j, z) \le C_\rho \max\{\rho(p_j,x), \rho(x,z)\} \le C r_j. \tag{4.220}$$

Having established (4.220), we next write formula (4.218) for $z \in E$ as in (4.219) and make use of fact that $f \in \dot{\mathscr{C}}^\beta(E,\rho)$, along with the properties of $\{\varphi_j\}_{j\in\mathbb{N}}$, to estimate

$$\begin{aligned} |(\mathscr{E} f)(x) - (\mathscr{E} f)(y)| &\le \sum_{j\in\mathbb{N}} |f(p_j) - f(z)||\varphi_j(x) - \varphi_j(y)| \\ &\le C\,\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta \sum_{j\in\mathbb{N}} \rho(p_j,z)^\beta \|\varphi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \big[\mathbf{1}_{B_\rho(x_j,\lambda' r_j)}(x) + \mathbf{1}_{B_\rho(x_j,\lambda' r_j)}(y)\big] \\ &\le C\,\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta \big(A_x + A_y\big), \end{aligned} \tag{4.221}$$

where

$$A_x := \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x\in B_\rho(x_j,\lambda' r_j)}} \rho(p_j,z)^\beta\, r_j^{-\beta} \quad \text{and} \quad A_y := \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ y\in B_\rho(x_j,\lambda' r_j)}} \rho(p_j,z)^\beta\, r_j^{-\beta}. \tag{4.222}$$

Now, (4.214) and (4.220) give that $A_x \le C$ for some finite constant $C = C(\rho) \ge 0$. To derive a similar estimate for A_y, assume that

$$j \in \mathbb{N} \ \text{ is such that } \ y \in B_\rho(x_j, \lambda' r_j). \tag{4.223}$$

Then, by (4.216), (4.223), and the fact that $B_\rho(x_j, \Lambda r_j) \cap E \neq \emptyset$, we have

$$\mathrm{dist}_\rho(x,E) \le \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big)\,\mathrm{dist}_\rho(y,E) \le \Lambda\Big(\frac{C_\rho^2}{1 - \varepsilon C_\rho}\Big) r_j. \tag{4.224}$$

In turn, (4.224) permits us to deduce that

$$\begin{aligned}\rho(x,x_j) &\leq C_\rho \max\{\rho(x,y),\rho(y,x_j)\} \leq C_\rho \max\Big\{\varepsilon\,\mathrm{dist}_\rho(x,E)\,,\,\lambda' r_j\Big\}\\ &\leq C_\rho\cdot\max\Big\{\varepsilon\,\Lambda\Big(\tfrac{C_\rho^2}{1-\varepsilon C_\rho}\Big)\,,\,\lambda'\Big\}\cdot r_j\\ &< \lambda r_j,\end{aligned}\tag{4.225}$$

where the last inequality is a consequence of the fact that $\lambda' C_\rho < \lambda$ and the way ε has been chosen in (4.215). Estimate (4.225) shows that

$$\text{if } j \text{ is as in (4.223), then } x \in B_\rho(x_j,\lambda r_j).\tag{4.226}$$

With (4.226) in hand, a reference to (4.220) then gives

$$\text{if } j \text{ is as in (4.223), then } \rho(z,p_j) \leq C r_j \text{ whenever } z \in E \text{ is as in (4.219)}\tag{4.227}$$

for some finite $C = C(\rho) \geq 0$. With (4.227) having been proved, the estimate $A_y \leq C$ for some $C = C(\rho) < +\infty$ follows as in the case of A_x, already treated. All together, this proves that $A_x + A_y \leq C = C(\rho) < +\infty$, which, in combination with (4.221), shows that $|(\mathscr{E} f)(x) - (\mathscr{E} f)(y)| \leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta$ under the hypotheses specified in Subcase I. This bound is of the right order, and this completes the treatment of Subcase I.

`Subcase II`: *With the parameter* $\varepsilon > 0$ *as in Subcase I, assume that* $x, y \in X\setminus E$ *are such that* $\rho(x,y) \geq \varepsilon\,\mathrm{dist}_\rho(x,E)$. Consider a point $z \in E$ as in (4.219) and note that, in the current situation, this forces $\rho(x,z) \leq 2\,\mathrm{dist}_\rho(x,E) \leq 2\varepsilon^{-1}\rho(x,y)$. Hence, we also have $\rho(y,z) \leq C_\rho \max\{\rho(x,y),\rho(x,z)\} \leq C\rho(x,y)$. Consequently,

$$\begin{aligned}\big|(\mathscr{E} f)(x) - (\mathscr{E} f)(y)\big| &\leq \big|(\mathscr{E} f)(x) - (\mathscr{E} f)(z)\big| + \big|(\mathscr{E} f)(z) - (\mathscr{E} f)(y)\big|\\ &\leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,z)^\beta + C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(z,y)^\beta\\ &\leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta,\end{aligned}\tag{4.228}$$

by what we have established in the first part of the proof (i.e., using (4.213) twice, once for $x \in X\setminus E$ and $z \in E$ and a second time for $y \in X\setminus E$ and $z \in E$).

In summary, the analysis so far proves that there exists $C = C(\rho) \in (0,+\infty)$ with the property that for every $f \in \dot{\mathscr{C}}^\beta(E,\rho)$ we have

$$\big|(\mathscr{E} f)(x) - (\mathscr{E} f)(y)\big| \leq C\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)}\,\rho(x,y)^\beta, \qquad \forall\, x,y \in X.\tag{4.229}$$

This shows that the operator (4.50) is well defined, linear, and bounded.

At this stage, it remains to prove that the operator $\mathscr{E}$ defined in (4.201) has the property that

$$\mathscr{E} f \text{ is continuous on } X \text{ whenever } f : E \to \mathbb{R} \text{ is continuous.} \tag{4.230}$$

To this end, fix an arbitrary continuous function $f : E \to \mathbb{R}$ and note that, by design, $\mathscr{E} f$ is continuous on the open set $X \setminus E$ (since the sum in (4.201) is locally finite and the functions φ_j are continuous). It remains to show that $\mathscr{E} f$ is continuous at any point in E. Furthermore, since (as seen from Theorem 3.46) the topology $\tau_{\mathbf{q}}$ is metrizable, we may use the sequential characterization of continuity. Fix $z \in E$ and assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X that converges to z in the topology $\tau_{\mathbf{q}}$. Introduce $N_1 := \{n \in \mathbb{N} : x_n \in E\}$ and $N_2 := \{n \in \mathbb{N} : x_n \in X \setminus E\}$. Then, on the one hand,

$$\lim_{N_1 \ni n \to \infty} (\mathscr{E} f)(x_n) = \lim_{N_1 \ni n \to \infty} f(x_n) = f(z) \tag{4.231}$$

since f is continuous on E. On the other hand, for each $n \in N_2$, much as in (4.213), we may estimate

$$\big|(\mathscr{E} f)(x_n) - (\mathscr{E} f)(z)\big| \leq \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x_n \in B_\rho(x_j, \lambda' r_j)}} |f(z) - f(p_j)|. \tag{4.232}$$

Let us also note that the version of (4.212) in the notation currently employed reads

$$j \in \mathbb{N} \text{ and } x_n \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(z, p_j) \leq C\rho(x_n, z) \tag{4.233}$$

for some finite $C = C(\rho) > 0$, independent of $n \in N_2$. Fix an arbitrary $\varepsilon > 0$ and, based on the continuity of f at z, pick $\delta > 0$ with the property that

$$|f(z) - f(w)| < \varepsilon \quad \text{whenever} \quad w \in E \quad \text{is such that} \quad \rho(z, w) < \delta. \tag{4.234}$$

Since $\lim_{N_2 \ni n \to \infty} x_n = z$, it follows that there exists $m \in \mathbb{N}$ with the property that

$$\rho(x_n, z) < \delta/C \text{ for each } n \in N_2 \text{ with the property that } n \geq m, \tag{4.235}$$

where the constant C is as in (4.233). Thus,

$$\big|(\mathscr{E} f)(x_n) - (\mathscr{E} f)(z)\big| \leq M\varepsilon, \qquad \text{for every} \quad n \in N_2 \quad \text{with} \quad n \geq m, \tag{4.236}$$

by (4.232)–(4.234) and (4.214). Since $\varepsilon > 0$ was arbitrary, it follows from (4.231) and (4.236) that $\mathscr{E} f$ is continuous at z. This completes the justification of (4.230) and completes the proof of Theorem 4.11. □

We conclude this section by discussing a version of Theorem 4.17 for closed subsets of geometrically doubling quasimetric spaces that makes critical use of the tools developed in the first part of this section and is more in line with [113, Theorem 2, p. 171], dealing with the case of closed subsets of an Euclidean space.

Theorem 4.23. *Suppose that* $(X, \mathbf{q})$ *is a geometrically doubling quasimetric space and that* E *is a nonempty, proper, closed subset of* $(X, \tau_{\mathbf{q}})$. *In addition, suppose that* $\rho \in \mathbf{q}$ *and* $\beta \in \mathbb{R}$ *are such that* $0 < \beta \le [\log_2 C_\rho]^{-1}$. *Let*

$$X \setminus E = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j) \tag{4.237}$$

be the Whitney decomposition given by Theorem 4.21 of the open, nonempty, proper subset $X \setminus E$ *of* $(X, \tau_{\mathbf{q}})$, *and denote by* $(\varphi_j)_{j \in \mathbb{N}}$ *the Whitney partition of unity associated with the decomposition* (4.237) *of the open set* $X \setminus E$ *(in the manner described in Comment 4.22). Finally, pick an arbitrary number* $\gamma \in \mathbb{R}$ *and define*

$$\Delta_{E,\gamma}(x) := \sum_{j \in \mathbb{N}} r_j^\gamma \varphi_j(x), \qquad \forall\, x \in X \setminus E. \tag{4.238}$$

Then the function $\Delta_{E,\gamma} : X \setminus E \to [0, +\infty)$ *satisfies the following properties:*

(i) *There exist two constants* $c_0, c_1 \in (0, +\infty)$ *that depend only on* C_ρ *and* β, γ *such that*

$$c_0 \left[\mathrm{dist}_\rho(x, E)\right]^\gamma \le \Delta_{E,\gamma}(x) \le c_1 \left[\mathrm{dist}_\rho(x, E)\right]^\gamma, \qquad \forall\, x \in X \setminus E. \tag{4.239}$$

(ii) *One has* $\Delta_{E,\gamma} \in \dot{\mathscr{C}}^\beta_{loc}(X \setminus E, \rho)$ *in the quantitative sense that for every* $\varepsilon \in (0, C_\rho^{-1})$ *there exists* $C \in (0, +\infty)$, *depending only on* C_ρ, β, γ *and* ε, *such that*

$$\sup \left\{ \frac{|\Delta_{E,\gamma}(x) - \Delta_{E,\gamma}(y)|}{\rho(x, y)^\beta} \ :\ x, y \in B_\rho\Big(z, \varepsilon\, \mathrm{dist}_\rho(z, E)\Big),\ x \ne y \right\}$$
$$\le C \left[\mathrm{dist}_\rho(z, E)\right]^{\gamma - \beta}, \qquad \forall\, z \in X \setminus E. \tag{4.240}$$

Proof. Fix two constants λ, λ' as in Comment 4.22. To prove (i), observe that based on properties (2)–(4) from Theorem 4.21 we may write for each $x \in X \setminus E$ that

$$\Delta_{E,\gamma}(x) = \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda r_j)}} r_j^\gamma \varphi_j(x) \le \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda r_j)}} r_j^\gamma \le C \left[\mathrm{dist}_\rho(x, E)\right]^\gamma \tag{4.241}$$

for some finite constant $C > 0$ depending only on C_ρ and γ. In the opposite direction, for every $x \in X \setminus E$ there exists $j_o \in \mathbb{N}$ such that $x \in B_\rho(x_{j_o}, r_{j_o})$. Then $r_{j_o} \approx \mathrm{dist}_\rho(x, E)$ by Theorem 4.21 and $\varphi_{j_o}(x) \ge 1/C$ by the last condition in (4.133). Consequently, since all functions φ_j are nonnegative,

$$C^{-1} \left[\mathrm{dist}_\rho(x, E)\right]^\gamma \le r_{j_o}^\gamma / C \le r_{j_o}^\gamma \varphi_{j_o}(x) \le \sum_{j \in \mathbb{N}} r_j^\gamma \varphi_j(x) = \Delta_{E,\gamma}(x). \tag{4.242}$$

Now (4.239) follows from (4.241) and (4.242), completing the proof of (i). As regards (ii), if $\varepsilon \in (0, C_\rho^{-1})$ and $z \in X \setminus E$ are fixed, then (4.111), (4.118), and (4.117) give that

$$\begin{aligned} &\operatorname{dist}_\rho(x, E) \approx \operatorname{dist}_\rho(z, E) \approx \operatorname{dist}_\rho(y, E), \\ &\text{uniformly for } \; x, y \in B_\rho\big(z, \varepsilon \operatorname{dist}_\rho(z, E)\big), \end{aligned} \tag{4.243}$$

with the proportionality constants independent of z. Based on this, and on Theorems 4.18 and 4.21, for every $x, y \in B_\rho\big(z, \varepsilon \operatorname{dist}_\rho(z, E)\big)$ we may write

$$\begin{aligned} |\Delta_{E,\gamma}(x) - \Delta_{E,\gamma}(y)| &\leq \sum_{j\in\mathbb{N}} r_j^\gamma |\varphi_j(x) - \varphi_j(y)| \\ &\leq C\rho(x, y)^\beta \sum_{j\in\mathbb{N}} r_j^\gamma \cdot r_j^{-\beta}\big[\mathbf{1}_{B_\rho(x_j,\lambda r_j)}(x) + \mathbf{1}_{B_\rho(x_j,\lambda r_j)}(y)\big] \\ &\leq C\rho(x, y)^\beta \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x\in B_\rho(x_j,\lambda r_j)}} r_j^{\gamma-\beta} + C\rho(x, y)^\beta \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ y\in B_\rho(x_j,\lambda r_j)}} r_j^{\gamma-\beta} \\ &\leq C\rho(x, y)^\beta \big[\operatorname{dist}_\rho(z, E)\big]^{\gamma-\beta}, \end{aligned} \tag{4.244}$$

where $C \in (0, +\infty)$ depends only on C_ρ, β, and ε. This concludes the justification of (4.240) and completes the proof of the theorem. □

4.7 Smoothness Indexes of a Quasimetric Space

The goal here is to introduce some new, natural concepts of lower and upper smoothness indexes for a quasimetric space (along with some other related notions) and to highlight the basic role they play in describing the structural richness of Hölder spaces. We indicate how these indexes compare to one another and study their relationship with Assouad's convexity index introduced in [10], as well as with Rolewicz's modulus of concavity for locally bounded topological vector spaces from [104].

To get started, assume that X is a fixed set of cardinality ≥ 2. For each $\rho \in \mathfrak{Q}(X)$ we then define

$$c_\rho := \inf_{\substack{x,y\in X \\ x\neq y}} \left(\sup_{z\in X} \frac{\rho(x, y)}{\max\{\rho(x, z), \rho(z, y)\}} \right). \tag{4.245}$$

Recall the constant C_ρ introduced earlier in (4.2). It is then readily seen from definitions that

$$\forall\, \rho \in \mathfrak{Q}(X) \implies c_\rho \in [1, C_\rho], \tag{4.246}$$

$$\rho \text{ ultrametric on } X \implies c_\rho = 1, \tag{4.247}$$

$$\forall\, \rho \in \mathfrak{Q}(X), \;\; \forall\, \beta \in (0, +\infty) \implies c_{\rho^\beta} = (c_\rho)^\beta. \tag{4.248}$$

Let us point out that, in general, it may happen that $\rho \in \mathfrak{Q}(X)$ is such that $c_\rho < C_\rho$; see the discussion in Comment 4.29 below. To make the relationship between c_ρ and C_ρ more transparent, we introduce the following definition.

Definition 4.24. Let X be a given set of cardinality ≥ 2. For each $\rho \in \mathfrak{Q}(X)$ define

$$\rho_{\rm mid}(x,y) := \inf_{z\in X}\big(\max\{\rho(x,z),\rho(z,y)\}\big), \qquad \forall\, x,y \in X, \tag{4.249}$$

and refer to the function $\rho_{\rm mid} : X \times X \to [0,+\infty)$ defined as in (4.249) as the `midpoint version` of the quasidistance ρ.

The terminology introduced in this definition is suggested by formula (4.250) below.

Proposition 4.25. *Let X be a given set of cardinality ≥ 2, and assume that $\rho \in \mathfrak{Q}(X)$. Then the midpoint version of the quasidistance ρ satisfies*

$$\rho_{\rm mid}(x,y) = \inf\big\{r > 0 : \ B_\rho(x,r) \cap B_\rho(y,r) \neq \emptyset\big\}, \qquad \forall\, x,y \in X, \tag{4.250}$$

and

$$(\rho^\beta)_{\rm mid} = (\rho_{\rm mid})^\beta, \qquad \forall\, \beta \in (0,+\infty). \tag{4.251}$$

Also, $\rho_{\rm mid} \in \mathfrak{Q}(X)$ and $\rho_{\rm mid} \approx \rho$. More precisely, with c_ρ and C_ρ as in (4.245) and (4.2), respectively, there holds

$$c_\rho \leq \frac{\rho(x,y)}{\rho_{\rm mid}(x,y)} \leq C_\rho, \qquad \forall\, x,y \in X \text{ with } x \neq y, \tag{4.252}$$

and the constants involved are optimal in this context, in the sense that

$$c_\rho = \inf_{\substack{x,y\in X\\ x\neq y}} \left(\frac{\rho(x,y)}{\rho_{\rm mid}(x,y)}\right) \quad \text{and} \quad C_\rho = \sup_{\substack{x,y\in X\\ x\neq y}} \left(\frac{\rho(x,y)}{\rho_{\rm mid}(x,y)}\right). \tag{4.253}$$

Proof. All claims are straightforward consequences of definitions. □

Definition 4.26. The `upper smoothness index` of a given quasimetric space $(X,\mathbf{q})$ is defined as

$$\operatorname{Ind}(X,\mathbf{q}) := \inf\big\{[\log_2 c_\rho]^{-1} : \ \rho \in \mathbf{q}\big\} \in [0,+\infty], \tag{4.254}$$

where, for every $\rho \in \mathfrak{Q}(X)$, the constant c_ρ was introduced in (4.245). In addition, define the `lower smoothness index` of $(X,\mathbf{q})$ by setting

$$\operatorname{ind}(X,\mathbf{q}) := \sup\big\{[\log_2 C_\rho]^{-1} : \ \rho \in \mathbf{q}\big\} \in (0,+\infty], \tag{4.255}$$

where, for every $\rho \in \mathfrak{Q}(X)$, the constant C_ρ was introduced in (4.2).

Finally, if X is an arbitrary set of cardinality ≥ 2 and $\rho \in \mathfrak{Q}(X)$, then abbreviate $\mathrm{Ind}\,(X, \rho) := \mathrm{Ind}\,(X, [\rho])$ and $\mathrm{ind}\,(X, \rho) := \mathrm{ind}\,(X, [\rho])$.

Some elementary properties of the upper and lower smoothness indexes are recorded next.

Proposition 4.27. *For any set X (of cardinality ≥ 2)*

$$\rho \text{ ultrametric on } X \implies \mathrm{ind}\,(X, \rho) = +\infty, \tag{4.256}$$

$$\rho \text{ distance on } X \implies \mathrm{ind}\,(X, \rho) \geq 1, \tag{4.257}$$

$$X \text{ finite and } \rho \in \mathfrak{Q}(X) \implies \mathrm{ind}\,(X, \rho) = +\infty. \tag{4.258}$$

Furthermore, for each quasidistance $\rho \in \mathfrak{Q}(X)$ and any $\alpha \in (0, +\infty)$ one has

$$\mathrm{ind}\,(X, \rho^\alpha) = \tfrac{1}{\alpha}\,\mathrm{ind}\,(X, \rho) \ \text{ and } \ \mathrm{Ind}\,(X, \rho^\alpha) = \tfrac{1}{\alpha}\,\mathrm{Ind}\,(X, \rho), \tag{4.259}$$

and the following bounds hold:

$$[\log_2 C_\rho]^{-1} \leq \mathrm{ind}\,(X, \rho), \qquad \mathrm{Ind}\,(X, \rho) \leq [\log_2 c_\rho]^{-1}. \tag{4.260}$$

Moreover, given a quasimetric space structure $\mathbf{q}$ on X, one has $\mathrm{Ind}\,(X, \mathbf{q}) = +\infty$ if and only if for every $\rho \in \mathbf{q}$ there exist two sequences $\{x_j\}_{j\in\mathbb{N}}, \{y_j\}_{j\in\mathbb{N}} \subseteq X$ such that $x_j \neq y_j$ for each $j \in \mathbb{N}$ and

$$\sup_{z\in X} \frac{\rho(x_j, y_j)}{\max\{\rho(x_j, z), \rho(z, y_j)\}} \searrow 1 \quad \text{as} \quad j \to +\infty. \tag{4.261}$$

Consequently, given $\rho \in \mathfrak{Q}(X)$, one has

$$\mathrm{Ind}\,(X, \rho) = +\infty \ \text{ if } \ \exists\, x_*, y_* \in X \text{ such that } \inf_{\substack{x,y\in X\\ \text{not equal}}} \rho(x, y) = \rho(x_*, y_*) > 0. \tag{4.262}$$

In particular,

$$X \text{ finite and } \rho \in \mathfrak{Q}(X) \implies \mathrm{Ind}\,(X, \rho) = +\infty. \tag{4.263}$$

Proof. These are all more or less straightforward consequences of definitions (for (4.262), note that (4.261) is satisfied with $x_j := x_*$ and $y_j := y_*$ for each $j \in \mathbb{N}$). □

Proposition 4.28. *Let $N \in \mathbb{N}$ be fixed and assume that (X_i, ρ_i), $1 \leq i \leq N$, are quasimetric spaces. Define $X := X_1 \times \cdots \times X_N$ and consider $\rho : X \times X \to [0, +\infty)$ given by*

$$\rho(x, y) := \max_{1\leq i\leq N} \rho_i(x_i, y_i) \ \text{ for all } \ x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in X. \tag{4.264}$$

Then $\rho \in \mathfrak{Q}(X)$ *and*

$$\mathrm{ind}\,(X,\rho) = \min_{1\le i\le N} \mathrm{ind}\,(X_i,\rho_i). \tag{4.265}$$

Proof. The fact that $\rho \in \mathfrak{Q}(X)$ is clear from definitions. Let us also note that whenever $\rho \in \mathfrak{Q}(X)$ is related to $\rho_i \in \mathfrak{Q}(X_i)$, $1 \le i \le N$, as in (4.264), then

$$C_\rho = \max_{1\le i\le N} C_{\rho_i}. \tag{4.266}$$

To see why this is the case, set $C_* := \max_{1\le i\le N} C_{\rho_i}$ and note that for every $x, y, z \in X$ with $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N)$, and $z = (z_1,\dots,z_N)$ we have

$$\begin{aligned}
\rho(x,y) &= \max_{1\le i\le N} \rho_i(x_i,y_i) \le \max_{1\le i\le N} C_{\rho_i} \max\{\rho_i(x_i,z_i),\rho_i(y_i,z_i)\}\\
&\le C_* \max_{1\le i\le N} \max\{\rho_i(x_i,z_i),\rho_i(y_i,z_i)\}\\
&= C_* \max\Big\{\max_{1\le i\le N} \rho_i(x_i,z_i),\ \max_{1\le i\le N} \rho_i(y_i,z_i)\Big\}\\
&= C_* \max\big\{\rho(x,z),\rho(z,y)\big\}.
\end{aligned} \tag{4.267}$$

Thus, on the one hand, $C_\rho \le C_*$. On the other hand, if $i_o \in \{1,\dots,N\}$ is a fixed, arbitrary number and if $w = (w_1,\dots,w_N) \in X$ is a fixed point, then for every $x_{i_o}, y_{i_o}, z_{i_o} \in X_{i_o}$ we have

$$\begin{aligned}
\rho_{i_o}(x_{i_o},y_{i_o}) &= \rho\big((w_1,\dots,w_{i_o-1},x_{i_o},w_{i_o+1},\dots,w_N),(w_1,\dots,w_{i_o-1},y_{i_o},w_{i_o+1},\dots,w_N)\big)\\
&\le C_\rho \max\Big\{\rho\big((w_1,\dots,w_{i_o-1},x_{i_o},w_{i_o+1},\dots,w_N),(w_1,\dots,w_{i_o-1},z_{i_o},w_{i_o+1},\dots,w_N)\big),\\
&\qquad \rho\big((w_1,\dots,w_{i_o-1},z_{i_o},w_{i_o+1},\dots,w_N),(w_1,\dots,w_{i_o-1},y_{i_o},w_{i_o+1},\dots,w_N)\big)\Big\}\\
&= C_\rho \max\big\{\rho_{i_o}(x_{i_o},z_{i_o}),\rho_{i_o}(z_{i_o},y_{i_o})\big\},
\end{aligned} \tag{4.268}$$

which proves that $C_{\rho_{i_o}} \le C_\rho$. As a result, $C_* = \max_{1\le i\le N} C_{\rho_i} \le C_\rho$, and (4.266) follows. In turn, (4.266) entails

$$\big(\log_2 C_\rho\big)^{-1} = \big(\log_2 C_*\big)^{-1} = \Big(\max_{1\le i\le N}\big(\log_2 C_{\rho_i}\big)\Big)^{-1} = \min_{1\le i\le N}\big(\log_2 C_{\rho_i}\big)^{-1}. \tag{4.269}$$

After these preparations, we are ready to tackle (4.265). Specifically, for each number $i \in \{1,\dots,N\}$, consider a sequence $\tilde\rho_i^{(n)} \in \mathfrak{Q}(X_i)$, $n \in \mathbb{N}$, such that $\tilde\rho_i^{(n)} \approx \rho_i$, $n \in \mathbb{N}$, and

$$\big(\log_2 C_{\tilde\rho_i^{(n)}}\big)^{-1} \longrightarrow \mathrm{ind}\,(X_i,\rho_i) \qquad \text{as} \quad n \to \infty. \tag{4.270}$$

Fix $n \in \mathbb{N}$ and define

$$\tilde{\rho}^{(n)}(x, y) := \max_{1 \le i \le N} \tilde{\rho}_i^{(n)}(x_i, y_i) \tag{4.271}$$

for every $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in X$. Then, as noted before, we have $\tilde{\rho}^{(n)} \in \mathfrak{Q}(X)$, $\tilde{\rho}^{(n)} \approx \rho$ and, by the reasoning that lead to (4.269),

$$\big(\log_2 C_{\tilde{\rho}^{(n)}}\big)^{-1} = \min_{1 \le i \le N} \big(\log_2 C_{\tilde{\rho}_i^{(n)}}\big)^{-1}. \tag{4.272}$$

Hence, we may write, based on (4.255), (4.272), (4.270), and the continuity of the minimum function, that

$$\begin{aligned} \operatorname{ind}(X, \rho) &\ge \big(\log_2 C_{\tilde{\rho}^{(n)}}\big)^{-1} \\ &= \min_{1 \le i \le N} \big(\log_2 C_{\tilde{\rho}_i^{(n)}}\big)^{-1} \to \min_{1 \le i \le N} \operatorname{ind}(X_i, \rho_i) \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.273}$$

Thus, $\operatorname{ind}(X, \rho) \ge \min_{1 \le i \le N} \operatorname{ind}(X_i, \rho_i)$, and to complete the proof of the proposition there remains to establish the opposite inequality. To this end, pick a sequence $\rho^{(n)} \in \mathfrak{Q}(X)$, $n \in \mathbb{N}$, with the property that

$$\rho^{(n)} \approx \rho \ \text{ for each } \ n \in \mathbb{N}, \ \text{ and } \ \big(\log_2 C_{\rho^{(n)}}\big)^{-1} \longrightarrow \operatorname{ind}(X, \rho) \quad \text{as} \quad n \to \infty. \tag{4.274}$$

To proceed, fix an arbitrary point $w = (w_1, \dots, w_N) \in X$. Then, if for each $n \in \mathbb{N}$ and $i \in \{1, \dots, N\}$ we define

$$\rho_i^{(n)}(x_i, y_i) := \rho^{(n)}\big((w_1, \dots, w_{i-1}, x_i, w_{i+1}, \dots, w_N), (w_1, \dots, w_{i-1}, y_i, w_{i+1}, \dots, w_N)\big) \tag{4.275}$$

for every $x_i, y_i \in X_i$, then it follows from (4.275) and the fact that $\rho^{(n)} \approx \rho$ that $\rho_i^{(n)} \in \mathfrak{Q}(X_i)$ and $\rho_i^{(n)} \approx \rho_i$ for each $n \in \mathbb{N}$ and $i \in \{1, \dots, N\}$. Moreover, as a direct consequence of (4.275), we also have that

$$C_{\rho_i^{(n)}} \le C_{\rho^{(n)}}, \qquad \text{for each } n \in \mathbb{N} \text{ and each } i \in \{1, \dots, N\}. \tag{4.276}$$

In turn, this implies that for each $n \in \mathbb{N}$ and each $i \in \{1, \dots, N\}$ we have

$$\operatorname{ind}(X_i, \rho_i) \ge \big(\log_2 C_{\rho_i^{(n)}}\big)^{-1} \ge \big(\log_2 C_{\rho^{(n)}}\big)^{-1}. \tag{4.277}$$

In concert, (4.277) and (4.274) imply that

$$\min_{1 \le i \le N} \operatorname{ind}(X_i, \rho_i) \ge \big(\log_2 C_{\rho^{(n)}}\big)^{-1} \longrightarrow \operatorname{ind}(X, \rho) \quad \text{as} \quad n \to \infty. \tag{4.278}$$

This shows that $\min_{1\le i\le N} \operatorname{ind}(X_i, \rho_i) \ge \operatorname{ind}(X, \rho)$, completing the proof of the proposition. □

Comment 4.29. Recall from (4.260) that, by design, for any $\rho \in \mathfrak{Q}(X)$ we have the lower bound $\operatorname{ind}(X, \rho) \ge [\log_2 C_\rho]^{-1}$. However, as a simple example shows, it may be the case that $\rho \in \mathfrak{Q}(X)$ is such that $\operatorname{ind}(X, \rho)$ is substantially larger than $[\log_2 C_\rho]^{-1}$. To see this, for each fixed parameter $\lambda > 0$, consider the case when $X := \{x_1, x_2, x_3\}$ equipped with the quasidistance $\rho_\lambda : X \times X \to [0, +\infty)$ given by

$$\begin{aligned}\rho_\lambda(x_1, x_2) &:= \rho_\lambda(x_2, x_1) := \lambda,\\ \rho_\lambda(x_1, x_3) &:= \rho_\lambda(x_3, x_1) := 1,\\ \rho_\lambda(x_2, x_3) &:= \rho_\lambda(x_3, x_2) := 1,\\ \rho_\lambda(x_1, x_1) := \rho_\lambda(x_2, x_2) &:= \rho_\lambda(x_3, x_3) := 0.\end{aligned} \tag{4.279}$$

Hence, corresponding to $\lambda = 1$, ρ_1 is the discrete metric on X, i.e., $\rho_1(x_j, x_k) = 1 - \delta_{jk}$ for $j, k \in \{1, 2, 3\}$ (where δ_{jk} denotes the usual Kronecker symbol). Then, on the one hand, $\rho_\lambda \approx \rho_1$ for every $\lambda > 0$, which, in light of (4.256), gives that for each $\lambda > 0$ one has $\operatorname{ind}(X, \rho_\lambda) = \operatorname{ind}(X, \rho_1) = +\infty$. On the other hand, if $\lambda > 1$, then $C_{\rho_\lambda} = \lambda$, and, hence, $[\log_2 C_{\rho_\lambda}]^{-1} \to 0$ as $\lambda \to +\infty$. In the same setting, it may be verified that $c_\rho = 1$ whenever the quasidistance $\rho \in \mathfrak{Q}(X)$ is such that $\rho \approx \rho_\lambda$. Note that this entails $c_{\rho_\lambda} < C_{\rho_\lambda}$ when $\lambda > 1$. ■

The upper smoothness index is explicitly computed below in the case where the ambient set is a union of two disjoint, nonadjacent intervals on the real line. In this scenario, the aforementioned index turns out to be $+\infty$. This should be contrasted with the fact that the upper smoothness index of an interval on the real line is 1 (as proved later, in (4.302)) and goes to show that the upper smoothness index is sensitive to the presence of "holes" in the ambient set.

Comment 4.30. Assume that a, b, c, d are four real numbers with the property that $a < b < c < d$. Consider the set $X := [a, b] \cup [c, d] \subseteq \mathbb{R}$, equipped with the distance $\rho(x, y) := |x - y|$ for each $x, y \in X$. Taking $x_j := b$ and $y_j := c$ for each $j \in \mathbb{N}$, a straightforward calculation shows thats

$$\sup_{z\in X} \frac{\rho(x_j, y_j)}{\max\{\rho(x_j, z), \rho(z, y_j)\}} = 1, \qquad \forall\, j \in \mathbb{N}. \tag{4.280}$$

Hence, condition (4.261) is satisfied and, as such,

$$\operatorname{Ind}\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = +\infty. \tag{4.281}$$

Moreover, a similar reasoning yields

$$\operatorname{Ind}\big((a, b) \cup (c, d), |\cdot - \cdot|\big) = +\infty, \tag{4.282}$$

this time verifying (4.261) by taking $x_j \nearrow b$ and $y_j \searrow c$ as $j \to \infty$. ■

Comment 4.31. We claim that if C_3 denotes the Cantor ternary set obtained by repeatedly deleting the open middle thirds of a collection of line segments, starting with [0, 1], then

$$\operatorname{ind}\big(C_3, |\cdot - \cdot|\big) = +\infty. \tag{4.283}$$

To justify this claim, it is convenient to use the following analytic description of the Cantor ternary set:

$$C_3 := \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \tfrac{x_i}{3^i} \text{ where } x_i \in \{0,2\} \text{ for each } i \in \mathbb{N} \right\}. \tag{4.284}$$

That is, C_3 consists of numbers from [0, 1] whose digital writing in base 3 uses only 0s and 2s. Given two such numbers, say $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$ and $y = \sum_{i=1}^{\infty} \frac{y_i}{3^i}$, define the quantity $L(x, y) := \inf\{i \in \mathbb{N} : x_i \neq y_i\}$, with the understanding that $\inf \emptyset := +\infty$, and set

$$d(x,y) := 3^{-L(x,y)}. \tag{4.285}$$

Since $L(x, y) \geq \min\{L(x,z), L(z,y)\}$ for every $x, y, z \in C_3$, it follows that d is an ultrametric on C_3. Furthermore, for any $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$ and $y = \sum_{i=1}^{\infty} \frac{y_i}{3^i}$ in C_3 we have

$$x - y = \sum_{i=L(x,y)}^{\infty} \tfrac{x_i - y_i}{3^i} = \pm \tfrac{2}{3^{L(x,y)}} + \sum_{i=L(x,y)+1}^{\infty} \tfrac{x_i - y_i}{3^i} \tag{4.286}$$

with

$$\left| \sum_{i=L(x,y)+1}^{\infty} \tfrac{x_i - y_i}{3^i} \right| \leq \sum_{i=L(x,y)+1}^{\infty} \tfrac{2}{3^i} = \tfrac{1}{3^{L(x,y)}}. \tag{4.287}$$

In concert, (4.286) and (4.287) then prove that

$$d(x,y) = 3^{-L(x,y)} \leq |x - y| \leq 3^{1-L(x,y)} = 3\,d(x,y), \qquad \forall\, x, y \in C_3. \tag{4.288}$$

This shows that, when restricted to C_3, the standard distance on the real line is equivalent to the ultrametric d defined in (4.285). In turn, this and (4.256) allow us to conclude that (4.283) holds. ∎

In particular, (4.283) shows that, as opposed to, say, the case of $\big([0,1], |\cdot - \cdot|\big)$, where any Hölder function of order > 1 is constant, there are plenty of Hölder functions of any order on $(C_3, |\cdot - \cdot|)$.

Definition 4.32. Given two arbitrary quasimetric spaces $(X_j, \mathbf{q}_j)$, $j = 0, 1$, a mapping $\Phi : (X_0, \mathbf{q}_0) \to (X_1, \mathbf{q}_1)$ is called `bi-Lipschitz` provided for some (hence, any) $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, one has $\rho_1(\Phi(x), \Phi(y)) \approx \rho_0(x, y)$, uniformly for $x, y \in X_0$.

Of course, in the context of this definition, the specific choice of the quasidistances $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, used to define the notion of bi-Lipschitzianity is immaterial. In particular, being bi-Lipschitz is a quality that intrinsically depends only on the quasimetric space structures of the spaces involved. Let us also note that, given a set X and $\rho_0, \rho_1 \in \mathfrak{Q}(X)$, we have

$$\rho_0 \approx \rho_1 \iff \text{identity} : (X, \rho_0) \longrightarrow (X, \rho_1) \text{ is bi-Lipschitz.} \tag{4.289}$$

This explains why the terminology "ρ_0 *is bi-Lipschitz equivalent to* ρ_1" is sometimes used (cf., e.g., [59]) to signify that, in our sense, $\rho_0 \approx \rho_1$.

Our next result shows (among other things) that the lower smoothness index is invariant under bi-Lipschitz homeomorphisms.

Proposition 4.33. *If* $(X_j, \mathbf{q}_j)$, $j = 0, 1$, *are quasimetric spaces for which there is a bi-Lipschitz mapping* $\Phi : (X_0, \mathbf{q}_0) \to (X_1, \mathbf{q}_1)$, *then* $\operatorname{ind}(X_0, \mathbf{q}_0) \geq \operatorname{ind}(X_1, \mathbf{q}_1)$. *Consequently, if two quasimetric spaces are bi-Lipschitz homeomorphic, then they have the same lower smoothness index.*

As a corollary, if (X, ρ) *is a quasimetric space and* $Y \subseteq X$ *has cardinality* ≥ 2, *then* $\operatorname{ind}(Y, \rho) \geq \operatorname{ind}(X, \rho)$.

Proof. For every $\rho \in \mathbf{q}_1$ define $\tilde{\rho} : X_0 \times X_0 \to [0, +\infty)$ by setting $\tilde{\rho}(x, y) := \rho(\Phi(x), \Phi(y))$ for each $x, y \in X_0$. The fact that Φ is bi-Lipschitz implies that $\tilde{\rho} \in \mathbf{q}_0$. Also, by design, $C_{\tilde{\rho}} \leq C_\rho$. Hence,

$$\operatorname{ind}(X_0, \mathbf{q}_0) = \operatorname{ind}(X_0, \tilde{\rho}) \geq [\log_2 C_{\tilde{\rho}}]^{-1} \geq [\log_2 C_\rho]^{-1}. \tag{4.290}$$

Taking the supremum over all $\rho \in \mathbf{q}_1$ then yields $\operatorname{ind}(X_0, \mathbf{q}_0) \geq \operatorname{ind}(X_1, \mathbf{q}_1)$.

To prove the last claim in the statement of the proposition, observe that the canonical inclusion map $\iota : (Y, \rho) \hookrightarrow (X, \rho)$ is bi-Lipschitz. Consequently, $\operatorname{ind}(Y, \rho) \geq \operatorname{ind}(X, \rho)$ by what we have proved in the first part. □

We continue by discussing how the upper and lower smoothness indexes are related. As a preamble, we first establish the following characterization of the lower smoothness index.

Proposition 4.34. *If* X *is an arbitrary set of cardinality* ≥ 2, *then for every quasidistance* $\rho \in \mathfrak{Q}(X)$ *one has*

$$\begin{aligned} \operatorname{ind}(X, \rho) &= \sup\left\{[\log_2 C_{\rho'}]^{-1} : \rho' \in [\rho]\right\} \\ &= \sup_{\substack{\theta : X \times X \to \mathbb{R} \\ 0 < \inf \theta \leq \sup \theta < +\infty}} \left[\log_2\left(\sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{(\theta\rho)(x, y)}{\max\{(\theta\rho)(x, z), (\theta\rho)(y, z)\}}\right)\right]^{-1}. \end{aligned} \tag{4.291}$$

Proof. The first equality in (4.291) is just a rewriting of the definition (4.255). Also, the fact that the first supremum in (4.291) is dominated by the second supremum in (4.291) follows from conclusion (i) of Remark 3.16. Consider next a function $\theta : X \times X \to \mathbb{R}$ with the property that $0 < \inf \theta \leq \sup \theta < +\infty$ and set $\rho' := \theta\rho$. Then, by conclusion (6) in Theorem 3.46, $(\rho')_{\text{sym}}$ is a quasidistance on X with

$C_{(\rho')_{\rm sym}} = C_{\rho'}$. Also, by conclusion (3) in Theorem 3.46, we have $(\rho')_{\rm sym} \approx \rho' \approx \rho$. Consequently, $\operatorname{ind}(X, \rho) \geq [\log_2 C_{\rho'}]^{-1}$, which proves that the second supremum in (4.291) is majorized by $\operatorname{ind}(X, \rho)$. The desired conclusion follows. □

Here is the theorem alluded to previously; it asserts that, for any quasimetric space, the lower smoothness index is less than or equal to the upper smoothness index.

Theorem 4.35. *For any quasimetric space* $(X, \mathbf{q})$ *one has* $\operatorname{ind}(X, \mathbf{q}) \leq \operatorname{Ind}(X, \mathbf{q})$.

Proof. Fix some $\rho \in \mathbf{q}$ and consider a function $\theta : X \times X \to \mathbb{R}$ with the property that $0 < \inf \theta \leq \sup \theta < +\infty$. Thus, if we set $M := \sup\{\theta(x, y) : x, y \in X,\ x \neq y\}$, then it follows that $0 < M < +\infty$. Fix an arbitrary $\varepsilon \in (0, 1)$ and select $x_\varepsilon, y_\varepsilon \in X$ such that $x_\varepsilon \neq y_\varepsilon$ and $\theta(x_\varepsilon, y_\varepsilon) > M - \varepsilon$. Use the definition of c_ρ from (4.245) to conclude that there exists some $z_\varepsilon \in X$ for which

$$\frac{\rho(x_\varepsilon, y_\varepsilon)}{\max\{\rho(x_\varepsilon, z_\varepsilon), \rho(z_\varepsilon, y_\varepsilon)\}} > c_\rho - \varepsilon. \tag{4.292}$$

In the case when $z_\varepsilon \neq x_\varepsilon$, $z_\varepsilon \neq y_\varepsilon$, we have $0 < \theta(x_\varepsilon, z_\varepsilon) \leq M$ and $0 < \theta(z_\varepsilon, y_\varepsilon) \leq M$, which, together with the previous observations, allow us to estimate

$$\begin{aligned}\frac{(\theta\rho)(x_\varepsilon, y_\varepsilon)}{\max\{(\theta\rho)(x_\varepsilon, z_\varepsilon), (\theta\rho)(z_\varepsilon, y_\varepsilon)\}} &> \frac{(M-\varepsilon)\rho(x_\varepsilon, y_\varepsilon)}{M\ \max\{\rho(x_\varepsilon, z_\varepsilon), \rho(z_\varepsilon, y_\varepsilon)\}} \\ &> \Big(1 - \frac{\varepsilon}{M}\Big)(c_\rho - \varepsilon).\end{aligned} \tag{4.293}$$

This implies that

$$\frac{(\theta\rho)(x_\varepsilon, y_\varepsilon)}{\max\{(\theta\rho)(x_\varepsilon, z_\varepsilon), (\theta\rho)(z_\varepsilon, y_\varepsilon)\}} > \Big(1 - \frac{\varepsilon}{M}\Big)(c_\rho - \varepsilon) \tag{4.294}$$

whenever $z_\varepsilon \notin \{x_\varepsilon, y_\varepsilon\}$. However, in the case when $z_\varepsilon = x_\varepsilon$, or $z_\varepsilon = y_\varepsilon$, estimate (4.294) follows directly from (4.292). Thus, (4.294) holds in all cases and we may conclude, based on it, that

$$\sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{(\theta\rho)(x, y)}{\max\{(\theta\rho)(x, z), (\theta\rho)(y, z)\}} \geq c_\rho. \tag{4.295}$$

Hence, since θ was arbitrarily chosen from among the class of functions with the specified properties, it follows from (4.295) that

$$\sup_{\substack{\theta : X\times X \to \mathbb{R} \\ 0 < \inf\theta \leq \sup\theta < +\infty}} \left[\log_2\Big(\sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{(\theta\rho)(x, y)}{\max\{(\theta\rho)(x, z), (\theta\rho)(y, z)\}}\Big)\right]^{-1} \leq [\log_2 c_\rho]^{-1}. \tag{4.296}$$

Having established this, Proposition 4.34 then gives $\operatorname{ind}(X,\rho) \le [\log_2 c_\rho]^{-1}$, which further yields $\operatorname{ind}(X,\rho) \le \operatorname{Ind}(X,\rho)$ after taking the infimum over all $\rho \in \mathbf{q}$. □

An equivalent way of expressing the conclusion in Theorem 4.35 is to say that

$$c_{\rho_1} \le C_{\rho_2} \qquad \text{for all} \quad \rho_1, \rho_2 \in \mathbf{q}. \tag{4.297}$$

In particular, one always has $\operatorname{Ind}(X,\mathbf{q}) > 0$. Also, it follows from (4.256) and (4.257) and Theorem 4.35 that for any set X we have

$$\rho \text{ ultrametric on } X \implies \operatorname{Ind}(X,\rho) = +\infty, \tag{4.298}$$

$$\rho \text{ distance on } X \implies \operatorname{Ind}(X,\rho) \ge 1. \tag{4.299}$$

Other significant consequences are discussed in the following corollary.

Corollary 4.36. *If $(X,\mathbf{q})$ is a quasimetric space with the property that there exists $\rho \in \mathbf{q}$ such that*

$$\forall\, x,y \in X \;\; \exists\, z \in X \;\; \text{with} \;\; \rho(x,z) \le \frac{1}{2}\rho(x,y) \;\; \text{and} \;\; \rho(z,y) \le \frac{1}{2}\rho(x,y), \tag{4.300}$$

then $\operatorname{Ind}(X,\mathbf{q}) \le 1$. *In particular, if $(X,\|\cdot\|)$ is a normed vector space and if $\mathbf{q}$ stands for the quasimetric space structure induced by the norm $\|\cdot\|$, then*

$$\begin{gathered}\operatorname{ind}(Y,\mathbf{q}) = \operatorname{Ind}(Y,\mathbf{q}) = 1 \\ \text{for any convex subset } Y \text{ of } X \text{ of cardinality} \ge 2.\end{gathered} \tag{4.301}$$

As a consequence, for any $n \in \mathbb{N}$ one has

$$\begin{aligned}\operatorname{ind}(\mathbb{R}^n, |\cdot - \cdot|) &= \operatorname{Ind}(\mathbb{R}^n, |\cdot - \cdot|) = 1, \\ \operatorname{ind}([0,1]^n, |\cdot - \cdot|) &= \operatorname{Ind}([0,1]^n, |\cdot - \cdot|) = 1,\end{aligned} \tag{4.302}$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

Proof. If $\rho \in \mathbf{q}$ satisfies (4.300), then $\rho_{\text{mid}} \le \frac{1}{2}\rho$. In concert with (4.253), this forces $c_\rho \ge 2$; hence, further, $\operatorname{Ind}(X,\mathbf{q}) \le 1$. This proves the first claim in the statement of the corollary. Note that, when used together with Theorem 4.35, this gives that $\operatorname{ind}(X,\mathbf{q}) \le \operatorname{Ind}(X,\mathbf{q}) \le 1$. Next, if Y is a convex subset of a normed vector space $(X,\|\cdot\|)$, then for any $x,y \in Y$ we have $\frac{x+y}{2} \in Y$ and $\|x - (\frac{x+y}{2})\| = \|y - (\frac{x+y}{2})\| = \frac{1}{2}\|x-y\|$. This shows that, if $\mathbf{q}$ denotes the quasimetric space structure induced by the norm, then $\operatorname{Ind}(X,\mathbf{q}) \le 1$ by what we have proved so far. When combined with Theorem 4.35 and (4.257), this shows that $\operatorname{ind}(X,\mathbf{q}) = \operatorname{Ind}(X,\mathbf{q}) = 1$, as claimed. Finally, (4.302) is an obvious consequence of the more general situation considered earlier in the proof. □

Corollary 4.37. *Let $(X, \|\cdot\|)$ be a normed vector space, and assume that $Y \subseteq X$ contains a convex subset Z of cardinality ≥ 2. Then, if $\mathbf{q}$ is the quasimetric space structure induced by the norm $\|\cdot\|$ on Y, then one has*

$$\text{ind}\,(Y, \mathbf{q}) = 1. \tag{4.303}$$

Proof. On the one hand, from the last part of Proposition 4.33 and (4.301) we see that $\text{ind}\,(Y, \mathbf{q}) \leq \text{ind}\,(Z, \mathbf{q}) = 1$. On the other hand, since $d(x, y) := \|x - y\|$ for each $x, y \in X$ is a distance, we have $\text{ind}\,(Y, \mathbf{q}) = \text{ind}\,(Y, d) \geq 1$ by (4.257), and the desired conclusion follows. □

Comment 4.38. Let a, b, c, d be four real numbers with the property that $a < b < c < d$. Then, on the one hand, it was shown in (4.281) that $\text{Ind}\,\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = +\infty$. On the other hand, Corollary 4.37 gives that

$$\text{ind}\,\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = 1. \tag{4.304}$$

This shows that the inequality $\text{ind}\,(X, \mathbf{q}) \leq \text{Ind}\,(X, \mathbf{q})$ proved in Theorem 4.35 for any quasimetric space $(X, \mathbf{q})$ can be strict. ■

The next definition is a natural adaptation of a similar concept introduced by David and Semmes (cf. [38]) in the setting of metric spaces.

Definition 4.39. Call a quasimetric space $(X, \mathbf{q})$ `uniformly disconnected` if one can find $\rho \in \mathbf{q}$ and $\varepsilon_o > 0$ with the following significance. For no pair of distinct points $x, y \in X$ does there exist a finite chain, $z_1, \ldots, z_N \in X$, such that $z_1 = x$ and $z_N = y$, with the property that $\rho(z_i, z_{i+1}) < \varepsilon_o \rho(x, y)$ for all $i \in \{1, \ldots, N-1\}$.

For example, when equipped with the metric space structure induced by the standard metric space structure on the real line, the Cantor ternary set is uniformly disconnected, whereas the set $\{n^{-1} : n \in \mathbb{N}\}$ is not.

The equivalence in the theorem below generalizes [38, Proposition 15.7, p. 161], which deals with metric spaces.

Theorem 4.40. *For any quasimetric space $(X, \mathbf{q})$ one has*

$$\begin{aligned}(X, \mathbf{q}) \text{ is uniformly disconnected} &\Longleftrightarrow \mathbf{q} \text{ contains an ultrametric} \\ &\Longrightarrow \text{ind}\,(X, \mathbf{q}) = +\infty \\ &\Longrightarrow \text{Ind}\,(X, \mathbf{q}) = +\infty.\end{aligned} \tag{4.305}$$

Proof. If $(X, \mathbf{q})$ is uniformly disconnected, then, from Definition 4.39, we deduce that there exist $\rho \in \mathbf{q}$ and $\varepsilon_o > 0$ with the property that if $x, y \in X$ are any two distinct points, then

$$\forall\, z_1,\dots,z_N \in X \text{ such that } x_1 = x \text{ and } z_N = y \\ \Longrightarrow \max_{1\le i\le N-1} \rho(z_i, z_{i+1}) \ge \varepsilon_o \rho(x,y). \tag{4.306}$$

To proceed, recall the function $\rho_\infty : X \times X \to [0,+\infty)$ from (3.495) and note that (4.306) entails

$$\varepsilon_o \rho(x,y) \le \rho_\infty(x,y), \qquad \forall\, x,y \in X. \tag{4.307}$$

Based on this, (3.498), and (3.512), we deduce that ρ_∞ is an ultrametric on X. Furthermore, from (3.496) and (4.307) we also deduce that $\varepsilon_o\rho \le \rho_\infty \le \rho$, so $\rho \approx \rho_\infty$. Hence, ultimately, $\rho_\infty \in \mathbf{q}$. Conversely, if $\rho \in \mathbf{q}$ is an ultrametric, then for any $x, y \in X$ and any $z_1,\dots,z_N \in X$ such that $x_1 = x$, $z_N = y$, it follows that

$$\rho(x,y) \le \max_{1\le i\le N-1} \rho(z_i, z_{i+1}). \tag{4.308}$$

This shows that the conditions stipulated in Definition 4.39 hold if, e.g., we take $\varepsilon_o := 1$. Thus, $(X,\mathbf{q})$ is uniformly disconnected, completing the proof of the first equivalence in (4.305). Next, the fact that $\mathbf{q}$ contains an ultrametric implies $\operatorname{ind}(X,\mathbf{q}) = +\infty$ is contained in (4.256). Also, $\operatorname{ind}(X,\mathbf{q}) = +\infty$ forces $\operatorname{Ind}(X,\mathbf{q}) = +\infty$ by Theorem 4.35. This completes the proof of the theorem. □

By an `embedding` we will understand a mapping between two topological spaces that is a homeomorphism onto its image.

Corollary 4.41. *If the interval* $[0,1]$ *may be bi-Lipschitzly embedded into the quasimetric space* $(X,\mathbf{q})$*, then* $\operatorname{ind}(X,\mathbf{q}) \le 1$.

Also, if $(X,\mathbf{q})$ *is a quasimetric space with the property that* $\operatorname{ind}(X,\mathbf{q}) < 1$*, then* $(X,\mathbf{q})$ *cannot be bi-Lipschitzly embedded into some* $\mathbb{R}^n$*,* $n \in \mathbb{N}$.

Proof. Both claims in the statement of the corollary are consequences of the last part of Proposition 4.33 and (4.302). □

Moving on, we recall the notion of convexity index (*l'indice de convexité*) introduced by Assouad in [10, Définition 3, p. 732].

Definition 4.42. Assouad's `convexity index` of a quasimetric space $(X,\mathbf{q})$ is defined as

$$\operatorname{Cv}(X,\mathbf{q}) := \inf\big\{p \in (0,+\infty) : \exists\, \rho \in \mathbf{q} \text{ such that} \\ \rho^{1/p} \text{ is equivalent to a distance on } X\big\}. \tag{4.309}$$

Also, if X is a set and $\rho \in \mathfrak{Q}(X)$, then set $\operatorname{Cv}(X,\rho) := \operatorname{Cv}(X,[\rho])$.

Our next result is the following theorem asserting that, for any quasimetric space, our lower smoothness index coincides with the reciprocal of Assouad's convexity index.

Theorem 4.43. *For any quasimetric space* $(X, \mathbf{q})$ *one has*

$$\operatorname{ind}(X, \mathbf{q}) = \frac{1}{\operatorname{Cv}(X, \mathbf{q})}. \tag{4.310}$$

Proof. Let $p \in (0, +\infty)$ be such that there exists $\rho \in \mathbf{q}$ and a distance d on X with the property that $\rho^{1/p} \approx d$. It follows that $d^p \in \mathbf{q}$ and $C_{d^p} = (C_d)^p \leq 2^p$ since $C_d \leq 2$ given that d is a distance. This proves that $\operatorname{ind}(X, \mathbf{q}) \geq [\log_2 C_{d^p}]^{-1} \geq 1/p$ hence, ultimately, $\operatorname{ind}(X, \mathbf{q}) \geq [\operatorname{Cv}(X, \mathbf{q})]^{-1}$. In the opposite direction, pick $\rho \in \mathbf{q}$ and assume that $0 < \beta < [\log_2 C_\rho]^{-1}$. Then part (12) in Theorem 3.46 shows that there exists a distance d on X with the property that $d^{1/\beta} \in \mathbf{q}$. In turn, after unraveling definitions, this condition readily implies $1/\beta \geq \operatorname{Cv}(X, \mathbf{q})$. This forces $[\log_2 C_\rho]^{-1} \leq [\operatorname{Cv}(X, \mathbf{q})]^{-1}$, hence $\operatorname{ind}(X, \mathbf{q}) \leq [\operatorname{Cv}(X, \mathbf{q})]^{-1}$. All in all, (4.310) follows. □

Given a metric space (X, d) and a number $\varepsilon \in (0, 1)$, call the metric space (X, d^{ε}) the ε-`snowflaked version` of (X, d). This terminology is suggested by the fact that for each $\varepsilon \in (\frac{1}{2}, 1)$ the quasimetric space $(\mathbb{R}, |\cdot - \cdot|^{\varepsilon})$ admits a bi-Lipschitz embedding into $\mathbb{R}^2$ whose image is reminiscent of the boundary of a domain depicting a snowflake (recall that an embedding is a mapping between two topological spaces that is a homeomorphism onto its image).

Corollary 4.44. *A quasimetric space is bi-Lipschitz homeomorphic to a snowflaked version of a metric space if and only if its lower smoothness index is* > 1.

Proof. Assume the quasimetric space $(X, \mathbf{q})$ and the metric space (Y, d) are such that there exists $\varepsilon \in (0, 1)$ with the property that $(X, \mathbf{q})$ and (Y, d^{ε}) are bi-Lipschitz homeomorphic. Then Proposition 4.33 gives that $\operatorname{ind}(X, \mathbf{q}) = \operatorname{ind}(Y, d^{\varepsilon}) > \varepsilon^{-1}$, where the last inequality follows from (4.255), after observing that $C_{d^\varepsilon} = (C_d)^{\varepsilon} \leq 2^{\varepsilon}$. This proves the right-pointing implication in the statement of the corollary. As regards the opposite implication, note that if $(X, \mathbf{q})$ is a quasimetric space satisfying $\operatorname{ind}(X, \mathbf{q}) > 1$, then, by Theorem 4.43, we have $\operatorname{Cv}(X, \mathbf{q}) < 1$. Hence, upon recalling (4.309), it follows that there exist $\rho \in \mathbf{q}$, $p \in (0, 1)$, and a distance d on X with the property that $\rho^{1/p} \approx d$. Thus, the identity mapping is a bi-Lipschitz homeomorphism of $(X, \mathbf{q})$ onto the snowflaked version (X, d^p) of the metric space (X, d). □

Our next theorem contains an intrinsic description of the lower smoothness index of a quasimetric space.

Theorem 4.45. *Assume that* X *is an arbitrary set and that* $\rho \in \mathfrak{Q}(X)$. *Denote by* $P(X, \rho)$ *the infimum of all numbers* $p \in (0, +\infty)$ *for which the following condition is satisfied: there exists* $c_p > 0$ *such that for each* $x, y \in X$ *one has*

$$\inf \left(\sum_{i=1}^{N} \rho(z_i, z_{i+1})^{1/p} \right)^p \geq c_p\, \rho(x, y), \tag{4.311}$$

where the infimum is taken over all $N \in \mathbb{N}$ *and all* $z_1, \dots, z_{N+1} \in X$ *with* $z_1 = x$ *and* $z_{N+1} = y$*. Then*

$$\mathrm{ind}\,(X, \rho) = \frac{1}{P(X, \rho)}. \tag{4.312}$$

Proof. Thanks to (4.310), it suffices to show that $P(X, \rho) = \mathrm{Cv}\,(X, \rho)$. To this end, assume first that $p \in (0, +\infty)$ is such that there exists $\rho' \in \mathfrak{Q}(X)$ satisfying $\rho' \approx \rho$ and such that $(\rho')^{1/p}$ is equivalent to a distance d on X. Then one can find a constant $c \in [1, +\infty)$ with the property that

$$c^{-1}\rho(x, y)^{1/p} \le d(x, y) \le c\rho(x, y)^{1/p}, \qquad \forall\, x, y \in X. \tag{4.313}$$

In turn, given two arbitrary points $x, y \in X$, for any $N \in \mathbb{N}$ and any $z_1, \dots, z_{N+1} \in X$ with $z_1 = x$, $z_{N+1} = y$, (4.313) allows us to estimate

$$\left(\sum_{i=1}^{N} \rho(z_i, z_{i+1})^{1/p}\right)^p \ge c^{-p}\left(\sum_{i=1}^{N} d(z_i, z_{i+1})\right)^p \ge c^{-p} d(x, y)^p \ge c^{-2p}\rho(x, y). \tag{4.314}$$

This proves that (4.311) holds, with $c_p := c^{-2p} > 0$, for each $x, y \in X$. Hence, based on this, (4.309), and definitions, we obtain that $\mathrm{Cv}(X, \rho) \ge P(X, \rho)$. To establish the opposite inequality, suppose that $p \in (0, +\infty)$ is such that there exists $c_p > 0$ with the property that (4.311) holds for each $x, y \in X$. Using notation introduced in (3.494), the latter condition amounts to having $\rho_{1/p}(x, y) \ge c_p\, \rho(x, y)$ for all $x, y \in X$. Since, by (3.496), we always have $\rho_{1/p}(x, y) \le \rho(x, y)$ for all $x, y \in X$, we arrive at the conclusion that

$$\rho_{1/p} \approx \rho. \tag{4.315}$$

In particular, $(\rho_{1/p})^{-1}(\{0\}) = \mathrm{diag}(X)$. In turn, in combination with (3.512) and (3.497), this implies that $(\rho_{1/p})^{1/p}$ is a distance on X. Consequently, from the definitions of $\mathrm{Cv}(X, \rho)$ and $P(X, \rho)$ we obtain $\mathrm{Cv}(X, \rho) \le P(X, \rho)$. This completes the proof of the theorem. □

Corollary 4.46. *For any quasimetric space* (X, ρ) *one has*

$$\begin{aligned}\mathrm{ind}\,(X, \rho) &= \sup\left\{\alpha \in (0, +\infty) : \inf_{\substack{x, y \in X \\ \text{not equal}}} \left(\frac{\rho_\alpha(x, y)}{\rho(x, y)}\right) > 0\right\} \\ &= \sup\big\{\alpha \in (0, +\infty) : \ \rho_\alpha \approx \rho\big\},\end{aligned} \tag{4.316}$$

where ρ_α *is defined as in* (3.494)*. Moreover,*

$$\begin{aligned}&\textit{the set } \big\{\alpha \in (0, +\infty) : \ \rho_\alpha \approx \rho\big\} \textit{ is either} \\ &\textit{the interval } \big(0, \mathrm{ind}\,(X, \rho)\big) \textit{ or the interval } \big(0, \mathrm{ind}\,(X, \rho)\big].\end{aligned} \tag{4.317}$$

In particular,

$$\rho_\alpha \approx \rho \quad \textit{for each } \alpha \in \big(0, \operatorname{ind}(X,\rho)\big). \tag{4.318}$$

Proof. The first equality in (4.316) follows upon observing that, by (4.311) and (4.312),

$$\operatorname{ind}(X,\rho) = \sup\big\{\tfrac{1}{p} :\ p \in (0,+\infty) \text{ such that } \rho_{1/p} \geq c_p\,\rho, \text{ for some } c_p > 0\big\}. \tag{4.319}$$

In turn, the second equality in (4.316) follows from the first equality in (4.316) and the last formula in (3.103). Finally, the claim in (4.317) is a consequence of the first equality in (4.316) and the first formula in (3.103), whereas the claim in (4.318) readily follows from (4.317). □

Comment 4.47. It is instructive to note that

$$\operatorname{ind}\Big(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\Big) = \min\{1,p\} \quad \text{for any} \quad p \in (0,+\infty] \tag{4.320}$$

and that the supremum that defines the index on the left-hand side (cf. the general definition in (4.255)) is attained. To see that this is the case, consider first the situation where $p \in (0,1]$. In this scenario, we have $\operatorname{ind}\Big(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\Big) = p$, and if we set $\rho(f,g) := \|f-g\|_{L^p(\mathbb{R})}$ for every $f,g \in L^p(\mathbb{R})$, then $(\log_2 C_\rho)^{-1} = p$. Indeed, all these claims follow from the discussion just after the statement of Theorem 3.27 and Corollary 4.46. The claim for the case $p \in [1,+\infty]$ follows from (4.301) and the fact that, this time, $(\log_2 C_\rho)^{-1} = 1$. Of course, similar properties hold for the sequence space ℓ^p, if $p \in (0,+\infty]$. ■

The characterization of the lower smoothness index established in Corollary 4.46 suggests making the following definition.

Definition 4.48. Given a quasimetric space (X,ρ), set

$$\operatorname{ind}_0(X,\rho) := \inf\Big\{\alpha \in (0,+\infty) :\ \forall\, x,y \in X \ \text{ and } \ \forall\, \varepsilon > 0\ \ \exists\, \xi_1,\dots,\xi_{N+1} \in X$$
$$\text{such that } \xi_1 = x,\ \xi_{N+1} = y \text{ and } \sum_{i=1}^{N} \rho(\xi_i,\xi_{i+1})^\alpha < \varepsilon\Big\}, \tag{4.321}$$

with the agreement that $\inf\emptyset := +\infty$.

Hence, if (X,ρ) is an arbitrary quasimetric space and if ρ_α is defined as in (3.494), then

$$\operatorname{ind}_0(X,\rho) = \inf\big\{\alpha \in (0,+\infty) :\ \rho_\alpha = 0 \text{ on } X \times X\big\}. \tag{4.322}$$

In concert with Lemma 3.14 [which guarantees that $\rho \approx \rho' \Longrightarrow \rho_\alpha \approx \rho'_\alpha$ for any $\rho, \rho' \in \mathfrak{Q}(X)$ and $\alpha \in (0,+\infty]$], it follows that we may unambiguously define

$$\operatorname{ind}_0(X, [\rho]) := \operatorname{ind}_0(X, \rho), \qquad \forall\, \rho \in \mathfrak{Q}(X). \tag{4.323}$$

Let us also point out here that, together with the formula proved in part (9) of Lemma 3.14, (4.322) also makes it clear that

$$\operatorname{ind}_0(X, \rho^\beta) = \tfrac{1}{\beta} \operatorname{ind}_0(X, \rho), \qquad \forall\, \beta \in (0, +\infty). \tag{4.324}$$

Remark 4.49. Let a, b, c, d be four real numbers with the property that $a < b < c < d$. Then it follows from Definition 4.48 (cf. the convention in the last part of its statement) that

$$\operatorname{ind}_0\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = +\infty. \tag{4.325}$$

Let us also note here that while for any quasimetric space (X, ρ) we have $\rho_\alpha \approx \rho$ for any $\alpha \in \big(0, \operatorname{ind}(X, \rho)\big)$ and $\rho_\alpha = 0$ for any $\alpha \in \big(\operatorname{ind}_0(X, \rho), +\infty\big]$, in the case when $X = [a, b] \cup [c, d]$ and $\rho = |\cdot - \cdot|$ the α-regularization ρ_α of ρ becomes degenerate without vanishing identically when α belongs to the "transition" interval $\big(\operatorname{ind}(X, \rho), \operatorname{ind}_0(X, \rho)\big)$. Indeed, as may be readily seen from definitions, in this situation for each $\alpha \in (1, +\infty)$ (recall that $\operatorname{ind}\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = 1$), we have

$$\rho_\alpha(x, y) = \begin{cases} 0, & \text{if } x, y \in [a, b], \text{ or } x, y \in [c, d], \\ c - b, & \text{if } x \in [a, b] \text{ and } y \in [c, d], \text{ or } y \in [a, b] \text{ and } x \in [c, d]. \end{cases} \tag{4.326}$$

In what follows, we estimate the index introduced in Definition 4.48 for a Jordan curve (i.e., a non-self-intersecting continuous loop in the Euclidean space) equipped with the standard Euclidean distance.

Proposition 4.50. *Let $a, b \in \mathbb{R}$, $a < b$, and assume that $f : [a, b] \to \mathbb{R}^n$ is a continuous injective function. Then, if $\Sigma := f\big([a, b]\big) \subseteq \mathbb{R}^n$, it follows that*

$$1 \leq \operatorname{ind}_0\big(\Sigma, |\cdot - \cdot|\big) \leq n. \tag{4.327}$$

Proof. The first inequality in (4.327) is a direct consequence of the last part of Proposition 4.33 and (4.302). As far as the second inequality in (4.327) is concerned, according to (4.321) it suffices to show that for each $t_0, t_1 \in [a, b]$ with $t_0 < t_1$ and any $\varepsilon > 0$ there exist points $\xi_1, \ldots, \xi_{N+1} \in [a, b]$ with the property that $t_0 = \xi_1 < \xi_2 < \cdots < \xi_N < \xi_{N+1} = t_1$ and $\sum_{i=1}^{N} |f(\xi_i) - f(\xi_{i+1})|^n < \varepsilon$. When $t_0 = 0$ and $t_1 = 1$, this is a deep result of Besicovitch and Schoenberg (cf. [17, Theorem 3, p. 115]) in the two-dimensional setting (i.e., when $n = 2$), which was subsequently extended to arbitrary space dimensions by Katznelson (cf. the discussion in [11, Sect. 4.13, p. 447]). Of course, this result holds, via an affine rescaling, for arbitrary $t_0, t_1 \in [a, b]$ with $t_0 < t_1$, and the desired conclusion follows. □

Definition 4.51. Let (X, ρ) be a quasimetric space. Given any $N \in \mathbb{N}$, for each $x, y \in X$ define

$$\rho_{(N)}(x,y) := \inf\Big\{r > 0 : \exists\, \xi_1, \dots, \xi_{N+1} \in X \text{ such that}$$
$$\xi_1 = x,\ \xi_{N+1} = y \text{ and } \max_{1 \le i \le N} \rho(\xi_i, \xi_{i+1}) < r\Big\}. \tag{4.328}$$

Proposition 4.52. *Given a quasimetric space (X, ρ) and $N \in \mathbb{N}$, one has $\rho_{(N)} \in \mathfrak{Q}(X)$ and*

$$C_\rho^{1-N} \rho \le \rho_{(N)} \le \rho_{(N-1)} \le \cdots \le \rho_{(2)} = \rho_{\mathrm{mid}} \le \rho_{(1)} = \rho \ \ \text{on} \ \ X \times X, \tag{4.329}$$

where ρ_{mid} is as in (4.250). In particular,

$$\rho_{(N)} \approx \rho, \quad \forall\, N \in \mathbb{N}. \tag{4.330}$$

Proof. All the claims in the statement of the proposition are direct consequences of definitions. □

Definition 4.53. Let (X, ρ) be a quasimetric space. For each $N \in \mathbb{N}$ consider

$$c_\rho^{(N)} := \inf_{\substack{x,y \in X \\ x \neq y}} \left(\frac{\rho(x,y)}{\rho_{(N)}(x,y)} \right) \in \big[1, C_\rho^{N-1}\big], \tag{4.331}$$

then define

$$\mathrm{Ind}_0(X, \rho) := \inf\Big\{ \big(\log_N c_{\rho'}^{(N)}\big)^{-1} : \ N \in \mathbb{N} \text{ with } N \ge 2, \text{ and } \rho' \in [\rho] \Big\}. \tag{4.332}$$

Our next result may be regarded as a refinement of the inequality established earlier in Theorem 4.35.

Theorem 4.54. *For any quasimetric space (X, ρ) there holds*

$$\mathrm{ind}\,(X,\rho) \le \mathrm{ind}_0(X,\rho) \le \mathrm{Ind}_0(X,\rho) \le \mathrm{Ind}\,(X,\rho). \tag{4.333}$$

Furthermore,

$$\begin{gathered} \textit{the set } \big\{\alpha \in (0, +\infty) : \ \rho_\alpha = 0 \textit{ on } X \times X\big\} \textit{ is either} \\ \textit{the interval } \big(\mathrm{ind}_0(X,\rho), +\infty\big) \textit{ or the interval } \big[\mathrm{ind}_0(X,\rho), +\infty\big). \end{gathered} \tag{4.334}$$

Consequently,

$$\rho_\alpha = 0 \textit{ on } X \times X \quad \textit{for each } \alpha \in \mathbb{R} \textit{ with } \alpha > \mathrm{ind}_0(X,\rho) \tag{4.335}$$

and, as a corollary of (4.335) *and* (4.333),

$$\rho_\alpha = 0 \text{ on } X \times X \quad \text{for each } \alpha \in \mathbb{R} \text{ with } \alpha > \operatorname{Ind}_0(X, \rho). \tag{4.336}$$

Proof. Consider first the middle inequality in (4.333). If $\operatorname{Ind}_0(X, \rho) = +\infty$, then there is nothing to prove, so assume that $\operatorname{Ind}_0(X, \rho) < +\infty$. To proceed, suppose that $\beta \in \mathbb{R}$ is such that $\beta > \operatorname{Ind}_0(X, \rho)$. Then (cf. (4.332)) there exist $N \in \mathbb{N}$ with $N \geq 2$ and $\rho' \in [\rho]$ with the property that $\beta > \big(\log_N c_{\rho'}^{(N)}\big)^{-1} \in [0, +\infty)$. Hence, $N^{1/\beta} < c_{\rho'}^{(N)}$, which means that we can select $\gamma \in (0, \beta)$ with the property that $N^{1/\gamma} < c_{\rho'}^{(N)}$. Based on this, (4.331), and (4.328), it follows that if $\kappa := N^{-1/\gamma}$, then

$$\begin{aligned} &\forall\, (x, y) \in X \times X \setminus \operatorname{diag}(X) \quad \exists\, \xi_1, \dots, \xi_{N+1} \in X \text{ distinct such that} \\ &\qquad \xi_1 = x,\ \xi_{N+1} = y \ \text{ and } \ \max\nolimits_{1 \leq i \leq N} \rho'(\xi_i, \xi_{i+1}) \leq \kappa \rho'(x, y). \end{aligned} \tag{4.337}$$

Consider now an arbitrary pair of distinct points in X, say $x, y \in X$ with $x \neq y$. For each number $M \in \mathbb{N}$, making repeated use of (4.337) we obtain a family of points $\{z_i \in X : 1 \leq i \leq N^M + 1\}$ satisfying

$$z_1 = x,\ z_{N^M+1} = y \ \text{ and } \ \rho'(z_i, z_{i+1}) \leq \kappa^M \rho'(x, y) \ \text{ if } \ 1 \leq i \leq N^M. \tag{4.338}$$

Next, using (4.338) and the fact that $\rho' \approx \rho$, we may write

$$\sum_{i=1}^{N^M} \rho(z_i, z_{i+1})^\beta \leq C \sum_{i=1}^{N^M} \rho'(z_i, z_{i+1})^\beta = C\rho'(x, y)^\beta (N\kappa^\beta)^M. \tag{4.339}$$

Note that the fact that $0 < \gamma < \beta$ forces $N\kappa^\beta = N^{1-\beta/\gamma} \in (0, 1)$ and, hence, $(N\kappa^\beta)^M \to 0$ as $M \to +\infty$. Thus, β is a participant in the infimum process described in (4.321) and, as such, we necessarily have $\beta \geq \operatorname{ind}_0(X, \rho)$. Given that this happens for each $\beta \in \mathbb{R}$ satisfying $\beta > \operatorname{Ind}_0(X, \rho)$, we conclude that the middle inequality in (4.333) holds.

Moving on, using the fact that $c_{\rho'}^{(2)} = c_{\rho'}$ for any $\rho' \in \mathfrak{Q}(X)$ [which is seen by comparing (4.253) with (4.331)], we deduce from (4.254) and (4.332) that the last inequality in (4.333) is valid. Next, the first formula in (3.103), together with (4.322), readily yields the claim in (4.334), which in turn proves (4.335). Finally, the first inequality in (4.333) follows from (4.335) and (4.317). □

Remark 4.55. Let $(X, \mathbf{q})$ be a quasimetric space with $\operatorname{ind}(X, \mathbf{q}) = \operatorname{Ind}(X, \mathbf{q}) < +\infty$. Then, if $\alpha_X \in (0, +\infty)$ denotes the common value of these indexes, it follows from Corollary 4.46 and Theorem 4.54 that, for each $\rho \in \mathbf{q}$,

$$\rho_\alpha \approx \rho \text{ for each } \alpha \in (0, \alpha_X) \text{ and } \rho_\alpha \equiv 0 \text{ for each } \alpha \in (\alpha_X, +\infty), \tag{4.340}$$

where ρ_α is defined as in (3.494).

The following comment complements the analysis in Comment 4.47.

Comment 4.56. For any $p \in (0, +\infty)$ one has

$$\mathrm{ind}_0\Big(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\Big) = \min\{1, p\}. \tag{4.341}$$

Indeed, granted (4.320), (4.333), (4.301), and (4.322), it suffices to show that if $p \in (0, 1)$ and $\rho(f, g) := \|f - g\|_{L^p(\mathbb{R})}$ for each $f, g \in L^p(\mathbb{R})$, then

$$\rho_\alpha \equiv 0 \quad \text{for any } \alpha > p. \tag{4.342}$$

With this goal in mind, let $f \in L^p(\mathbb{R})$ be given, and fix an arbitrary $\varepsilon > 0$. By Lebesgue's dominated convergence theorem, there exists $r > 0$ large enough so that

$$\big\|f - f \cdot \mathbf{1}_{[-r,r]}\big\|_{L^p(\mathbb{R})} < \varepsilon/3, \tag{4.343}$$

and there exists $M > 0$ large enough so that

$$\big\|f \cdot \mathbf{1}_{\{x\in[-r,r]:\,|f(x)|>M\}}\big\|_{L^p(\mathbb{R})} < \varepsilon/3. \tag{4.344}$$

Fix $r, M > 0$ such that (4.343) and (4.344) hold. For each $N \in \mathbb{N}$ consider the family of functions $\{f_i\}_{0\leq i\leq N+1}$ given by

$$f_i := \begin{cases} f - f \cdot \mathbf{1}_{[-r,r]} & \text{if } i = 0, \\ f \cdot \mathbf{1}_{\{x\in[-r+(i-1)\frac{2r}{N},-r+i\frac{2r}{N}]:\,|f(x)|\leq M\}} & \text{if } 1 \leq i \leq N, \\ f \cdot \mathbf{1}_{\{x\in[-r,r]:\,|f(x)|>M\}} & \text{if } i = N+1. \end{cases} \tag{4.345}$$

It is straightforward to check that $f = \sum_{i=0}^{N+1} f_i$ and that, in the notation introduced in (4.345), estimates (4.343) and (4.344) imply $\|f_0\|_{L^p(\mathbb{R})} < \varepsilon/3$ and $\|f_{N+1}\|_{L^p(\mathbb{R})} < \varepsilon/3$. From (4.345) we also have

$$\|f_i\|_{L^p(\mathbb{R})} \leq \frac{M(2r)^{1/p}}{N^{1/p}}, \qquad \forall\, i \in \{1, \dots, N\}, \tag{4.346}$$

and consequently

$$\Big(\sum_{i=0}^{N+1} \|f_i\|^\alpha\Big)^{1/\alpha} \leq \Big(2\big(\tfrac{\varepsilon}{3}\big)^\alpha + M \cdot (2r)^{1/p} \cdot N^{1-\alpha/p}\Big)^{1/\alpha}. \tag{4.347}$$

In particular, since $\alpha > p$, it follows that $\Big(\sum_{i=0}^{N+1} \|f_i\|^\alpha\Big)^{1/\alpha} \leq \varepsilon$ if N is large enough. This shows that for each $f \in L^p(\mathbb{R})$ there holds

$$\inf\left\{\Big(\sum_{i=1}^{N}\|f_i\|^{\alpha}\Big)^{\frac{1}{\alpha}}: N\in\mathbb{N},\ f_i\in L^p(\mathbb{R}),\ 1\le i\le N,\ \text{ with } \sum_{i=1}^{N} f_i=f\right\}=0. \tag{4.348}$$

Now the claim (4.342) follows immediately from (4.348) and the fact that for each $\alpha>0$ the quasidistance ρ_α is translation invariant. ■

Definition 4.57. The `Hölder index` of a given a quasimetric space $(X,\mathbf{q})$ is defined as

$$\mathrm{ind}_{\mathrm{H}}(X,\mathbf{q}) := \sup\big\{\alpha\in(0,+\infty): \dot{\mathscr{C}}^{\alpha}(X,\mathbf{q})\neq\mathbb{R}\big\}\in(0,+\infty]. \tag{4.349}$$

Moreover, given an arbitrary set X of cardinality ≥ 2 and given $\rho\in\mathfrak{Q}(X)$, abbreviate $\mathrm{ind}_{\mathrm{H}}(X,\rho):=\mathrm{ind}_{\mathrm{H}}(X,[\rho])$.

A couple of comments are in order here.

Remark 4.58. (i) From Theorem 4.6 we know that the space $\dot{\mathscr{C}}^{\alpha}(X,\mathbf{q})$ contains nonconstant functions if $\alpha>0$ is small enough. Hence, the Hölder index defined in (4.349) is well defined (recall that, unless otherwise noted, we always assume that the cardinality of X is ≥ 2).
(ii) It is clear from Definition 4.57 that for any quasimetric space (X,ρ) there holds

$$\mathrm{ind}_{\mathrm{H}}(X,\rho^{\beta})=\tfrac{1}{\beta}\,\mathrm{ind}_{\mathrm{H}}(X,\rho),\qquad\forall\,\beta\in(0,+\infty). \tag{4.350}$$

Next, we propose to study the significance of the upper and lower smoothness indexes in relation to the structural richness of the space of Hölder functions on a quasimetric space. Our main result pertaining to the notion of Hölder index is contained in the theorem below.

Theorem 4.59. *For any quasimetric space (X,ρ) there holds*

$$\mathrm{ind}_{\mathrm{H}}(X,\rho)=\mathrm{ind}_{0}(X,\rho). \tag{4.351}$$

Proof. As a first step we will show that

$$\alpha\in(0,+\infty)\ \text{ and }\ \rho_\alpha=0\ \text{ on }\ X\times X\implies\dot{\mathscr{C}}^{\alpha}(X,\rho)=\mathbb{R}. \tag{4.352}$$

Seeking a contradiction, assume that there exists $\alpha\in(0,+\infty)$ such that $\rho_\alpha=0$ on $X\times X$ and yet $\dot{\mathscr{C}}^{\alpha}(X,\rho)\neq\mathbb{R}$. Then there exists $f\in\dot{\mathscr{C}}^{\alpha}(X,\rho)$ that is not constant, say $f(x)\neq f(y)$ for some $x,y\in X$. Set $C:=\|f\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)}\in(0,+\infty)$ and introduce $\varepsilon:=\big[C^{-1}|f(x)-f(y)|\big]^{1/\alpha}>0$. Then for every collection of points $\xi_1,\dots,\xi_{N+1}\in X$ such that $\xi_1=x$ and $\xi_{N+1}=y$ we may write

$$C\varepsilon^{\alpha}=|f(x)-f(y)|\le\sum_{i=1}^{N}|f(\xi_i)-f(\xi_{i+1})|\le\sum_{i=1}^{N}C\rho(\xi_i,\xi_{i+1})^{\alpha}. \tag{4.353}$$

In turn, this implies $\Big(\sum_{i=1}^{N}\rho(\xi_i,\xi_{i+1})^{\alpha}\Big)^{1/\alpha}\geq\varepsilon$, which, after taking the infimum over all ξ_i's as above, yields $\rho_\alpha(x,y)\geq\varepsilon>0$. This contradicts the fact that $\rho_\alpha=0$ on $X\times X$, completing the proof of (4.352). Having established (4.352), we have from this and (4.334) that $\mathrm{ind}_{\mathrm{H}}(X,\rho)\leq\mathrm{ind}_0(X,\rho)$.

Hence, to complete the proof of the theorem, it remains to prove the opposite inequality. With this in mind, fix an arbitrary number $\alpha>0$ such that $\mathrm{ind}_0(X,\rho)>\alpha$. Then (4.334) entails that ρ_α does not vanish identically on $X\times X$. In particular, there exist $x_o,y_o\in X$ such that $\rho_\alpha(x_o,y_o)>0$. To proceed, observe that

$$0\leq\rho_\alpha\leq\rho\ \text{ on }\ X\times X, \tag{4.354}$$

$$\rho_\alpha\equiv 0\ \text{ on }\ \mathrm{diag}\,(X), \tag{4.355}$$

$$\rho_\alpha\ \text{ is symmetric}, \tag{4.356}$$

$$\big[\rho_\alpha(x,y)\big]^{\alpha}\leq\big[\rho_\alpha(x,z)\big]^{\alpha}+\big[\rho_\alpha(z,y)\big]^{\alpha}\ \text{ for all }\ x,y,z\in X. \tag{4.357}$$

Indeed, (4.354) follows from property (1) in Lemma 3.14, (4.355) is implied by (4.354), (4.356) is based on property (1) in Lemma 3.22, and (4.357) is a consequence of property (3) in Lemma 3.14.

Let us now define $f:X\to\mathbb{R}$ by setting $f(x):=\big[\rho_\alpha(x,y_o)\big]^{\alpha}$ for every $x\in X$. Then, by design, $f(x_o)=\big[\rho_\alpha(x_o,y_o)\big]^{\alpha}>0$, while $f(y_o)=0$ by (4.355). In particular, the function f is not constant. Moreover, for every $x,y\in X$, we have

$$|f(x)-f(y)|=\big|\big[\rho_\alpha(x,y_o)\big]^{\alpha}-\big[\rho_\alpha(y,y_o)\big]^{\alpha}\big|\leq\big[\rho_\alpha(x,y)\big]^{\alpha}\leq\big[\rho(x,y)\big]^{\alpha}, \tag{4.358}$$

by (4.356), (4.357), and (4.354). This proves that $f\in\dot{\mathscr{C}}^{\alpha}(X,\rho)$. All together, this analysis establishes the existence of a nonconstant function in $\dot{\mathscr{C}}^{\alpha}(X,\rho)$. As such, we conclude that $\mathrm{ind}_{\mathrm{H}}(X,\rho)\geq\alpha$. With this in hand, upon letting $\alpha\nearrow\mathrm{ind}_0(X,\rho)$ yields $\mathrm{ind}_{\mathrm{H}}(X,\rho)\geq\mathrm{ind}_0(X,\rho)$. The two inequalities for indexes then combine to give (4.351). □

Corollary 4.60. *For any quasimetric space* $(X,\mathbf{q})$ *there holds*

$$\mathrm{ind}\,(X,\mathbf{q})\leq\mathrm{ind}_{\mathrm{H}}(X,\mathbf{q})\leq\mathrm{Ind}_0(X,\mathbf{q})\leq\mathrm{Ind}\,(X,\mathbf{q}). \tag{4.359}$$

As a consequence, if $\beta>\mathrm{Ind}_0(X,\mathbf{q})$, *then* $\dot{\mathscr{C}}^{\beta}(X,\mathbf{q})$ *contains only constant functions; in particular, this is the case whenever* $\beta>\mathrm{Ind}\,(X,\mathbf{q})$.

Proof. This is a direct consequence of Theorems 4.59 and 4.54. □

Recall from [59, Sect. 11.1, p. 88] that a *metric space* (X,d) is said to be `uniformly perfect` provided there exists $C\in(1,+\infty)$ so that

$$X\setminus B_d(x,r)\neq\emptyset\implies B_d(x,r)\setminus B_d(x,r/C)\neq\emptyset,\quad\forall\,x\in X,\ \forall\,r\in(0,+\infty). \tag{4.360}$$

Note that the condition of uniform perfectness forbids the existence of islands that are too isolated from the rest of the space, relative to their size, in a scale-invariant fashion. This suggests introducing the following definition.

Definition 4.61. Call a quasimetric space $(X, \mathbf{q})$ `imperfect` provided there exist a quasidistance $\rho \in \mathbf{q}$, a point $x_o \in X$, and a number $r \in (0, +\infty)$, with the property that

$$X \setminus B_\rho(x_o, r) \neq \emptyset \quad \text{and} \quad \inf_{y \in X \setminus B_\rho(x_o,r),\, z \in B_\rho(x_o,r)} \rho(y,z) > 0. \tag{4.361}$$

Proposition 4.62. *If* $(X, \mathbf{q})$ *is an imperfect quasimetric space, then* $\operatorname{ind}_0(X, \mathbf{q}) = +\infty$. *In particular,* $\operatorname{Ind}(X, \mathbf{q}) = +\infty$ *in this case.*

Proof. Seeking a contradiction, suppose that $(X, \mathbf{q})$ is an imperfect quasimetric space for which $\operatorname{ind}_0(X, \mathbf{q}) < +\infty$. It is then possible to pick $\beta \in \mathbb{R}$ such that $\beta > \operatorname{ind}_0(X, \mathbf{q})$. Also, there exist $\rho \in \mathbf{q}$, $x_o \in X$, and $r \in (0, +\infty)$ with the property that (4.361) holds. Then, on the one hand, the function

$$f : X \longrightarrow \mathbb{R}, \quad f \equiv 1 \text{ on } B_\rho(x_o, r) \text{ and } f \equiv 0 \text{ on } X \setminus B_\rho(x_o, r) \tag{4.362}$$

is, thanks to (4.361), a nonconstant function in $\dot{\mathscr{C}}^\beta(X, \rho)$. On the other hand, from Corollary 4.60 we know that $\dot{\mathscr{C}}^\beta(X, \rho)$ contains only constant functions. This contradiction proves the proposition. □

Given a quasimetric space $(X, \mathbf{q})$, consider next the question whether the supremum that defines the lower smoothness index in (4.255) is attained. That is, the issue is whether

$$\exists\, \rho \in \mathbf{q} \text{ such that } [\log_2 C_\rho]^{-1} = \operatorname{ind}(X, \mathbf{q}). \tag{4.363}$$

A first result in this direction is offered by the following theorem, according to which, given a quasimetric space $(X, \mathbf{q})$, a necessary and sufficient condition for the supremum that defines the lower smoothness index of $(X, \mathbf{q})$ to be attained is that

$$\rho_{\operatorname{ind}(X,\mathbf{q})} \approx \rho \text{ for some (or any) } \rho \in \mathbf{q}. \tag{4.364}$$

Theorem 4.63. *Let* $(X, \mathbf{q})$ *be a quasimetric space and set* $\alpha_* := \operatorname{ind}(X, \mathbf{q}) \in (0, +\infty]$. *Then the following statements are equivalent:*

(1) $\rho_{\alpha_*} \approx \rho$ *for each quasidistance* $\rho \in \mathbf{q}$.
(2) There exists a quasidistance $\rho \in \mathbf{q}$ *with the property that* $\rho_{\alpha_*} \approx \rho$.
(3) The supremum that defines the lower smoothness index in (4.255) *is attained in the sense that* (4.363) *holds.*

Proof. Obviously, (1) ⇒ (2). To prove that (2) ⇒ (3), suppose that $\rho \in \mathbf{q}$ is a quasidistance with the property that $\rho_{\alpha_*} \approx \rho$. Then $C_{\rho_{\alpha_*}} \leq 2^{1/\alpha_*}$ by (3.499); hence, further, $\alpha_* \leq [\log_2 C_{\rho_{\alpha_*}}]^{-1} \leq \operatorname{ind}(X, \mathbf{q}) = \alpha_*$, where the second inequality uses (4.255), and the last equality is just the definition of α_*. Thus, (4.363) holds (for the choice $\rho := \rho_{\alpha_*}$), completing the proof of the fact that (2) $\Rightarrow$ (3). Finally, consider the implication (3) $\Rightarrow$ (1). To set the stage, assume that there exists $\rho \in \mathbf{q}$ such that $[\log_2 C_\rho]^{-1} = \alpha_*$ and pick an arbitrary $\rho' \in \mathbf{q}$. Then, on the one hand, $\rho_{\alpha_*} \approx \rho$ by (3.515) and (3.516), while on the other hand, $\rho' \approx \rho$. Consequently, $(\rho')_{\alpha_*} \approx \rho_{\alpha_*} \approx \rho \approx \rho'$ by (3.124), which shows that, ultimately, $(\rho')_{\alpha_*} \approx \rho'$. Thus, (1) holds, and this completes the proof of the theorem. □

A moment's reflection shows that (4.363) is further equivalent to the question of whether

$$\inf\{C_\rho : \rho \in \mathbf{q}\} \quad \text{is attained.} \tag{4.365}$$

Later on, in Corollary 4.69, we will show that there exist quasimetric spaces for which the question posed in (4.365) has a negative answer. For now, our goal is to establish a related positive result, stated in Theorem 4.64 below. As a preamble, we make some comments motivating the setting in the aforementioned theorem. Specifically, let X be an arbitrary set. For each fixed $\lambda \in [1, +\infty)$ consider the following (symmetric and reflexive) relation on $\mathfrak{Q}(X)$:

$$\rho \overset{\lambda}{\approx} \rho' \overset{\text{def}}{\Longleftrightarrow} \lambda^{-1}\rho(x,y) \leq \rho'(x,y) \leq \lambda\rho(x,y) \quad \text{for all} \quad x, y \in X. \tag{4.366}$$

Also, given $\rho \in \mathfrak{Q}(X)$, define

$$[\![\rho]\!]_\lambda := \Big\{\rho' \in \mathfrak{Q}(X) : \rho' \overset{\lambda}{\approx} \rho\Big\}. \tag{4.367}$$

Hence,

$$[\![\rho]\!]_{\lambda_1} \subseteq [\![\rho]\!]_{\lambda_2} \quad \text{whenever} \quad 1 \leq \lambda_1 \leq \lambda_2 < +\infty. \tag{4.368}$$

Recall next that if $\rho \in \mathfrak{Q}(X)$, then $[\rho]$ denotes the equivalence class of ρ with respect to the equivalence relation introduced in Definition 3.15, and note that

$$[\rho] = \bigcup_{\lambda \geq 1} [\![\rho']\!]_\lambda \quad \text{for all} \quad \rho, \rho' \in \mathfrak{Q}(X) \quad \text{satisfying} \quad \rho \approx \rho'. \tag{4.369}$$

In particular, given $\rho \in \mathfrak{Q}(X)$, if $\alpha := (\log_2 C_\rho)^{-1}$ and $\rho_\alpha \in \mathfrak{Q}(X)$ is as in (3.494), then

$$[\rho] = \bigcup_{\lambda \geq 1} [\![\rho_\alpha]\!]_\lambda, \tag{4.370}$$

by (3.516) and (4.369). As a consequence of this and (4.368), we therefore have

$$\inf\{C_{\rho'} : \rho' \in [\![\rho_\alpha]\!]_\lambda\} \searrow \inf\{C_{\rho'} : \rho' \in [\rho]\} \quad \text{as} \quad \lambda \nearrow +\infty. \tag{4.371}$$

The theorem below addresses the issue of whether the individual infima in the left-hand side of (4.371) are attained for each given $\lambda \in [1, +\infty)$.

Theorem 4.64. *Suppose that* (X, ρ) *is a quasimetric space with the property that* (X, τ_ρ) *is a separable topological space. Set* $\alpha := (\log_2 C_\rho)^{-1} \in (0, +\infty]$ *and consider* $\rho_\alpha \in \mathfrak{Q}(X)$ *as in* (3.494). *Then*

$$\inf\big\{C_{\rho'} : \rho' \in [\![\rho_\alpha]\!]_\lambda\big\} \quad \textit{is attained for every} \quad \lambda \in [1, +\infty). \tag{4.372}$$

Proof. Fix some $\lambda \in [1, +\infty)$ and abbreviate $A := \inf\big\{C_{\rho'} : \rho' \in [\![\rho_\alpha]\!]_\lambda\big\} \in [1, +\infty)$. Hence $C_{\rho_\alpha} \geq A$, and if $C_{\rho_\alpha} = A$, then there is nothing to prove (since $\rho_\alpha \in [\![\rho_\alpha]\!]_\lambda$). It remains to consider the case when $A < C_{\rho_\alpha}$. Note that, in this scenario, (4.6) forces $C_\rho \geq C_{\rho_\alpha} > 1$, so that, necessarily, $\alpha \in (0, +\infty)$. Next, select a sequence

$$\{\rho_j\}_{j\in\mathbb{N}} \subseteq [\![\rho_\alpha]\!]_\lambda \ \text{ such that } \ C_{\rho_\alpha} > C_{\rho_j} \searrow A \ \text{ as } \ j \to \infty. \tag{4.373}$$

In particular, $\lambda^{-1}\rho_\alpha \leq \rho_j \leq \lambda\rho_\alpha$ on $X \times X$, for every $j \in \mathbb{N}$, which by (3.501), (3.503), (3.504), and (3.496) further entails

$$\lambda^{-1}\rho_\alpha \leq (\rho_j)_\alpha \leq \lambda\rho_\alpha \leq \lambda\rho \quad \text{on } X \times X \quad \text{for every } \ j \in \mathbb{N}. \tag{4.374}$$

In addition, by (4.373), for each $j \in \mathbb{N}$ and every $x, y, z \in X$ we have

$$\rho_j(x, y) \leq C_{\rho_j} \max\big\{\rho_j(x, z), \rho_j(z, y)\big\} < C_\rho \max\big\{\rho_j(x, z), \rho_j(z, y)\big\}. \tag{4.375}$$

Fix $\beta \in \big(0, \min\{\alpha, 1\}\big)$. We then deduce from (4.375), (3.541), and (4.374) that for each $j \in \mathbb{N}$ and any $x, y, z, w \in X$ we have

$$\begin{aligned}
&\big|(\rho_j)_\alpha(x, y) - (\rho_j)_\alpha(w, z)\big| \\
&\leq \tfrac{1}{\beta} \max\big\{(\rho_j)_\alpha(x, y)^{1-\beta}, (\rho_j)_\alpha(w, z)^{1-\beta}\big\}\Big(\big[(\rho_j)_\alpha(x, w)\big]^\beta + \big[(\rho_j)_\alpha(y, z)\big]^\beta\Big) \\
&\leq \tfrac{\lambda}{\beta} \max\big\{\rho(x, y)^{1-\beta}, \rho(w, z)^{1-\beta}\big\}\big(\rho(x, w)^\beta + \rho(y, z)^\beta\big) \\
&\leq \tfrac{\lambda}{\beta} \max\Big\{\rho(x, y)^{1-\beta}, C_\rho \max\big\{\rho(w, x)^{1-\beta}, \rho(x, z)^{1-\beta}\big\}\Big\}\big(\rho(x, w)^\beta + \rho(y, z)^\beta\big) \\
&\leq \tfrac{\lambda}{\beta} \max\Big\{C_\rho^2\rho(x, y)^{1-\beta}, C_\rho\rho(x, w)^{1-\beta}, C_\rho^2\rho(y, z)^{1-\beta}\Big\}\big(\rho(x, w)^\beta + \rho(y, z)^\beta\big).
\end{aligned} \tag{4.376}$$

Having established this, we may then conclude that the family of functions

$$(\rho_j)_\alpha : \big(X \times X, \tau_\rho \times \tau_\rho\big) \longrightarrow [0, +\infty), \qquad j \in \mathbb{N}, \tag{4.377}$$

is equicontinuous. In addition, the family $\big\{(\rho_j)_\alpha\big\}_{j\in\mathbb{N}}$ is also pointwise bounded on $X \times X$, thanks to (4.374), and our hypotheses imply that $\big(X \times X, \tau_\rho \times \tau_\rho\big)$ is a separable topological space. As such, Ascoli's theorem (cf., e.g., [106, Theorem 33,

p. 179]) may be invoked to conclude that there exists a subsequence $\{(\rho_{j_k})_\alpha\}_{k\in\mathbb{N}}$ of $\{(\rho_j)_\alpha\}_{j\in\mathbb{N}}$ and a continuous function

$$\rho_* : (X \times X, \tau_\rho \times \tau_\rho) \longrightarrow [0, +\infty) \tag{4.378}$$

with the property that

$$\begin{gathered}(\rho_{j_k})_\alpha \longrightarrow \rho_* \quad \text{as} \quad k \to +\infty \\ \text{uniformly on compact subsets of } \ (X \times X, \tau_\rho \times \tau_\rho).\end{gathered} \tag{4.379}$$

In particular,

$$\lim_{k\to\infty} (\rho_{j_k})_\alpha(x,y) = \rho_*(x,y), \qquad \forall\, x,y \in X. \tag{4.380}$$

Collectively, (4.380), (4.374), (4.375), and (4.373) then imply that

$$\rho_* \in [\![\rho_\alpha]\!]_\lambda \quad \text{and} \quad \rho_*(x,y) \le A \max\{\rho_*(x,z), \rho_*(z,y)\}, \quad \forall\, x,y,z \in X. \tag{4.381}$$

Hence, $A \le C_{\rho_*} \le A$, which shows that $A = C_{\rho_*} \in \{C_{\rho'} : \rho' \in [\![\rho_\alpha]\!]_\lambda\}$, as desired. □

Our strategy for proving the existence of quasimetric spaces for which the question formulated in (4.365) has a negative answer (cf. Corollary 4.69) involves relating our lower smoothness index associated with a quasinormed vector space to the so-called modulus of concavity of the space in question. This is accomplished in Theorem 4.68 below, whose statement and proof require some preliminaries, which we dispense with first.

Let X be a vector space. Following [104, p. 89], call a set $U \subseteq X$ `starlike` if $\lambda U \subseteq U$ for every $\lambda \in (0,1]$. The `modulus of concavity` of a starlike set $U \subseteq X$ is then defined as

$$c(U) := \inf\{\lambda \in (0,+\infty) : U + U \subseteq \lambda U\}, \tag{4.382}$$

with the convention that $\inf \emptyset := +\infty$. Call a starlike set $U \subseteq X$ `pseudoconvex` provided $U \neq \emptyset$ and $c(U) < +\infty$.

Lemma 4.65. *Let X be a vector space.*

(i) For any pseudoconvex set $U \subseteq X$ one has

$$U + U \subseteq c(U)U. \tag{4.383}$$

(ii) For any pseudoconvex set $U \subseteq X$ such that $U \neq \{0\}$ one has $c(U) \ge 2$.
(iii) If $U \subseteq X$ is convex and $U \neq \{0\}$, then $c(U) = 2$.
(iv) If X is a topological vector space and $U \subseteq X$ is a starlike set that is either closed or open and satisfies $c(U) = 2$, then U is convex.

Proof. The aforementioned properties follow easily from definitions (cf. also [104, pp. 89–90] for more details). □

Recall that a topological vector space is said to be locally bounded provided there exists a (topologically) bounded neighborhood of the zero vector (i.e., the set in question is absorbed by each neighborhood of zero). Also, a set U in a vector space is said to be balanced if $\lambda U \subseteq U$ for every scalar λ with $|\lambda| \leq 1$. It follows that any nonempty, open, (topologically) bounded, balanced set $U \subseteq X$ is pseudoconvex.

Definition 4.66. Assume that (X, τ) is a locally bounded topological vector space. Define `Rolewicz's logarithmic modulus of concavity` of the space (X, τ) as

$$\operatorname{ind}_{\mathrm{R}}(X,\tau) := \sup\Big\{\big(\log_2 c(U)\big)^{-1} : 0 \in U \subseteq X, \\ U \text{ open, (topologically) bounded, and balanced}\Big\}. \tag{4.384}$$

The rationale for the piece of terminology is that, up to taking the logarithm and reciprocating, the index introduced in Definition 4.66 coincides with the notion of `modulus of concavity` of a locally bounded topological vector space (X, τ) defined by Rolewicz in [104, p. 96] as

$$c(X,\tau) := \inf\Big\{c(U) : 0 \in U \subseteq X \\ U \text{ open, (topologically) bounded, and balanced}\Big\}. \tag{4.385}$$

See also [74, p. 161] in this regard. Given a locally bounded topological vector space (X, τ), the significance of the number $\operatorname{ind}_{\mathrm{R}}(X, \tau)$ stems from the observation (cf., e.g., [74, (5) p. 161]) that

$$\text{for each } p \in \big(0, \operatorname{ind}_{\mathrm{R}}(X,\tau)\big) \text{ there exists a } p\text{-norm} \\ \text{on } X \text{ that yields the same topology on } X \text{ as } \tau. \tag{4.386}$$

Definition 4.67. Given a vector space X, call a quasidistance $\rho \in \mathfrak{Q}(X)$ `translation invariant` provided

$$\rho(x+z, y+z) = \rho(x,y) \quad \text{for any} \quad x, y, z \in X. \tag{4.387}$$

Also, call $\rho \in \mathfrak{Q}(X)$ `homogeneous` provided

$$\rho(\lambda x, \lambda y) = |\lambda|\rho(x,y) \quad \text{for any} \quad \lambda \in \mathbb{R}. \tag{4.388}$$

Moving on, assume that $(X, \|\cdot\|)$ is a quasinormed vector space. Then the function

$$\rho_{\|\cdot\|} : X \times X \to [0, +\infty), \quad \rho_{\|\cdot\|}(x,y) := \|x-y\|, \quad \forall\, x, y \in X, \tag{4.389}$$

is a quasidistance on X [i.e., $\rho_{\|\cdot\|} \in \mathfrak{Q}(X)$] and has the property that $\tau_{\rho_{\|\cdot\|}}$ coincides with the topology $\tau_{\|\cdot\|}$ canonically induced by the quasinorm $\|\cdot\|$ on X. Furthermore, $\rho_{\|\cdot\|}$ is translation invariant and homogeneous (in the sense of (4.387) and (4.388)).

Theorem 4.68. *For any quasinormed vector space* $(X, \|\cdot\|)$ *there holds*

$$\text{ind}\big(X, \rho_{\|\cdot\|}\big) = \text{ind}_{\text{R}}(X, \tau_{\|\cdot\|}) \in (0, 1]. \tag{4.390}$$

Furthermore, the supremum defining the lower smoothness index on the left-hand side of (4.390) *(compare with* (4.255)*) is attained if and only if the supremum defining Rolewicz's logarithmic modulus of concavity on the right-hand side of* (4.390) *(cf.* (4.384)*) is attained.*

Proof. That $\text{ind}_{\text{R}}(X, \tau_{\|\cdot\|}) \in (0, 1]$ is clear from (4.384) and item (*ii*) in Lemma 4.65. We divide the remaining part of the proof into a number of steps.

`Step 1.` *Assume that* (X, τ) *is a topological vector space and suppose that* $U \subseteq X$ *is an open, (topologically) bounded, balanced set (in particular, a neighborhood of the zero vector* $0 \in X$*). If* $\|\cdot\|_U$ *stands for the Minkowski gauge function of* U*, i.e.,*

$$\|x\|_U := \inf\big\{\lambda \in (0, +\infty) : x/\lambda \in U\big\}, \quad \forall\, x \in X, \tag{4.391}$$

then

$$x \in U \iff \|x\|_U < 1, \quad \forall\, x \in X, \tag{4.392}$$

$$\|x + y\|_U \le c \cdot \max\big\{\|x\|_U, \|y\|_U\big\},\ \forall\, x, y \in X \iff c \ge c(U), \tag{4.393}$$

$$\|\cdot\|_U \text{ is a quasinorm on } X, \text{ and } \|\cdot\|_U \approx \|\cdot\|, \tag{4.394}$$

$$C_{\rho_{\|\cdot\|_U}} = c(U), \tag{4.395}$$

where $\rho_{\|\cdot\|_U}$ *is as in* (4.389) *corresponding to the quasinorm* (4.391) *(recall that, in general,* C_ρ *was introduced in* (4.2)*).*

To justify the first claim, fix an arbitrary number $x \in X$ and define $f : \mathbb{R} \to X$ by setting $f(\lambda) := \lambda x$ for all $\lambda \in \mathbb{R}$. Then, since (X, τ) is a topological vector space, it follows that f is continuous. Moreover, if $x \in U$, then $f(1) \in U$, and since U is open in (X, τ), we deduce that there exists $\varepsilon \in (0, 1)$ such that $f(\lambda) \in U$ for every $\lambda \in (1 - \varepsilon, 1 + \varepsilon)$. In particular, there exists $\lambda > 1$ such that $\lambda x \in U$, which shows that $\|x\|_U \le \lambda^{-1} < 1$, as desired. Conversely, if $\|x\|_U < 1$, pick $\lambda \in \big(\|x\|_U, 1\big)$. Then $\|x\|_U < \lambda$ implies the existence of some $\lambda_o \in (0, \lambda)$ such that $x/\lambda_o \in U$. Now, the fact that U is balanced entails $x = \lambda_o(x/\lambda_0) \in U$, given that $\lambda_0 \in (0, 1)$. This completes the proof of (4.392).

Assume next that $x, y \in X$ are given, and fix an arbitrary number $\lambda_o > \max\big\{\|x\|_U, \|y\|_U\big\}$. Then there exists $\lambda \in (0, \lambda_o)$ with the property that $x/\lambda, y/\lambda \in$

U. In turn, this implies $(x+y)/\lambda \in U+U \subseteq c(U)U$, so that $(x+y)/(\lambda c(U)) \in U+U \subseteq U$. Consequently, $\|x+y\|_U \leq \lambda c(U) < \lambda_0 c(U)$. Since the number $\lambda_o > \max\{\|x\|_U, \|y\|_U\}$ was arbitrarily chosen, this proves that

$$\|x+y\|_U \leq c(U)\,\max\{\|x\|_U, \|y\|_U\}, \qquad \forall\, x, y \in X. \tag{4.396}$$

In particular, $\|x+y\|_U \leq c\,\max\{\|x\|_U, \|y\|_U\}$ for all $x, y \in X$, provided $c \geq c(U)$. Assume next that $c \in (0, +\infty)$ is such that $\|x+y\|_U \leq c\,\max\{\|x\|_U, \|y\|_U\}$ for all $x, y \in X$. Taking $x, y \in U$ arbitrary, it follows from (4.392) that $\|x\|_U < 1$ and $\|y\|_U < 1$. Also, by the assumption just made, $\|x+y\|_U \leq c$. Hence, if $c' > c$ is arbitrary, then we deduce from this and (4.392) that $(x+y)/c' \in U$ or, equivalently, $x+y \in c'U$. Since $x, y \in U$ were arbitrarily chosen, this proves that $U+U \subseteq c'U$. With this in hand, (4.382) gives $c(U) \leq c'$. Since $c' > c$ was arbitrary, we arrive at the conclusion that $c \geq c(U)$, as desired. This completes the proof of (4.393).

Going further, we claim that

$$\|\lambda x\|_U = |\lambda|\|x\|_U, \quad \forall\, x \in X,\ \forall\, \lambda \in \mathbb{R}. \tag{4.397}$$

Indeed, this is clear if $\lambda = 0$, whereas for $\lambda \in \mathbb{R} \setminus \{0\}$ we may write (for each $x \in X$)

$$\begin{aligned} \|\lambda x\|_U &= \inf\{\eta > 0 : \lambda x/\eta \in U\} = \inf\{\eta > 0 : x/(\eta/\lambda) \in U\} \\ &= \inf\{\eta > 0 : x/(\eta/|\lambda|) \in U\} = |\lambda|\,\inf\{\xi > 0 : x/\xi \in U\} \\ &= |\lambda|\|x\|_U, \end{aligned} \tag{4.398}$$

where the first equality is a consequence of (4.391), the second one is obvious, the third one uses the fact that U is balanced, and the last one is seen by introducing $\xi := \eta/|\lambda| > 0$. This proves (4.397).

Consider next $B := \{x \in X : \|x\| < 1\}$. Then B is a bounded neighborhood of the zero vector and, since U is (topologically) bounded, there exists $M \in (0, +\infty)$ such that $U/M \subseteq B$. In turn, this entails $\|x\| \leq M$ for every $x \in U$. Fix now $x \in X$. Then, if $\lambda > \|x\|_U$ is arbitrary, it follows that there exists $\lambda_o \in (0, \lambda)$ such that $x/\lambda_o \in U$. Hence, by what we have just proved, $\|x/\lambda_o\| \leq M$ or, equivalently, $\|x\| \leq \lambda_o M$. Hence, $\|x\| \leq \lambda M$, and since $\lambda > \|x\|_U$ was arbitrarily chosen, we may conclude that $\|x\| \leq M\|x\|_U$. Conversely, since U is a neighborhood of the origin, there exists $\varepsilon > 0$ with the property that $\{x \in X : \|x\| < 2\varepsilon\} \subseteq U$. Thus, for any $x \in X \setminus \{0\}$ we have $\varepsilon x/\|x\| \in U$, which, in light of (4.391), further implies that $\|x\|_U \leq \varepsilon^{-1}\|x\|$. At this stage, the results established so far permit us to conclude that the claims made in (4.394) hold.

We are left with proving (4.395). In one direction, observe that for each triplet $x, y, z \in X$ we have

$$\begin{aligned}\rho_{\|\cdot\|_U}(x,y) &= \|x-y\|_U = \|(x-z)-(y-z)\|_U \\ &\le c(U)\cdot\max\{\|x-z\|_U, \|y-z\|_U\} \\ &= c(U)\cdot\max\{\rho_{\|\cdot\|_U}(x,z), \rho_{\|\cdot\|_U}(z,y)\},\end{aligned} \tag{4.399}$$

which proves that $C_{\rho_{\|\cdot\|_U}} \le c(U)$. In the opposite direction, start from the fact that $\rho_{\|\cdot\|_U}(x,y) \le C_{\rho_{\|\cdot\|_U}} \max\{\rho_{\|\cdot\|_U}(x,z), \rho_{\|\cdot\|_U}(z,y)\}$ for each $x, y, z \in X$. Specializing this to the case where $z = 0$ and y is replaced by $-y$ yields

$$\|x+y\|_U \le C_{\rho_{\|\cdot\|_U}} \cdot \max\{\|x\|_U, \|y\|_U\}, \qquad \forall\, x, y \in X. \tag{4.400}$$

With this in hand, (4.393) then gives $C_{\rho_{\|\cdot\|_U}} \ge c(U)$, which completes the proof of (4.395).

`Step 2.` *Assume that X is a vector space and suppose that $\rho \in \mathfrak{Q}(X)$ is a quasidistance that is quasi translation invariant and quasihomogeneous, in the sense that there exists $C \in [1, +\infty)$ such that*

$$\rho(x+z, y+z) \le C\rho(x,y) \quad \text{and} \quad \rho(\lambda x, \lambda y) \le C|\lambda|\rho(x,y) \tag{4.401}$$

for every $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then

$$\rho_\star(x,y) := \sup_{z\in X,\, \lambda\in\mathbb{R}\setminus\{0\}} \Big[|\lambda|\rho\big(\lambda^{-1}x+z, \lambda^{-1}y+z\big)\Big], \qquad \forall\, x, y \in X, \tag{4.402}$$

satisfies (cf. Definition 4.67)

$$\rho_\star \in \mathfrak{Q}(X), \qquad \rho_\star \approx \rho, \qquad C_{\rho_\star} \le C_\rho, \text{ and} \tag{4.403}$$

$$\rho_\star \text{ is translation invariant and homogeneous.} \tag{4.404}$$

From the definition of $\rho_\star$ and (4.401) one readily obtains that $\rho \le \rho_\star \le C^2\rho$ on $X \times X$. Moreover, since ρ is symmetric, so is $\rho_\star$, and, as such, it follows that $\rho_\star \in \mathfrak{Q}(X)$ and $\rho_\star \approx \rho$. Next, for each $x, y, z, w \in X$ and $\lambda \in \mathbb{R} \setminus \{0\}$ we write

$$\begin{aligned}&|\lambda|\rho\big(\lambda^{-1}x+z, \lambda^{-1}y+z\big) \\ &\quad \le C_\rho \max\big\{|\lambda|\rho\big(\lambda^{-1}x+z, \lambda^{-1}w+z\big), |\lambda|\rho\big(\lambda^{-1}w+z, \lambda^{-1}y+z\big)\big\} \\ &\quad \le C_\rho \max\big\{\rho_\star(x,w), \rho_\star(w,y)\big\}.\end{aligned} \tag{4.405}$$

Taking the supremum over all $z \in X$ and $\lambda \in \mathbb{R} \setminus \{0\}$ then yields

$$\rho_\star(x,y) \le C_\rho \max\{\rho_\star(x,w), \rho_\star(w,y)\}, \qquad \forall\, x,y,w \in X, \tag{4.406}$$

hence $C_{\rho_\star} \le C_\rho$, as desired. To prove that $\rho_\star$ is translation invariant, note that for each $x, y, w \in X$ we have

$$\begin{aligned}\rho_\star(x+w, y+w) &= \sup_{z\in X,\,\lambda\in\mathbb{R}\setminus\{0\}} \Big[|\lambda|\rho\big(\lambda^{-1}x + (\lambda^{-1}w+z), \lambda^{-1}y + (\lambda^{-1}w+z)\big)\Big] \\ &= \sup_{v\in X,\,\lambda\in\mathbb{R}\setminus\{0\}} \Big[|\lambda|\rho\big(\lambda^{-1}x + v, \lambda^{-1}y + v\big)\Big] \\ &= \rho_\star(x,y),\end{aligned} \tag{4.407}$$

as desired. Finally, the fact that $\rho_\star$ is homogeneous follows by writing, for each $x, y \in X$ and each $\eta \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned}\rho_\star(\eta x, \eta y) &= \sup_{z\in X,\,\lambda\in\mathbb{R}\setminus\{0\}} \Big[|\lambda|\rho\big((\lambda/\eta)^{-1}x + z, (\lambda/\eta)^{-1}y + z\big)\Big] \\ &= |\eta| \sup_{z\in X,\,\xi\in\mathbb{R}\setminus\{0\}} \Big[|\xi|\rho\big(\xi^{-1}x + z, \xi^{-1}y + z\big)\Big] \\ &= |\eta|\rho_\star(x,y).\end{aligned} \tag{4.408}$$

This completes the treatment of the claims made in Step 2.

Step 3. *Let X be a vector space and assume that $\rho \in \mathfrak{Q}(X)$ is a translation-invariant and homogeneous quasidistance. Then*

$$\rho_\# \text{ is also translation invariant and homogeneous.} \tag{4.409}$$

This follows directly from the definition of $\rho_\#$ from (3.528), (3.494), and (3.495), by observing that, given any two vectors $x, y \in X$, if $\xi_1, \dots, \xi_{N+1} \in X$ is a "chain" between x and y (i.e., $\xi_1 = x$ and $\xi_{N+1} = y$), then for any $z \in X$ the family $\xi_1 + z, \dots, \xi_{N+1} + z$ constitutes a chain between $x + z$ and $y + z$, while for every $\lambda \in \mathbb{R}$ the family $\lambda\xi_1, \dots, \lambda\xi_{N+1}$ forms a chain between λx and λy.

Step 4. *For any quasinormed vector space $(X, \|\cdot\|)$ there holds*

$$\operatorname{ind}\big(X, \rho_{\|\cdot\|}\big) \ge \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}). \tag{4.410}$$

To justify this claim, observe that if $U \subseteq X$ is an open, (topologically) bounded, balanced neighborhood of the zero vector $0 \in X$, then $\rho_{\|\cdot\|_U} \in \mathfrak{Q}(X)$, $\rho_{\|\cdot\|_U} \approx \rho_{\|\cdot\|}$ by (4.394), hence

$$\operatorname{ind}\big(X, \rho_{\|\cdot\|}\big) \ge \big(\log_2 C_{\rho_{\|\cdot\|_U}}\big)^{-1} = \big(\log_2 c(U)\big)^{-1}, \tag{4.411}$$

by (4.395). Then (4.410) follows by taking the supremum in (4.411) over all such U.

`Step 5.` *For any quasinormed vector space* $(X, \|\cdot\|)$ *there holds*

$$\operatorname{ind}\big(X, \rho_{\|\cdot\|}\big) \leq \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}). \tag{4.412}$$

To see this, pick an arbitrary $\rho' \in \mathfrak{Q}(X)$ with the property that $\rho' \approx \rho_{\|\cdot\|}$. Since $\rho_{\|\cdot\|}$ is translation invariant and homogeneous, it follows that ρ' is quasi translation invariant and quasihomogeneous. Consider $(\rho'_\star)_\# \in \mathfrak{Q}(X)$, the regularization as in (3.528), (3.494), and (3.495), of the quasidistance $\rho'_\star$ (in turn, defined according to formula (4.402), relative to ρ'); then introduce

$$\|x\|' := (\rho'_\star)_\#(x, 0), \qquad \forall\, x \in X. \tag{4.413}$$

From the results proved in Steps 2–3 it follows that $\|\cdot\|'$ is a quasinorm on X and, thanks to (3.530), (4.403), definitions, and assumptions, we have

$$\|x\|' = (\rho'_\star)_\#(x, 0) \approx \rho'_\star(x, 0) \approx \rho'(x, 0) \approx \rho_{\|\cdot\|}(x, 0) = \|x\| \tag{4.414}$$

uniformly in $x \in X$. Hence,

$$\|\cdot\|' \approx \|\cdot\|. \tag{4.415}$$

At this stage, introduce

$$U := \{x \in X : \|x\|' < 1\} \subseteq X. \tag{4.416}$$

Since $\|\cdot\|'$ is a quasinorm that is equivalent to the original $\|\cdot\|$, it follows that U is a (topologically) bounded, balanced subset of X, which contains the zero vector. Also, since the function $(\rho'_\star)_\# : X \times X \to \mathbb{R}$ is continuous when $X \times X$ is equipped with the product topology $\tau_{\|\cdot\|} \times \tau_{\|\cdot\|}$ (cf. (3.540)), and since $U = (\rho'_\star)_\#(\cdot, 0)^{-1}(-\infty, 1)$, we deduce that U is open in $(X, \tau_{\|\cdot\|})$. Moreover,

$$\|\cdot\|_U = \|\cdot\|' \tag{4.417}$$

since

$$\begin{aligned} \|x\|_U &= \inf\{\lambda > 0 : x/\lambda \in U\} = \inf\{\lambda > 0 : \|x/\lambda\|' < 1\} \\ &= \inf\{\lambda > 0 : \|x\|' < \lambda\} = \|x\|', \qquad \forall\, x \in X, \end{aligned} \tag{4.418}$$

and

$$\rho_{\|\cdot\|'} = (\rho'_\star)_\# \tag{4.419}$$

since by (4.413) and (4.409)

$$\rho_{\|\cdot\|'}(x, y) = \|x - y\|' = (\rho'_\star)_\#(x - y, 0) = (\rho'_\star)_\#(x, y), \quad \forall\, x, y \in X. \tag{4.420}$$

Consequently,

$$c(U) = C_{\rho_{\|\cdot\|_U}} = C_{\rho_{\|\cdot\|'}} = C_{(\rho'_\star)_\#} \leq C_{\rho'_\star} \leq C_{\rho'}, \tag{4.421}$$

by (4.395), (4.417), (4.419), (4.14), and (4.403). As a result,

$$\operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}) \geq \big(\log_2 c(U)\big)^{-1} \geq \big(\log_2 C_{\rho'}\big)^{-1}, \tag{4.422}$$

and taking the supremum over all $\rho' \in \mathfrak{Q}(X)$ with the property that $\rho' \approx \rho_{\|\cdot\|}$ yields (4.412).

Together, (4.410) and (4.412) prove (4.390).

`Step 6.` *The proof of the last claim in the statement of Theorem 4.68.* Suppose that the supremum defining Rolewicz's logarithmic modulus of concavity on the right-hand side of (4.390) is attained. Thus, by (4.384), there exists a U_* open, (topologically) bounded, balanced neighborhood of the zero vector in X, with the property that

$$\big(\log_2 c(U_*)\big)^{-1} = \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}). \tag{4.423}$$

Then, based on this, (4.390), and (4.411), it follows that

$$\begin{aligned} \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}) &= \operatorname{ind}(X, \rho_{\|\cdot\|}) \geq \big(\log_2 C_{\rho_{\|\cdot\|_{U_*}}}\big)^{-1} \\ &= \big(\log_2 c(U_*)\big)^{-1} = \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}). \end{aligned} \tag{4.424}$$

Hence, necessarily,

$$\big(\log_2 C_{\rho_{\|\cdot\|_{U_*}}}\big)^{-1} = \operatorname{ind}(X, \rho_{\|\cdot\|}), \tag{4.425}$$

which, given that $\rho_{\|\cdot\|_{U_*}} \in \mathfrak{Q}(X)$ satisfies $\rho_{\|\cdot\|_{U_*}} \approx \rho_{\|\cdot\|}$ (cf. (4.394)), shows that the supremum defining the lower smoothness index on the left-hand side of (4.390) (cf. (4.255)) is also attained.

Conversely, suppose that the supremum defining the lower smoothness index on the left-hand side of (4.390) is attained. Then there exists $\rho' \in \mathfrak{Q}(X)$ satisfying $\rho' \approx \rho_{\|\cdot\|}$ and with the property that

$$\big(\log_2 C_{\rho'}\big)^{-1} = \operatorname{ind}(X, \rho_{\|\cdot\|}). \tag{4.426}$$

If U is as in (4.416) (with $\|\cdot\|'$ as in (4.413)), then it follows that U is an open, (topologically) bounded, balanced neighborhood of the zero vector in X. Then, based on (4.390), (4.426), and (4.422), we may write

$$\operatorname{ind}(X, \rho_{\|\cdot\|}) = \operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}) \geq \big(\log_2 c(U)\big)^{-1} \geq \big(\log_2 C_{\rho'}\big)^{-1} = \operatorname{ind}(X, \rho_{\|\cdot\|}). \tag{4.427}$$

This proves that, necessarily, $\operatorname{ind}_{\mathrm{R}}(X, \tau_{\|\cdot\|}) = \big(\log_2 c(U)\big)^{-1}$, and the desired conclusion follows. □

Having dealt with Theorem 4.68 we may now return to the question raised in (4.365) and prove the following result.

Corollary 4.69. *There exists a quasimetric space* $(X, \mathbf{q})$ *for which the question posed in* (4.365) *has a negative answer.*

Proof. For each given $p_o \in (0, 1]$, an example of a locally bounded topological vector space (X, τ) was constructed in [104, Proposition 3.4.8, p. 115], which has the following property (recall (4.382) and (4.385)):

$$c(X,\tau) = 2^{1/p_o}, \text{ but } c(U) > 2^{1/p_o} \text{ for any open, bounded, starlike set } U \subseteq X. \tag{4.428}$$

See also the discussion in [74, p. 162] in this regard. Granted this, the fact that the question posed in (4.365) has a negative answer in general follows from the last part in the statement of Theorem 4.68. □

In the last part of this section we study the correlation between the size of the lower smoothness index of a quasimetric space and the Hausdorff dimension of the space itself or the Hausdorff dimension of continuous paths joining various points in the space in question. We begin this discussion by establishing the following definition.

Definition 4.70. Let (X, ρ) be a quasimetric space, and fix $d \geq 0$. Given a set $E \subseteq X$, for every $\delta > 0$ define

$$\mathcal{H}^d_{X,\rho,\delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} r_j^d : E \subseteq \bigcup_{j=1}^{\infty} B_\rho(x_j, r_j) \text{ and } r_j \leq \delta \text{ for every } j \right\} \tag{4.429}$$

(with the convention that $\inf \emptyset := +\infty$); then define the d-`dimensional Hausdorff outer measure` in (X, ρ) of the set E as

$$\mathcal{H}^d_{X,\rho}(E) := \lim_{\delta \to 0+} \mathcal{H}^d_{X,\rho,\delta}(E) = \sup_{\delta > 0} \mathcal{H}^d_{X,\rho,\delta}(E) \in [0, +\infty]. \tag{4.430}$$

Also, define the `Hausdorff dimension` in (X, ρ) of the set E by the formula

$$\dim^{\mathcal{H}}_{X,\rho}(E) := \inf \big\{ d \geq 0 : \mathcal{H}^d_{X,\rho}(E) = 0 \big\}, \tag{4.431}$$

again with the convention that $\inf \emptyset := +\infty$.

In the context of Definition 4.70, observe that for each $\delta > 0$ and $\gamma > 0$ we have

$$\mathcal{H}^d_{X,\rho^\gamma,\delta}(E) = \mathcal{H}^{d\gamma}_{X,\rho,\delta/\gamma}(E), \qquad \forall\, E \subseteq X; \tag{4.432}$$

hence,

$$\mathcal{H}^{d}_{X,\rho^\gamma}(E) = \mathcal{H}^{d\gamma}_{X,\rho}(E), \qquad \forall\, E \subseteq X. \tag{4.433}$$

Recall next that a topological space (X, τ) is called `pathwise connected` provided that for any $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. We will refer to the set $\Gamma := f\big([0, 1]\big) \subseteq X$ as a continuous path joining x and y.

Theorem 4.71. *Let (X, ρ) be a quasimetric space with the property that there exists $d \in [0, +\infty)$ satisfying*

$$\forall\, x, y \in X \ \ \exists\, \Gamma \ \ \textit{continuous path joining } x \textit{ and } y \textit{ with } \dim^{\mathcal{H}}_{X,\rho}(\Gamma) \le d \tag{4.434}$$

(thus, in particular, the topological space (X, τ_ρ) is pathwise connected). Then

$$\mathrm{ind}_0(X, \rho) \le d. \tag{4.435}$$

In the proof of this theorem the following lemma will be useful.

Lemma 4.72. *Assume that $(O_i)_{i\in I}$ is a family of open subsets of $\mathbb{R}$ with the property that $[0, 1] \subseteq \bigcup_{i\in I} O_i$. Then there exist $N \in \mathbb{N} \cup \{0\}$, $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1$, and a function $\sigma : \{1, \dots, N + 1\} \to I$ such that*

$$t_0 \in O_{\sigma(1)}, \quad t_{N+1} \in O_{\sigma(N+1)}, \quad t_j \in O_{\sigma(j)} \cap O_{\sigma(j+1)}, \quad \forall\, j \in \{1, \dots, N\}. \tag{4.436}$$

Proof. Invoking Lebesgue's number lemma (cf., e.g., [27]), it is possible to select a small $r > 0$ with the property that

$$\forall\, t \in [0, 1] \ \ \exists\, i_t \in I \ \ \text{such that} \ \ [t - r, t + r] \subseteq O_{i_t}. \tag{4.437}$$

Then take $N \in \mathbb{N}$ large enough so that there exist $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1$ satisfying

$$t_{j+1} - t_j < r, \qquad \forall\, j \in \{0, 1, \dots, N\}. \tag{4.438}$$

With the correspondence $[0, 1] \ni t \mapsto i_t \in I$ as in (4.437), set $\sigma(j) := i_{t_{j-1}}$ for each $j \in \{1, \dots, N + 1\}$. Thanks to (4.437) and (4.438), this ensures that

$$t_j \in [t_{j-1} - r, t_{j-1} + r] \subseteq O_{\sigma(j)}, \quad \forall\, j \in \{1, \dots, N + 1\}. \tag{4.439}$$

Since this forces $t_j \in O_{\sigma(j)} \cap O_{\sigma(j+1)}$ for each $j \in \{1, \dots, N\}$, the desired conclusion follows. □

After this preamble, we return to Theorem 4.71.

Proof of Theorem 4.71. Fix two distinct points $x, y \in X$ and pick an arbitrary number $\varepsilon > 0$. Also, let $f : [0, 1] \to (X, \tau_\rho)$ be a continuous function with the property that $f(0) = x$ and $f(1) = y$. Set $\Gamma := f\big([0,1]\big)$, fix an arbitrary number $d' > d$, and note that this forces $\mathcal{H}^{d'}_{X,\rho}(\Gamma) = 0$. Based on this and (4.430), we then obtain $\mathcal{H}^{d'}_{X,\rho,\delta}(\Gamma) = 0$ for every $\delta > 0$. In concert with (4.429), this allows us to conclude that, with some $\delta > 0$ having been fixed, there exist $\{x_j\}_{j\in\mathbb{N}} \subseteq X$ and $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, \delta]$ with the property that

$$\Gamma \subseteq \bigcup_{j=1}^{\infty} B_\rho(x_j, r_j) \quad \text{and} \quad \sum_{j=1}^{\infty} r_j^{d'} < \varepsilon / C_\rho^3. \tag{4.440}$$

Upon recalling the fact that $\rho_\# \leq \rho$, the first formula in (4.440) yields

$$\Gamma \subseteq \bigcup_{j=1}^{\infty} B_{\rho_\#}(x_j, r_j). \tag{4.441}$$

Hence, if for every $j \in \mathbb{N}$ we define $O_j := f^{-1}\big(B_{\rho_\#}(x_j, r_j)\big)$, then it follows that each O_j is a relatively open subset of $[0, 1]$ and $[0, 1] \subseteq \bigcup_{j\in\mathbb{N}} O_j$. Bring in Lemma 4.72 according to which there exist $N \in \mathbb{N} \cup \{0\}$, $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1$, and a function $\sigma : \{1, \ldots, N+1\} \to \mathbb{N}$ such that

$$t_0 \in O_{\sigma(1)}, \quad t_{N+1} \in O_{\sigma(N+1)}, \quad t_i \in O_{\sigma(i)} \cap O_{\sigma(i+1)}, \quad \forall\, i \in \{1, \ldots, N\}. \tag{4.442}$$

To proceed, introduce $\xi_i := f(t_i) \in X$ for each $i \in \{0, 1, \ldots, N+1\}$ so that, in particular, $\xi_0 = x$ and $\xi_{N+1} = y$. Moreover, for each $i \in \{1, \ldots, N\}$,

$$\xi_i \in f\big(O_{\sigma(i)}\big) \cap f\big(O_{\sigma(i+1)}\big) \subseteq B_{\rho_\#}(x_{\sigma(i)}, r_{\sigma(i)}) \cap B_{\rho_\#}(x_{\sigma(i+1)}, r_{\sigma(i+1)}), \tag{4.443}$$

so that

$$\xi_i,\ \xi_{i+1} \in B_{\rho_\#}(x_{\sigma(i+1)}, r_{\sigma(i+1)}), \qquad \forall\, i \in \{0, 1, \ldots, N\}. \tag{4.444}$$

Consider now the ordered sequence of balls

$$B_{\rho_\#}(x_{\sigma(1)}, r_{\sigma(1)}),\ B_{\rho_\#}(x_{\sigma(2)}, r_{\sigma(2)}), \ldots, B_{\rho_\#}(x_{\sigma(N+1)}, r_{\sigma(N+1)}). \tag{4.445}$$

Since the mapping $\sigma : \{1, \ldots, N+1\} \to \mathbb{N}$ is not necessarily injective, this may contain repetitions. Concretely, suppose that the integers i, j are such that $1 \leq i < j \leq N+1$ and $B_{\rho_\#}(x_{\sigma(i)}, r_{\sigma(i)}) = B_{\rho_\#}(x_{\sigma(j)}, r_{\sigma(j)})$. If this is the case, throw away $B_{\rho_\#}(x_{\sigma(j)}, r_{\sigma(j)})$ and discard the points $\xi_i, \xi_{i+1}, \ldots, \xi_{j-1}$. Repeat this process (finitely many times) until all repetitions have been eliminated, then relabel the remaining ξ_i in a consecutive manner. This yields a sequence of points $\xi_0, \xi_1, \ldots, \xi_{M+1} \in X$, where $M \in \{1, \ldots, N\}$, with the property that $\xi_0 = x$, $\xi_{M+1} = y$, and

$$\xi_i,\ \xi_{i+1}\in B_{\rho_\#}(x_{\tilde\sigma(i+1)},r_{\tilde\sigma(i+1)}),\qquad \forall\, i\in\{0,1,\dots,M\},\tag{4.446}$$

where, this time,

$$\tilde\sigma:\{1,\dots,M+1\}\to\mathbb{N}\quad\text{ is an injective function.}\tag{4.447}$$

Note that (4.446) entails

$$\begin{aligned}\rho(\xi_i,\xi_{i+1})&\le C_\rho^2\rho_\#(\xi_i,\xi_{i+1})\le C_\rho^3\max\bigl\{\rho_\#(\xi_i,x_{\tilde\sigma(i+1)}),\rho_\#(x_{\tilde\sigma(i+1)},\xi_{i+1})\bigr\}\\&\le C_\rho^3 r_{\tilde\sigma(i+1)},\quad \forall\, i\in\{0,1,\dots,M\}.\end{aligned}\tag{4.448}$$

Hence,

$$\sum_{i=0}^{M}\rho(\xi_i,\xi_{i+1})^{d'}\le C_\rho^3\sum_{i=0}^{M}\bigl(r_{\tilde\sigma(i+1)}\bigr)^{d'}\le C_\rho^3\sum_{j\in\mathbb{N}}r_j^{d'}\le\varepsilon,\tag{4.449}$$

by (4.448), (4.447), and (4.440). In turn, from (4.321) and (4.449) we deduce that $\mathrm{ind}_0(X,\rho)\le d'$, and since $d'>d$ was arbitrary, this ultimately yields (4.435). □

Definition 4.73. Call a quasimetric space (X,ρ) `upper` d`-Ahlfors regular` (for some $d\ge 0$) provided there exists a constant $C\in[0,+\infty)$ such that

$$\mathcal{H}^d_{X,\rho}\bigl(B_\rho(x,r)\bigr)\le C\,r^d,\qquad \forall\, x\in X,\ \ \forall\, r>0,\tag{4.450}$$

and call (X,ρ) `lower` d`-Ahlfors regular` provided there exists $c\in(0,+\infty)$ such that

$$c\,r^d\le\mathcal{H}^d_{X,\rho}\bigl(B_\rho(x,r)\bigr)\quad\text{for all }\ x\in X\quad\text{and all finite }\ r\in\bigl(0,\mathrm{diam}_\rho(X)\bigr].\tag{4.451}$$

Corollary 4.74. *(i) For any quasimetric space (X,ρ) satisfying (4.434) there holds*

$$\mathrm{ind}\,(X,\rho)\le d.\tag{4.452}$$

(ii) For any pathwise-connected quasimetric space (X,ρ) that is upper d-Ahlfors regular there holds $\mathrm{ind}_0(X,\rho)\le d$. In particular, (4.452) holds in this case as well.

Proof. Inequality (4.452) in part (i) is an immediate consequence of (4.435) and (4.333). As for the claim made in part (ii), if $\Gamma=f\bigl([0,1]\bigr)$ is an arbitrary continuous path in X, then it follows that Γ is a compact set in (X,τ_ρ). Hence, having fixed some point $x_o\in X$, we have that there exists $R\in(0,+\infty)$ with the property that $\Gamma\subseteq B_\rho(x_0,R)$. In light of condition (4.450), this implies that $\mathcal{H}^d_{X,\rho}(\Gamma)\le CR^d<+\infty$, which further entails $\mathrm{diam}^{\mathcal{H}}_{X,\rho}(\Gamma)\le d$. With this in hand, (4.435) gives $\mathrm{ind}_0(X,\rho)\le d$, as desired. □

Definition 4.75. Given a quasimetric space (X,ρ), a number $\gamma\in(0,+\infty)$, and two points $x,y\in X$, call $\Gamma\subseteq X$ a γ-`Hölder path connecting` x `and` y provided there exists a function $f:[0,1]\to X$ such that $f(0)=x$, $f(1)=y$, and $\Gamma=f\big([0,1]\big)$ and for which there exists a constant $C\in[0,+\infty)$ satisfying

$$\rho\big(f(t_1),f(t_2)\big)\leq C|t_1-t_2|^\gamma \quad \text{for every } \ t_1,t_2\in[0,1]. \tag{4.453}$$

Note that if (X,ρ) is actually a metric space, then any rectifiable path $\Gamma\subseteq X$ is 1-Hölder (taking $f(t):=s(Lt)$, $0\leq t\leq 1$, where L denotes the length of Γ and where s is the arc-length parameterization of Γ).

Corollary 4.76. *Let (X,ρ) be a quasimetric space with the property that there exists a number $\gamma\in(0,+\infty)$ such that*

$$\forall\, x,y\in X \ \ \exists\,\Gamma \ \ \gamma\text{-Hölder path joining } x \text{ and } y \tag{4.454}$$

(thus, in particular, the topological space (X,τ_ρ) is pathwise connected). Then

$$\operatorname{ind}_0(X,\rho)\leq\gamma^{-1}. \tag{4.455}$$

Hence, in particular, $\operatorname{ind}(X,\rho)\leq\gamma^{-1}$ *in this case.*

Proof. This follows from Theorem 4.71 upon observing that if Γ is as in Definition 4.75, then $\mathcal{H}^{1/\gamma}_{X,\rho}(\Gamma)\leq C^{1/\gamma}\mathcal{H}^{1/\gamma}_{\mathbb{R},|\cdot-\cdot|}\big([0,1]\big)<+\infty$, hence $\dim^{\mathcal{H}}_{X,\rho}(\Gamma)\leq 1/\gamma$. □

Recall from part (ii) of Corollary 4.74 that $\operatorname{ind}(X,\rho)\leq d$ for any pathwise-connected quasimetric space (X,ρ) that is upper d-Ahlfors regular. It turns out that the pathwise-connectivity assumption is indispensable for this type of inequality. Indeed, in Proposition 4.77 below we provide an example of an ultrametric space that is actually both upper and lower 1-Ahlfors regular. Hence, the lower smoothness index of such a space is $+\infty$, and the aforementioned inequality fails in this case. Looking for such an example in the category of ultrametric spaces is justified since these spaces happen to be totally disconnected (thus, in particular, they fail to be pathwise connected).

Proposition 4.77. *Let*

$$X:=\big\{a=(a^{(i)})_{i\in\mathbb{N}}:\ a^{(i)}\in\{0,1\}\ \ \textit{for each}\ \ i\in\mathbb{N}\big\}, \tag{4.456}$$

and define $d:X\times X\to[0,+\infty)$ by setting

$$\begin{gathered} d(a,b):=2^{-D(a,b)},\quad \forall\, a=(a^{(i)})_{i\in\mathbb{N}}\in X,\ \ \forall\, b=(b^{(i)})_{i\in\mathbb{N}}\in X,\\ \textit{where}\quad D(a,b):=\inf\big\{i\in\mathbb{N}:\ a^{(i)}\neq b^{(i)}\big\}, \end{gathered} \tag{4.457}$$

with the convention that $\inf\emptyset=+\infty$.

Then (X, d) *is a compact ultrametric space that is both upper and lower* 1*-Ahlfors regular. In particular,* (X, τ_d) *is totally disconnected and, as such, any continuous path in* (X, τ_d) *reduces to just a point.*

Proof. Note that since $D(a, b) \in \mathbb{N} \cup \{+\infty\}$, we have

$$d(a, b) \in \{0\} \cup \{2^{-k} : k \in \mathbb{N}\} \subseteq [0, 1/2] \quad \text{for every } a, b \in X. \tag{4.458}$$

Clearly d is symmetric and, for each $a, b \in X$, there holds $d(a, b) = 0$ if and only if $a = b$. Moreover,

$$D(a, b) \geq \min\{D(a, c), D(c, b)\}, \qquad \forall\, a, b, c \in X. \tag{4.459}$$

Indeed, this inequality is immediate if $c = a$ or $c = b$. Suppose now that $c \neq a$ and $c \neq b$. Without loss of generality we may assume that $D(a, c) \leq D(c, b)$. Then there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1 \leq k_2$ and $a^{(i)} = c^{(i)}$ for $i = 1, \dots, k_1 - 1$ and $a^{(k_1)} \neq c^{(k_1)}$, while $b^{(i)} = c^{(i)}$ for $i = 1, \dots, k_2 - 1$. Hence, $a^{(i)} = b^{(i)}$ for $i = 1, \dots, k_1 - 1$, so $D(a, b) \geq k_1 = D(a, c) = \min\{D(a, c), D(c, b)\}$. From this and (4.459) it follows that

$$d(a, b) \leq \max\{d(a, c), d(c, b)\} \quad \text{for every } a, b, c \in X. \tag{4.460}$$

Hence, d is an ultrametric on X. Also, from (4.458) we have $\mathrm{diam}_d(X) \leq \frac{1}{2}$.

Next, fix $a = (a^{(i)})_{i\in\mathbb{N}} \in X$. Then, given a number $r \in (0, 1/2)$, there exists a unique $k_o \in \mathbb{N}$ such that $\left(\frac{1}{2}\right)^{k_o+1} \leq r < \left(\frac{1}{2}\right)^{k_o}$. We claim that, in this scenario,

$$B_d(a, r) = \big\{b = (b^{(i)})_{i\in\mathbb{N}} \in X : b^{(i)} = a^{(i)},\ i = 1, \dots, k_o\big\}. \tag{4.461}$$

To justify this claim, observe that given any $b \in X$, we have $d(a, b) < r$ if and only if $2^{-D(a,b)} \leq 2^{-(k_o+1)}$ or, equivalently, $D(a, b) \geq k_o + 1$. From this, (4.461) readily follows.

The next claim we make is that

$$(X, d) \quad \text{is complete.} \tag{4.462}$$

To see this, suppose $\{a_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in X, with $a_j = (a_j^{(i)})_{i\in\mathbb{N}}$ for $j \in \mathbb{N}$. Thus, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(a_j, a_k) < \varepsilon$ for every $j, k \geq N_\varepsilon$. In particular, for each $\ell \in \mathbb{N}$ there exists $N'_\ell \in \mathbb{N}$ such that $d(a_j, a_k) < 2^{-\ell}$ whenever $j, k \geq N'_\ell$. Hence, if we now define $N_n := \max\{N'_1, \dots N'_n\}$ for each $n \in \mathbb{N}$, then it follows that

$$N_n \in \mathbb{N}, \quad N_{n+1} \geq N_n, \quad \text{and} \quad d(a_j, a_k) < 2^{-n-1}, \quad \forall\, n \in \mathbb{N}. \tag{4.463}$$

Combining (4.463) with (4.461) we obtain that

$$a_j^{(i)} = a_k^{(i)} \quad \text{for} \quad i = 1, \dots, n, \quad \forall\, j, k \geq N_n. \tag{4.464}$$

Next, define

$$a^{(n)} := a^{(n)}_{N_n}, \quad \forall\, n \in \mathbb{N}, \quad \text{and set} \quad a := (a^{(n)})_{n\in\mathbb{N}} \in X. \tag{4.465}$$

Note that since for each $i \in \mathbb{N}$ with $i \le n$ we have $N_i \le N_n$, from (4.465) we obtain

$$a^{(i)}_{N_i} = a^{(i)}_{N_n} \quad \text{for each} \quad i \in \{1, \dots, n\}. \tag{4.466}$$

Fix now an arbitrary number $n \in \mathbb{N}$. Then, on the one hand, (4.466) permits us to write

$$a = \big(a^{(1)}_{N_1}, a^{(2)}_{N_2}, \dots, a^{(n)}_{N_n}, a^{(n+1)}_{N_{n+1}}, \dots\big) = \big(a^{(1)}_{N_n}, a^{(2)}_{N_n}, \dots, a^{(n)}_{N_n}, a^{(n+1)}_{N_{n+1}}, \dots\big). \tag{4.467}$$

On the other hand, by once again relying on (4.464), for every $j \ge N_n$ we may write

$$a_j = \big(a^{(1)}_j, a^{(2)}_j, \dots, a^{(n)}_j, a^{(n+1)}_j, \dots\big) = \big(a^{(1)}_{N_n}, a^{(2)}_{N_n}, \dots, a^{(n)}_{N_n}, a^{(n+1)}_j, \dots\big). \tag{4.468}$$

Collectively, (4.467) and (4.468) imply that $D(a, a_j) \ge n + 1$ whenever $j \ge N_n$. Thus, $d(a, a_j) \le 2^{-n-1}$ for $j \ge N_n$. This proves that a_j converges to a as $j \to \infty$ in the topology τ_d on X, completing the proof of (4.462).

Moving on, we claim that

$$(X, d) \quad \text{is totally bounded.} \tag{4.469}$$

Recall that (4.469) amounts to showing that for every $\varepsilon > 0$ there exists a finite set $X_\varepsilon \subseteq X$ that is ε-dense in X, i.e., $\mathrm{dist}_d(a, X_\varepsilon) < \varepsilon$ for every $a \in X$. In turn, the latter condition will follow once we prove that for every $n \in \mathbb{N}$ there exists a finite set $X_n \subseteq X$ that is 2^{-n}-dense in X. With this goal in mind, fix $n \in \mathbb{N}$ and consider the set

$$X_n := \big\{(a^{(1)}, a^{(2)}, \dots, a^{(n)}, 0, 0, \dots) : a^{(i)} \in \{0, 1\},\ i = 1, \dots, n\big\} \subseteq X. \tag{4.470}$$

The cardinality of X_n is equal to 2^n; thus the set X_n is finite. In addition, given any point $a = (a^{(i)})_{i\in\mathbb{N}}$ in X, if we let $a_* := (a^{(1)}, a^{(2)}, \dots, a^{(n)}, 0, 0, \dots)$, then $a_* \in X_n$ and $D(a, a_*) \ge n + 1$, hence $d(a, a_*) \le 2^{-n-1} < 2^{-n}$, as desired. This completes the proof of the claim made in (4.469).

Given that the metric space (X, d) is complete and totally bounded, [27, Theorem 1.6.5, p. 14] implies that

$$(X, \tau_d) \quad \text{is compact.} \tag{4.471}$$

We now focus on showing that

$$(X, d) \quad \text{is both upper and lower 1-Ahlfors regular.} \tag{4.472}$$

To this end, let $\mathcal{H}^1_{X,d}$ be the 1-dimensional Hausdorff outer measure induced by d on X (cf. Definition 4.70). Also, fix $a \in X$ as well as $r \in (0, 1/2)$, and let N be the unique natural number for which $2^{-N-1} \leq r < 2^{-N}$. Fix some $M \in \mathbb{N}$ with $M > N$ and assume that the sequences $\{a_j\}_{j\in\mathbb{N}} \subseteq X$ and $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, 2^{-M})$ are such that $\overline{B_d(a,r)} \subseteq \bigcup_{j=1}^{\infty} B_d(a_j, r_j)$ (where the closure is taken in (X, τ_d)). By (4.471) we have that $\overline{B_d(a,r)}$ is compact; hence there exists a finite set $J \subseteq \mathbb{N}$ such that

$$\overline{B_d(a,r)} \subseteq \bigcup_{j\in J} B_d(a_j, r_j). \tag{4.473}$$

For each $j \in J$ pick $m_j \in \mathbb{N}$ such that $2^{-m_j-1} \leq r_j < 2^{-m_j}$ and set $m_0 := \max_{j\in J} m_j$. Hence, in particular, $m_0 \geq M > N$. Also, for each

$$\mathbf{b} := (b_1, \dots, b_{m_0-N}) \in \{0,1\}^{m_0-N} \tag{4.474}$$

define the following subset of X:

$$\mathcal{C}_{\mathbf{b}} := \big\{(a^{(1)}, \dots, a^{(N)}, b_1, \dots, b_{m_0-N}, c_{m_0+1}, c_{m_0+2}, \dots) : c_{m_0+k} \in \{0,1\},\ \forall\, k \in \mathbb{N}\big\}. \tag{4.475}$$

Hence, recalling (4.473) and (4.461) we have

$$\mathcal{C}_{\mathbf{b}} \subseteq B_d(a,r) \subseteq \bigcup_{j\in J} B_d(a_j, r_j) \qquad \text{for every } \mathbf{b} \text{ as in (4.474)}. \tag{4.476}$$

Assume now that some $j \in J$ has been fixed, and ask: for how may $\mathbf{b}$ as in (4.474) does the ball $B_d(a_j, r_j)$ intersect $\mathcal{C}_{\mathbf{b}}$? Since membership to $B_d(a_j, r_j)$ fully determines the first m_j components of a sequence in X (cf. (4.461)) whenever $B_d(a_j, r_j)$ intersects $\mathcal{C}_{\mathbf{b}}$, it follows that the components $b_1, \dots, b_{m_j-N}$ of $\mathbf{b}$ are predetermined (recall that $m_j > N$). Hence, the number of $\mathbf{b}$ as in (4.474) for which $\mathcal{C}_{\mathbf{b}}$ is intersected by $B_d(a_j, r_j)$ is bounded by the number of choices of the remaining components of $\mathbf{b}$, i.e., by the number of ways in which $b_{m_j-N+1}, \dots, b_{m_0-N} \in \{0,1\}$ may be selected. The conclusion is that each $B_d(a_j, r_j)$ can intersect at most $2^{m_0-m_j}$ sets of the form (4.475). Given that, by (4.476), the $B_d(a_j, r_j)$ with $j \in J$ intersect all the $\mathcal{C}_{\mathbf{b}}$ defined as in (4.475), we see that, necessarily,

$$\sum_{j\in J} 2^{m_0-m_j} \geq \text{cardinality of } \big\{\mathcal{C}_{\mathbf{b}} : \mathbf{b} \text{ as in (4.474)}\big\} = 2^{m_0-N}. \tag{4.477}$$

Hence, upon recalling that $r_j \geq 2^{-m_j-1}$ for every $j \in \mathbb{N}$, we conclude from (4.477) that

$$\sum_{j\in J} r_j \geq 2^{-m_0-1} \sum_{j\in J} 2^{m_0-m_j} \geq 2^{-N-1} \geq \frac{1}{2} r. \tag{4.478}$$

In light of (4.429), this shows that $\mathcal{H}^1_{X,d,2^{-M}}\big(\overline{B_d(a,r)}\big) \geq \frac{1}{2}r$, which ultimately implies (cf. (4.430))

$$\mathcal{H}^1_{X,d}\big(\overline{B_d(a,r)}\big) \geq \frac{1}{2}r. \tag{4.479}$$

Since d is a metric on X, it follows that $\overline{B_d(a,\lambda r)} \subseteq B_d(a,r)$ for every $\lambda \in (0,1)$. Consequently, for every $\lambda \in (0,1)$ we may write, based on (4.479) and the fact that $\lambda r \in (0,\frac{1}{2})$,

$$\mathcal{H}^1_{X,d}\big(B_d(a,r)\big) \geq \mathcal{H}^1_{X,d}\big(\overline{B_d(a,\lambda r)}\big) \geq \frac{\lambda}{2}r. \tag{4.480}$$

Upon letting $\lambda \nearrow 1$, this finally yields

$$\mathcal{H}^1_{X,d}\big(B_d(a,r)\big) \geq \frac{1}{2}r, \tag{4.481}$$

proving the lower 1-Ahlfors regularity condition stated in (4.472).

To establish the upper 1-Ahlfors regularity condition stated in (4.472), start by fixing a, r, N to be as before. Next, for each $M \in \mathbb{N}$ with $M > N$ consider the set

$$\mathcal{F}_M := \big\{B_d(c,2^{-M}) : c \in X,\ c^{(i)} = a^{(i)},\ i = 1,\dots,M, \text{ and } c^{(j)} = 0,\ j > M\big\}. \tag{4.482}$$

Then the cardinality of $\mathcal{F}_M$ is equal to 2^{M-N}, and we claim that

$$B_d(a,r) \subseteq \bigcup_{B \in \mathcal{F}_M} B. \tag{4.483}$$

To justify (4.483), pick an arbitrary $b \in B_d(a,r)$. If we then set

$$c := (a^{(1)}, a^{(2)}, \dots, a^{(N)}, b^{(N+1)}, \dots, b^{(M)}, 0, \dots, 0, \dots) \in X, \tag{4.484}$$

then $D(b,c) \geq M+1$; thus $d(b,c) \leq 2^{-M-1} < 2^{-M}$, so $b \in B_d(c,2^{-M}) \in \mathcal{F}_M$, as desired. Going further, note that

$$\sum_{B \in \mathcal{F}_M} \text{radius}(B) = (\text{cardinality of } \mathcal{F}_M) \cdot 2^{-M} = 2^{M-N} \cdot 2^{-M} = 2^{-N} \leq 2r. \tag{4.485}$$

From (4.485) and (4.429) we may then conclude that $\mathcal{H}^1_{X,d,2^{-M}}\big(B_d(a,r)\big) \leq 2r$. Consequently,

$$\mathcal{H}^1_{X,d}\big(B_d(a,r)\big) \leq 2r, \tag{4.486}$$

by what we have just proved and (4.430). This completes the proof of (4.472).

At this stage, to complete the proof of the theorem, we will show that ultrametric spaces are totally disconnected, i.e.,

$$\begin{aligned}&\text{if } (X,d) \text{ is an ultrametric space, then the only}\\&\text{connected sets in } (X,\tau_d) \text{ consist of singletons.}\end{aligned} \tag{4.487}$$

That any continuous path in (X, τ_d) is constant is then a simple consequence of (4.487) and the fact that any continuous function maps connected sets into connected sets.

To prove (4.487), reason by contradiction and assume that (X, d) is an ultrametric space and that $\mathcal{O}$ is a connected subset of (X, τ_d) containing two distinct points, x, y. Then $r := d(x, y) \in (0, +\infty)$, and we consider the family of all d-balls of radius r, say $\mathcal{F} := \{B_d(z, r) : z \in X\}$. From Zorn's lemma and the discussion in part (ii) of Remark 4.1 it is possible to find $Z \subseteq X$ such that

$$\begin{aligned}\mathcal{F} = \{B_d(z, r) : z \in Z\} \quad \text{and}\\ B_d(z_1, r) \cap B_d(z_2, r) = \emptyset \quad \forall\, z_1, z_2 \in Z, \text{ with } z_1 \neq z_2.\end{aligned} \tag{4.488}$$

Now, since $\mathcal{O}$ is a connected subset of (X, τ_d) and

$$\mathcal{O} \subseteq X = \bigcup_{B\in\mathcal{F}} B = \bigcup_{z\in Z} B_d(z, r), \tag{4.489}$$

we deduce (with the help of the last line in (4.488)) that there exists $z_o \in Z$ such that $\mathcal{O} \subseteq B_d(z_o, r)$. In turn, this forces $d(x, y) \leq \max\{d(x, z_o), d(z_o, y)\} < r$, contradicting the original definition of r. This proves (4.487). □

The example presented in Proposition 4.77 may be further refined to yield ultrametric spaces that are Ahlfors regular of any dimension, as discussed in the following corollary.

Corollary 4.78. *Assume that X, d are as in Proposition 4.77. Then for each number $\gamma \in (0, +\infty)$ it follows that (X, d^γ) is a compact ultrametric space (thus, in particular,* $\operatorname{ind}(X, \tau_{d^\gamma}) = +\infty$*), which is both upper and lower γ^{-1}-Ahlfors regular.*

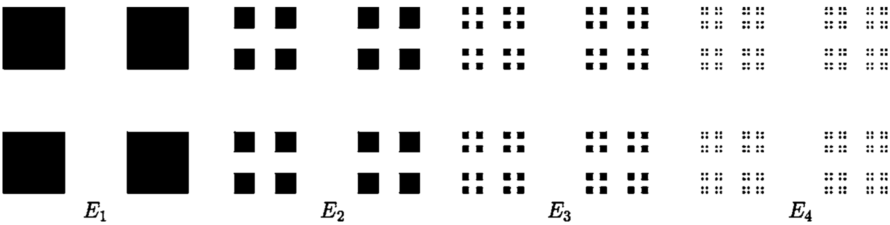
Proof. This is a direct consequence of the fact that any positive power of an ultrametric is still an ultrametric, the result proved in Proposition 4.77, and the observation that

$$\mathcal{H}^1_{X,d}(E) = \mathcal{H}^{1/\gamma}_{X,d^\gamma}(E), \qquad \forall\, E \subseteq X, \tag{4.490}$$

thanks to (4.433). □

We will describe next a phenomenon similar to that presented in Proposition 4.77, this time in the context of the so-called four-corner planar Cantor set. To this end, consider $E_0 := [0, 1]^2$, the unit square in $\mathbb{R}^2$, and let $\mathcal{C}_1$ be the set consisting of the four (closed) squares $\{Q_1^j\}_{j=1,\dots,4}$, of side length 4^{-1}, which are located in the corners of E_0, and set $E_1 := \bigcup_{j=1}^4 Q_1^j$. Iteratively, for each $n \in \mathbb{N}$ we let $\mathcal{C}_n$ denote the nth generation of squares defined as the collection of 4^n squares $\{Q_n^j\}_{j=1,\dots,4^n}$, of side length $\ell(Q_n^j) = 4^{-n}$, which are located in the corners of E_{n-1} (i.e., each Q_n^j, $j = 1, \dots, 4^n$, is located in one of the corners of the square Q_{n-1}^k for some $k \in \{1, \dots, 4^{n-1}\}$), and set $E_n := \bigcup_{j=1}^{4^n} Q_n^j$. Having introduced this notation, the `four-corner Cantor set` in $\mathbb{R}^2$ is then given by

$$E := \bigcap_{n=0}^{\infty} E_n. \tag{4.491}$$



The first four iterations in the construction of the four-corner planar Cantor set

We will prove the following result.

Proposition 4.79. *The four-corner planar Cantor set E from (4.491) is both upper and lower 1-Ahlfors regular. Moreover, while the Euclidean distance restricted to E is not an ultrametric, there exists an ultrametric on E which is equivalent to it.*

Proof. Using notation introduced in connection with the definition of the four-corner planar Cantor set, we start by making the claim that for each $N \in \mathbb{N} \cup \{0\}$ the following holds: for each $a \in \mathbb{R}^2$ we have

$$\begin{gathered} \#\Big\{j \in \{1, \dots, 4^n\} : \, Q_n^j \cap B(a, 4^{-N}) \neq \emptyset\Big\} \leq 4^{n-N} \\ \text{for every } n \in \mathbb{N} \cup \{0\} \text{ such that } n \geq N. \end{gathered} \tag{4.492}$$

In the preceding expression, $\#F$ stands for the cardinality of a set F, and all balls considered in this proof are with respect to the Euclidean distance. Having fixed $N \in \mathbb{N} \cup \{0\}$ arbitrary, we will prove the cardinality estimate in the first line of (4.492) by induction on $n \in \mathbb{N} \cup \{0\}$, $n \geq N$. Since squares in a given generation $\mathcal{C}_n$, $n \in \mathbb{N}$, are disjoint and have side length 4^{-n}, the cardinality estimate in (4.492) is clearly true when $n = N$. Assume next that the cardinality estimate in (4.492) holds for some $n \in \mathbb{N} \cup \{0\}$, $n \geq N$, and let $j \in \{1, \dots, 4^{n+1}\}$ be such that $Q_{n+1}^j \cap B(a, 4^{-N}) \neq \emptyset$. Then by construction $Q_{n+1}^j \subseteq Q_n^k$ for some $k \in \{1, \dots, 4^n\}$. In particular, we have $Q_n^k \cap B(a, 4^{-N}) \neq \emptyset$. Since each square from $\mathcal{C}_n$ gives rise to four squares in $\mathcal{C}_{n+1}$ (of which all four, or fewer, may intersect $B(a, 4^{-N})$), this discussion shows that

$$\begin{aligned} &\Big\{j \in \{1, \dots, 4^{n+1}\} : \, Q_{n+1}^j \cap B(a, 4^{-N}) \neq \emptyset\Big\} \\ &\qquad \leq 4 \cdot \#\Big\{k \in \{1, \dots, 4^n\} : \, Q_n^k \cap B(a, 4^{-N}) \neq \emptyset\Big\}. \end{aligned} \tag{4.493}$$

Based on this and using the induction hypothesis, we then obtain

$$\#\Big\{ j \in \{1, \dots, 4^{n+1}\} : \ Q \cap B(a, 4^{-N}) \neq \emptyset \Big\} \leq 4^{n+1-N}, \tag{4.494}$$

as desired. This completes the proof of the claim.

Going further, we claim that for each $N \in \mathbb{N} \cup \{0\}$ the following implication holds:

$$a \in E \Rightarrow \#\Big\{ j \in \{1, \dots, 4^n\} : Q_n^j \subseteq B(a, 4^{-N}) \Big\} \geq 4^{n-N-1}, \quad \forall\, n \in \mathbb{N}, n \geq N+1. \tag{4.495}$$

To see this, observe from (4.491) that, given $a \in E$, there exists a sequence of squares $\{Q_n^{j_n}\}_{n\in\mathbb{N}}$, such that $1 \leq j_n \leq 4^n$ and $Q_n^{j_n} \in \mathcal{C}_n$ for each $n \in \mathbb{N}$, that satisfy

$$Q_{n-1}^{j_{n-1}} \subseteq Q_n^{j_n}, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} Q_n^{j_n} = \{a\}. \tag{4.496}$$

Then

$$Q_n^{j_n} \subseteq B(a, 4^{-N}) \quad \text{whenever} \quad n > N + 4^{-1}. \tag{4.497}$$

Indeed if $n > N + 4^{-1}$, then, since $\ell(Q_n^{j_n}) = 4^{-n}$, it follows that $\sqrt{2}\,\ell(Q_n^{j_n}) < 4^{-N}$. In concert with the fact that $a \in Q_n^{j_n}$, this guarantees the inclusion in (4.497). In particular,

$$Q_{N+1}^{j_{N+1}} \subseteq B(a, 4^{-N}). \tag{4.498}$$

As such, for each integer $n \geq N+1$, the square $Q_{N+1}^{j_{N+1}}$ gives rise to 4^{n-N-1} squares in $\mathcal{C}_n$, all of which will also be contained in $B(a, 4^{-N})$. This completes the proof that for each $N \in \mathbb{N} \cup \{0\}$ implication (4.495) holds.

We turn now to proving that E is lower 1-Ahlfors regular, i.e.,

$$\exists\, c > 0 \quad \text{such that} \quad \mathcal{H}^1\big(B(a, r) \cap E\big) \geq c\,r \quad \forall\, a \in E \quad \text{and} \quad \forall\, r \in (0, 1), \tag{4.499}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff outer measure in $\mathbb{R}^2$, considered with respect to the standard Euclidean distance in the plane. Since for each $r > 0$ we trivially have $\mathcal{H}^1\big(B(a, r) \cap E\big) \geq \mathcal{H}^1\big(\overline{B(a, r/2)} \cap E\big)$, proving (4.499) reduces to showing that

$$\exists\, c > 0 \quad \text{such that} \quad \mathcal{H}^1\big(\overline{B(a, r)} \cap E\big) \geq c\,r \quad \forall\, a \in E \quad \text{and} \quad \forall\, r \in (0, 1). \tag{4.500}$$

To this end, recall from Definition 4.70 that, given a set $F \subseteq \mathbb{R}^2$, one has

$$\mathcal{H}^1(F) = \lim_{\delta \to 0^+} \mathcal{H}^1_\delta(F), \tag{4.501}$$

where

$$\mathcal{H}^1_\delta(F) = \inf\Big\{\sum_{j\in\mathbb{N}} r_j : F \subseteq \bigcup_{j\in\mathbb{N}} B(a_j, r_j), \text{ with } a_j \in \mathbb{R}^2, 0 < r_j \le \delta, j \in \mathbb{N}\Big\}. \tag{4.502}$$

Next, fix $\delta > 0$ and $r \in (0, 1)$ and let $\{a_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^2$ and $0 < r_j < \min\{\delta, 4^{-2}\}$, $j \in \mathbb{N}$, be such that $\overline{B(a, r)} \cap E \subseteq \bigcup_{j\in\mathbb{N}} B(a_j, r_j)$. Since the set $\overline{B(a, r)} \cap E$ is compact (given that, by design, the set E is closed), there exists a finite set $J \subseteq \mathbb{N}$ for which

$$\overline{B(a, r)} \cap E \subseteq \bigcup_{j\in J} B(a_j, r_j). \tag{4.503}$$

To proceed, for each $j \in J$ let $m_j \in \mathbb{N}$ be such that

$$4^{-m_j-1} \le r_j < 4^{-m_j}, \tag{4.504}$$

and set

$$m_o := \max_{j\in J} m_j. \tag{4.505}$$

In particular, $m_o \in \mathbb{N}$ and $m_o \ge 2$. On account of (4.504) and (4.492), for each number $N \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}\sum_{j\in\mathbb{N}} r_j &\ge \sum_{j\in J} r_j \ge \sum_{j\in J} 4^{-m_j-1} = 4^{-m_o-N-1}\sum_{j\in J} 4^{m_o-m_j+N} \\ &\ge 4^{-m_o-N-1}\sum_{j\in J} \#\Big\{k \in \{1,\dots,4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, 4^{-m_j}) \ne \emptyset\Big\}.\end{aligned} \tag{4.506}$$

Note that (4.504) entails $B(a_j, r_j) \subseteq B(a_j, 4^{-m_j})$, hence

$$\begin{aligned}&\Big\{k \in \{1,\dots,4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, r_j) \ne \emptyset\Big\} \\ &\qquad \subseteq \Big\{k \in \{1,\dots,4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, 4^{-m_j}) \ne \emptyset\Big\}\end{aligned} \tag{4.507}$$

for each $j \in J$ and $N \in \mathbb{N} \cup \{0\}$. Consequently,

$$\begin{aligned}&\#\Big\{k \in \{1,\dots,4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, r_j) \ne \emptyset\Big\} \\ &\qquad \le \#\Big\{k \in \{1,\dots,4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, 4^{-m_j}) \ne \emptyset\Big\}\end{aligned} \tag{4.508}$$

for each $j \in J$ and $N \in \mathbb{N} \cup \{0\}$. Going further, since $E \cap B(a,r) \subseteq \bigcup_{j \in J} B(a_j, r_j)$, for each $N \in \mathbb{N} \cup \{0\}$ we obtain that

$$\Big\{k \in \{1, \dots, 4^{m_o+N}\} : Q^k_{m_o+N} \subseteq B(a,r)\Big\} \subseteq \bigcup_{j \in J} \Big\{k \in \{1, \dots, 4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, r_j) \neq \emptyset\Big\}, \tag{4.509}$$

and thus

$$\#\Big\{k \in \{1, \dots, 4^{m_o+N}\} : Q^k_{m_o+N} \subseteq B(a,r)\Big\} \leq \sum_{j \in J} \#\Big\{k \in \{1, \dots, 4^{m_o+N}\} : Q^k_{m_o+N} \cap B(a_j, r_j) \neq \emptyset\Big\}. \tag{4.510}$$

Based on (4.506), (4.508), and (4.510), we may therefore conclude that for each number $N \in \mathbb{N} \cup \{0\}$ there holds

$$\sum_{j \in \mathbb{N}} r_j \geq 4^{-m_o-N-1} \cdot \#\Big\{k \in \{1, \dots, 4^{m_o+N}\} : Q^k_{m_o+N} \subseteq B(a,r)\Big\}. \tag{4.511}$$

Pick $N_r \in \mathbb{N} \cup \{0\}$ such that [recall that $r \in (0,1)$]

$$4^{-N_r-1} \leq r < 4^{-N_r}, \tag{4.512}$$

and note that this forces $B(a, 4^{-N_r-1}) \subseteq B(a,r)$. Hence, applying (4.511), (4.495), and (4.512) for the choice $n := m_o + N_r$ and $N := N_r + 1$ (note that $m_o \geq 2$ forces $n \geq N + 1$) yields

$$\begin{aligned} \sum_{j \in \mathbb{N}} r_j &\geq 4^{-m_o-N_r-1} \cdot \#\Big\{k \in \{1, \dots 4^{m_o+N_r}\} : Q^k_{m_o+N_r} \subseteq B(a, 4^{-N_r-1})\Big\} \\ &\geq 4^{-m_o-N_r-1} \cdot 4^{m_o+N_r-N_r-2} = 4^{-N_r-3} \geq 4^{-3} r. \end{aligned} \tag{4.513}$$

This shows that for any $r \in (0,1)$ and $\delta > 0$ there holds

$$\mathcal{H}^1_\delta(\overline{B(a,r)} \cap E) \geq 4^{-3} r. \tag{4.514}$$

Based on this and (4.501), the claim made in (4.500) readily follows. Thus (4.499) holds and, as such, E is lower 1-Ahlfors regular.

We claim next that the set E is also upper 1-Ahlfors regular, i.e.,

$$\exists\, C > 0 \quad \text{such that} \quad \mathcal{H}^1(B(a,r) \cap E) \leq C r, \quad \forall\, a \in E \text{ and } \forall\, r \in (0,1). \tag{4.515}$$

To prove this, fix $a \in E$, and for each $r \in (0,1)$ let $N_r \in \mathbb{N} \cup \{0\}$ be such that (4.512) is satisfied. Then, for each $n \geq N_r$ consider the set

$$\mathcal{F}_n := \Big\{ j \in \{1, \dots, 4^n\} : \ Q_n^j \cap B(a,r) \neq \emptyset \Big\}. \tag{4.516}$$

Since by (4.512) we have $B(a,r) \subseteq B(a, 4^{-N_r})$, it follows that

$$\mathcal{F}_n \subseteq \Big\{ j \in \{1, \dots, 4^n\} : \ Q_n^j \cap B(a, 4^{-N_r}) \neq \emptyset \Big\}, \tag{4.517}$$

and thus, based on (4.492), we may write

$$\begin{aligned} \#\mathcal{F}_n &\leq \#\Big\{ j \in \{1, \dots, 4^n\} : \ Q_n^j \cap B(a, 4^{-N_r}) \neq \emptyset \Big\} \\ &\leq 4^{n-N_r} \quad \text{whenever } n \in \mathbb{N} \text{ is such that } n \geq N_r. \end{aligned} \tag{4.518}$$

Going further, fix $n \geq N_r$, and for each $j \in \mathcal{F}_n$ consider the ball $B\big(x_{Q_n^j}, 4^{-n+\frac{1}{4}}\big)$, where $x_{Q_n^j}$ denotes the center of the square Q_n^j. Then, since $E \subseteq E_n := \bigcup_{j=1}^{4^n} Q_n^j$ and since $Q_n^j \subseteq B\big(x_{Q_n^j}, 4^{-n+\frac{1}{4}}\big)$ for each $j \in \{1, \dots, 4^n\}$, we may write

$$B(a,r) \cap E \subseteq B(a,r) \cap \Big(\bigcup_{j=1}^{4^n} Q_n^j \Big) \subseteq \bigcup_{j \in \mathcal{F}_n} Q_n^j \subseteq \bigcup_{j \in \mathcal{F}_n} B\big(x_{Q_n^j}, 4^{-n+\frac{1}{4}}\big), \tag{4.519}$$

where the second inclusion follows from discarding those squares from $\mathcal{C}_n$ (the nth generation of squares from the construction of E) that are disjoint from $B(a,r)$. This shows that the collection of balls $\{B(x_{Q_n^j}, 4^{-n+\frac{1}{4}})\}_{j \in \mathcal{F}_n}$ covers $B(a,r) \cap E$. At the same time, the sum of the radii of the balls in this collection satisfies, thanks to (4.518) and the fact that $n \geq N_r$,

$$\sum_{j \in \mathcal{F}_n} 4^{-n+\frac{1}{4}} = (\#\mathcal{F}_n) \cdot 4^{-n+\frac{1}{4}} \leq 4^{n-N_r} \cdot 4^{-n+\frac{1}{4}} = 4^{-N_r+\frac{1}{4}} \leq 4^{5/4} r. \tag{4.520}$$

Since the radii of the balls in $\{B(x_{Q_n^j}, 4^{-n+\frac{1}{4}})\}_{j \in \mathcal{F}_n}$ can be made arbitrarily small by taking n large enough, using the definition (4.502), we may conclude that for each $\delta > 0$ there holds

$$\mathcal{H}^1_\delta\big(B(a,r) \cap E\big) \leq 4^{5/4} r. \tag{4.521}$$

By once again appealing to (4.501), this shows that

$$\mathcal{H}^1\big(B(a,r) \cap E\big) \leq 4^{5/4} r. \tag{4.522}$$

Since $a \in E$ and $r \in (0,1)$ were arbitrarily chosen, (4.515) follows.

We turn our attention now to showing that, while the Euclidean distance $|\cdot - \cdot|$ in $\mathbb{R}^2$ restricted to E is not an ultrametric, there exists an ultrametric on E that is equivalent to $|\cdot - \cdot|\big|_E$. First, it is easy to see that

$$x := (0,0),\ \ y := (1,0) \quad \text{and} \quad z := (1/4, 3/4) \quad \text{satisfy} \quad x, y, z \in E \tag{4.523}$$

since they are corner points of squares from $\mathcal{C}_1$. However,

$$1 = |x - y| > \max\{|x - z|, |y - z|\}, \tag{4.524}$$

showing that $|\cdot - \cdot|\big|_E$ fails to be an ultrametric.

Next, introduce the function $d_\star : E \times E \to [0, \infty)$ given by

$$d_\star(x, y) := \inf\Big\{r > 0 : \exists\, \xi_1, \dots, \xi_{N+1} \in E,\ N \in \mathbb{N},\ \text{ such that} \tag{4.525}$$
$$x = \xi_1,\ y = \xi_{N+1} \text{ and } |\xi_i - \xi_{i+1}| < r,\ \ \forall\, i \in \{1, \dots, N\}\Big\}$$

for each $x, y \in E$. It is clear from this definition that

$$d_\star \text{ is nonnegative and symmetric and } d_\star(x, y) \le |x - y|, \quad \forall\, x, y \in E. \tag{4.526}$$

Going further, we claim that

$$d_\star(x, y) \le \max\{d_\star(x, z), d_\star(z, y)\}, \qquad \forall\, x, y, z \in E. \tag{4.527}$$

To justify (4.527), fix $x, y, z \in E$, and let $\{r_n\}_{n\in\mathbb{N}}, \{r_n'\}_{n\in\mathbb{N}} \subseteq (0, +\infty)$ be such that $r_n \searrow d_\star(x, z)$ and $r_n' \searrow d_\star(z, y)$ as $n \to \infty$, and with the property that for each $n \in \mathbb{N}$ there exist $N(n), N(n)' \in \mathbb{N}$ and $\xi_i^n, \eta_j^n \in E$, $i \in \{1, \dots, N(n)\}$ and $j \in \{1, \dots N'(n)\}$, satisfying

$$\begin{gathered} \xi_1^n = x,\quad \xi_{N(n)+1}^n = z, \quad \text{and} \quad |\xi_i^n - x_{i+1}^n| < r_n, \\ \forall\, i \in \{1, \dots, N(n)\} \quad \text{and} \quad \forall\, n \in \mathbb{N}, \end{gathered} \tag{4.528}$$

and

$$\begin{gathered} \eta_1^n = z,\ \ \eta_{N'(n)+1}^n = y, \quad \text{and} \quad |\eta_i^n - \eta_{i+1}^n| < r_n', \\ \forall\, i \in \{1, \dots, N'(n)\} \quad \text{and} \quad \forall\, n \in \mathbb{N}. \end{gathered} \tag{4.529}$$

For each $n \in \mathbb{N}$ consider then the sequence of points $\{\omega_j^n\}_{j\in\{1,\dots,N(n)+N'(n)+2\}} \subseteq E$ defined by

$$\begin{gathered} \omega_j^n := \xi_j^n, \quad \forall\, j \in \{1, \dots, N(n) + 1\}, \\ \text{and} \quad \omega_j^n = \eta_{j-N(n)-1}^n, \quad \forall\, j \in \{N(n) + 2, \dots, N(n) + N'(n) + 2\}. \end{gathered} \tag{4.530}$$

Then, using (4.528)–(4.530), we obtain

$$\begin{gathered} \omega_1^n = x, \quad \omega_{N(n)+N'(n)+2}^n = z, \quad \text{and} \\ |\omega_j^n - \omega_{j+1}^n| < \max\{r_n, r_n'\}, \quad \forall\, j \in \{1, \dots, N(n) + N'(n) + 1\}. \end{gathered} \tag{4.531}$$

Hence, combined with the definition of $d_\star$ from formula (4.525), this implies that $d_\star(x,y) \le \max\{r_n, r_n'\}$ for all $n \in \mathbb{N}$. By passing to the limit as $n \to \infty$ and using that $x, y, z \in E$ were arbitrarily chosen, (4.527) follows.

It remains to show that $d_\star$ bounds from below, up to a multiplicative constant, the Euclidean distance restricted to E, i.e.,

$$\exists\, c > 0 \quad \text{such that} \quad d_\star(x,y) \ge c|x-y|, \quad \forall\, x, y \in E. \tag{4.532}$$

Note that once (4.532) has been justified, it follows that

$$d_\star \ \text{ is nondegenerate, i.e., } \ d_\star^{-1}(\{0\}) = \operatorname{diag}(E). \tag{4.533}$$

Together, (4.526), (4.528), (4.527), (4.532), and (4.533) show that, as desired,

$$d_\star \ \text{ is an ultrametric on } E \text{ that is equivalent to } |\cdot - \cdot|\big|_E. \tag{4.534}$$

Turning our attention to establishing (4.532), start by fixing $x, y \in E$ such that $x \neq y$, and let $n \in \mathbb{N}$ be the largest subscript of a generation of squares $\mathcal{C}_n$ that has as an element a square containing both x and y. That is, suppose $n \in \mathbb{N}$ is such that

$$\begin{gathered} x, y \in Q_n^{j_o} \ \text{ for some } j_o \in \{1, \dots, 4^n\} \text{ and} \\ k \in \{1, \dots, 4^{n+1}\} \ \text{ does not exist such that } \ x, y \in Q_{n+1}^k. \end{gathered} \tag{4.535}$$

Since squares in the generation $\mathcal{C}_{n+1}$ separate the points x and y, and since by construction the distance between squares in this generation is $\ge 4^{-n-1}$, we deduce that

$$|x-y| \ge 4^{-n-1}. \tag{4.536}$$

Consider next $N \in \mathbb{N}$ and $\{\xi_i\}_{i=1,\dots,N+1} \subseteq E$ such that $\xi_1 = x$ and $\xi_{N+1} = y$. It follows that there exists $k \in \{1, \dots, N\}$ such that the points ξ_k and ξ_{k+1} are separated by squares in $\mathcal{C}_{n+1}$ [since, otherwise, both x and y would belong to some fixed square in $\mathcal{C}_{n+1}$, violating (4.535)]. Consequently, $|\xi_k - \xi_{k+1}| \ge 4^{-n-1}$. Since $N \in \mathbb{N}$ and $\{\xi_i\}_{i=2,\dots,N} \subseteq E$ were arbitrarily chosen (with the convention that this collection of points in E is an empty set when $N = 1$), it follows from definition (4.525) that, on the one hand,

$$d_\star(x,y) \ge 4^{-n-1}, \tag{4.537}$$

while on the other hand, since $x, y \in Q_n^{j_o}$, it follows that

$$|x-y| \le \operatorname{diam}(Q_n^{j_o}) = \sqrt{2}\,\ell(Q_n^{j_o}) = 4^{-n+\frac14}. \tag{4.538}$$

Combining (4.537) and (4.538) we obtain

$$d_\star(x,y) \ge 4^{-5/4}|x-y|. \tag{4.539}$$

Since $x, y \in E$ such that $x \neq y$ were arbitrarily chosen, and since obviously (4.539) also holds for $x, y \in E$ with $x = y$, the proof of (4.532) is completed, as $c := 4^{-5/4}$ does the job. This completes the proof of the proposition. □

Proposition 4.79 may be further refined as in the following corollary.

Corollary 4.80. *Let E be the four-corner planar Cantor set, and assume that $d_\star$ is the ultrametric on E defined as in* (4.525)*. Then, for any $\gamma > 0$ the space $(E, d_\star^\gamma)$ is a compact, ultrametric space that is both upper and lower $1/\gamma$-Ahlfors regular.*

Proof. This is established much as Corollary 4.78, using Proposition 4.79. □

Another example of an ultrametric on the four-corner Cantor set that is equivalent to the restriction of the Euclidean distance to this set is presented below.

Comment 4.81. Given a dyadic square Q in $\mathbb{R}^2$ (always considered to be closed), denote by $\widetilde{Q}$ the set consisting of Q with the upper horizontal and right vertical sides removed. In particular, for every $n \in \mathbb{Z}$ the plane $\mathbb{R}^2$ decomposes into the disjoint union of all $\widetilde{Q}$, where Q runs through the collection of all dyadic cubes with side length 2^{-n}. In turn, this observation may be used to justify that the function $\widetilde{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ given by

$$\widetilde{d}(x, y) := \inf\big\{\ell(Q) : \ Q \text{ dyadic cube such that } x, y \in \widetilde{Q}\big\}, \quad \forall\, x, y \in \mathbb{R}^2, \tag{4.540}$$

is well defined. In relation to this function, we make the following claims:

$$\widetilde{d} \text{ is an ultrametric in } \mathbb{R}^2, \tag{4.541}$$

and, with E denoting the four-corner planar Cantor set and with $d_\star$ as in (4.525),

$$\widetilde{d}\big|_E \approx d_\star. \tag{4.542}$$

To justify (4.541), we start by noting that, given any points $x, y, z \in \mathbb{R}^2$ and any dyadic cubes Q_1, Q_2, Q_3 in $\mathbb{R}^2$,

$$\left.\begin{array}{r} \ell(Q_1) \leq \ell(Q_2) \leq \ell(Q_3) \\ x, y \in \widetilde{Q}_1, \quad y, z \in \widetilde{Q}_2, \quad z, x \in \widetilde{Q}_3 \end{array}\right\} \Longrightarrow \ Q_1 \subseteq Q_2 \subseteq Q_3. \tag{4.543}$$

This follows from the general observation that for any dyadic cubes Q', Q'' in $\mathbb{R}^2$ with the property that $\widetilde{Q'} \cap \widetilde{Q''} \neq \emptyset$ one necessarily has either $Q' \subseteq Q''$ or $Q'' \subseteq Q'$. Having established (4.543), we deduce from this that whenever the points $x, y, z \in \mathbb{R}^2$ and dyadic cubes Q_1, Q_2, Q_3 in $\mathbb{R}^2$ are as on the left-hand side of (4.543), it follows that $x, y, z \in \widetilde{Q}_2$. Granted this, the ultrametric triangle inequality is readily verified for the function (4.540). Since this function is clearly symmetric and nondegenerate, the claim in (4.541) is proved. Let us also note here that for each $x, y \in \mathbb{R}^2$ contained in some $\widetilde{Q}$ one has $\ell(Q) \geq 2^{-1/2}\,\mathrm{diam}\,(Q) \geq 2^{-1/2}|x-y|$, hence

$$\widetilde{d}(x,y) \geq 2^{-1/2}|x-y|, \qquad \forall\, x, y \in \mathbb{R}^2. \tag{4.544}$$

As regards the claim in (4.542), observe that the estimates in (4.544) and (4.526) give $\widetilde{d}\big|_E \geq 2^{-1/2} d_\star$. Finally, the fact that there exists some finite constant $c > 0$ such that $\widetilde{d}\big|_E \leq c\, d_\star$ is a consequence of (4.535) and (4.536). This concludes the proof of (4.542). ∎

Of course, the claims established in Comment 4.81 have natural formulations in all space dimensions. In particular, a result related to the one-dimensional version reads as follows.

Proposition 4.82. *Let $X := [0,1)$, and for each $x, y \in X$ set*

$$d(x,y) := \begin{cases} \ell(x,y) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \tag{4.545}$$

where, for $x, y \in X$ such that $x \neq y$,

$$\begin{array}{l} \ell(x,y) \text{ is the length of the smallest dyadic interval } \big[\frac{k}{2^n}, \frac{k+1}{2^n}\big) \\ \text{containing both } x \text{ and } y, \text{ where } k \in \mathbb{N} \text{ is such that } 1 \leq k \leq 2^n - 1. \end{array} \tag{4.546}$$

Then (X, d) is an ultrametric space.

4.8 Distribution Theory on Quasimetric Spaces

Let $(X, \mathbf{q})$ be a quasimetric space. Then for each $\rho \in \mathbf{q}$ define the class of `test functions` on X as

$$\mathscr{D}(X,\rho) := \bigcap_{0<\beta<[\log_2 C_\rho]^{-1}} \dot{\mathscr{C}}_c^{\beta}(X,\rho), \tag{4.547}$$

equipped with a certain topology, $\tau_{\mathscr{D}}$, which we will describe momentarily. For now, we wish to note that $\mathscr{D}(X,\rho)$ is a rich set since, for example, the functions in $\mathscr{D}(X,\rho)$ separate the points in X (cf. the last part in Theorem 4.6, as well as Remark 4.8).

Turning to the issue of defining the topology $\tau_{\mathscr{D}}$ on $\mathscr{D}(X,\rho)$, fix a nested family $\{K_n\}_{n\in\mathbb{N}}$ of ρ-bounded subsets of X with the property that any ρ-ball is contained in one of the K_n sets. Hence, in particular, $\cup_{n\in\mathbb{N}} K_n = X$. Next, for each $n \in \mathbb{N}$, denote by $\mathscr{D}_n(X,\rho)$ the collection of functions from $\mathscr{D}(X,\rho)$ that vanish in $X \setminus K_n$. With $\|\cdot\|_\infty$ standing for the supremum norm on X, this becomes a Fréchet space when equipped with the topology τ_n induced by the family of norms

$$\left\{\|\cdot\|_\infty + \|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} : \beta \text{ rational number such that } 0 < \beta < [\log_2 C_\rho]^{-1}\right\}. \quad (4.548)$$

That is, $\mathscr{D}_n(X,\rho)$ is a Hausdorff topological space, whose topology is induced by a countable family of seminorms, and which is complete (as a uniform space with the uniformity canonically induced by the aforementioned family of seminorms as in Example 2.75 or, equivalently, as a metric space when endowed with a metric yielding the same topology as τ_n; cf. Theorem 1.1). Since for any $n \in \mathbb{N}$ the topology induced by τ_{n+1} on $\mathscr{D}_n(X,\rho)$ coincides with τ_n, we may turn $\mathscr{D}(X,\rho)$ into a topological space, $(\mathscr{D}(X,\rho), \tau_{\mathscr{D}})$, by regarding it as the strict inductive limit of the family of topological spaces $\left\{(\mathscr{D}_n(X,\rho), \tau_n)\right\}_{n\in\mathbb{N}}$.

Theorem 4.83. *Assume that $(X, \mathbf{q})$ is a quasimetric space. Then for each $\rho \in \mathbf{q}$ the class of test functions $\mathscr{D}(X,\rho)$, equipped with the topology $\tau_{\mathscr{D}}$ (defined as above), satisfies the following properties.*

(1) The topology $\tau_{\mathscr{D}}$ is independent of the particular choice of a family of sets $\{K_n\}_{n\in\mathbb{N}}$ with the properties specified above. Also, in general, $\tau_{\mathscr{D}}$ is not metrizable.

(2) $(\mathscr{D}(X,\rho), \tau_{\mathscr{D}})$ is a Hausdorff, locally convex, topological vector space. Also, for every $n \in \mathbb{N}$, the topology induced by $\tau_{\mathscr{D}}$ on $\mathscr{D}_n(X,\rho)$ coincides with τ_n.

(3) $(\mathscr{D}(X,\rho), \tau_{\mathscr{D}})$ has the Heine–Borel property [i.e., a subset of $\mathscr{D}(X,\rho)$ is compact in $\tau_{\mathscr{D}}$ if and only if it is closed and bounded].

(4) The topology $\tau_{\mathscr{D}}$ on $\mathscr{D}(X,\rho)$ is the final topology of the nested family of metrizable topological spaces $\left\{(\mathscr{D}_n(X,\rho), \tau_n)\right\}_{n\in\mathbb{N}}$ and, hence, $(\mathscr{D}(X,\rho), \tau_{\mathscr{D}})$ is an LF-space.

(5) A convex and balanced subset $\mathcal{O}$ of $\mathscr{D}(X,\rho)$ is open in $\tau_{\mathscr{D}}$ if and only if the set $\mathcal{O} \cap \mathscr{D}_n(X,\rho)$ is open in τ_n for every $n \in \mathbb{N}$, i.e., if and only if

$$\forall\, n \in \mathbb{N} \quad \exists\, \varepsilon > 0 \quad \exists\, \beta \in (0, [\log_2 C_\rho]^{-1}) \quad \textit{such that}$$
$$\left\{\phi \in \mathscr{D}(X,\rho) : \phi = 0 \textit{ on } X \setminus K_n \textit{ and } \|\phi\|_\infty + \|\phi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \varepsilon\right\} \subseteq \mathcal{O}. \quad (4.549)$$

(6) One has

$$\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}(X,\rho) \textit{ converges to zero in } \tau_{\mathscr{D}} \Longleftrightarrow \exists\, n \in \mathbb{N} \textit{ such that}$$
$$\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_n(X,\rho) \textit{ and } \{\phi_j\}_{j\in\mathbb{N}} \textit{ converges to zero in } \tau_n, \quad (4.550)$$

i.e., there exists $n \in \mathbb{N}$ with the property that $\phi_j = 0$ on $X \setminus K_n$ for every $j \in \mathbb{N}$ and $\lim_{j\to\infty}\left[\|\phi_j\|_\infty + \|\phi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\right] = 0$ whenever $0 < \beta < [\log_2 C_\rho]^{-1}$.

(7) A sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}(X,\rho)$ is Cauchy (in the sense of topological vector spaces) if and only if there exists $n \in \mathbb{N}$ with the property that $\phi_j = 0$ on $X \setminus K_n$ for every $j \in \mathbb{N}$ and whenever $0 < \beta < [\log_2 C_\rho]^{-1}$, one has $\|\phi_j - \phi_k\|_\infty + \|\phi_j - \phi_k\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \to 0$ as $j, k \to +\infty$.

(8) $\mathscr{D}(X,\rho)$ is sequentially complete, in the sense that any Cauchy sequence in $\mathscr{D}(X,\rho)$ converges to a (unique) function from $\mathscr{D}(X,\rho)$ in the topology $\tau_{\mathscr{D}}$.

(9) *A set* $\mathscr{B} \subseteq \mathscr{D}(X,\rho)$ *is bounded (i.e., any neighborhood of the origin in this topological vector space contains a positive dilate of* $\mathscr{B}$*) if and only if there exists* $n \in \mathbb{N}$ *with the property that*

$$\begin{aligned}
&\phi = 0 \ \text{ on } \ X \setminus K_n \ \text{ for each } \ \phi \in \mathscr{B}, \quad \text{and} \\
&\sup\big\{\|\phi\|_\infty + \|\phi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} : \phi \in \mathscr{B}\big\} < +\infty \qquad (4.551)\\
&\text{whenever } \beta \in \mathbb{R} \text{ satisfies } 0 < \beta < [\log_2 C_\rho]^{-1}.
\end{aligned}$$

Proof. This is proved along the lines of [108, Theorems 6.4–6.5, pp. 152–153]. □

Moving on, given a quasimetric space $(X,\mathbf{q})$, for each $\rho \in \mathbf{q}$ we define the `space of distributions` $\mathscr{D}'(X,\rho)$ on X as the (topological) dual of $\mathscr{D}(X,\rho)$ and denote by $\langle\cdot,\cdot\rangle$ the natural duality pairing between distributions in $\mathscr{D}'(X,\rho)$ and test functions in $\mathscr{D}(X,\rho)$.

Theorem 4.84. *Let* $(X,\mathbf{q})$ *be a quasimetric space and fix some* $\rho \in \mathbf{q}$. *Then for a linear mapping* $f : \mathscr{D}(X,\rho) \to \mathbb{R}$ *the following conditions are equivalent.*

(1) f *belongs to* $\mathscr{D}'(X,\rho)$.
(2) f *maps bounded subsets of the topological vector space* $(\mathscr{D}(X,\rho),\tau_{\mathscr{D}})$ *into bounded subsets of* $\mathbb{R}$.
(3) *If a sequence* $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}(X,\rho)$ *converges to zero in the topological vector space* $(\mathscr{D}(X,\rho),\tau_{\mathscr{D}})$, *then* $\langle f,\phi_j\rangle \to 0$ *as* $j \to +\infty$ *in* $\mathbb{R}$.
(4) *For each* $n \in \mathbb{N}$, *the restriction of* f *to* $(\mathscr{D}_n(X,\rho),\tau_n)$ *is continuous.*
(5) *For every* $n \in \mathbb{N}$ *there exist* $C \in (0,+\infty)$ *and* $\beta \in \big(0,[\log_2 C_\rho]^{-1}\big)$ *with the property that*

$$\big|\langle f,\phi\rangle\big| \le C\big(\|\phi\|_\infty + \|\phi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big), \qquad \forall\, \phi \in \mathscr{D}_n(X,\rho). \qquad (4.552)$$

Proof. This is proved by reasoning much as in [108, Theorems 6.6 on p. 155 and Theorem 6.8 on p. 156]. □

Given a quasimetric space $(X,\mathbf{q})$ and $\rho \in \mathbf{q}$, it follows that $\mathscr{D}'(X,\rho)$ has a natural vector space structure. We will equip this space with the weak-topology $\tau_{\mathscr{D}'}$, i.e., the topology induced by the family of seminorms $\{p_\phi\}_{\phi\in\mathscr{D}(X,\rho)}$ on $\mathscr{D}'(X,\rho)$, where for each $\phi \in \mathscr{D}(X,\rho)$ and $f \in \mathscr{D}'(X,\rho)$ we defined $p_\phi(f) := |\langle f,\phi\rangle|$. Thus, for a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'(X,\rho)$ and a distribution $f \in \mathscr{D}'(X,\rho)$,

$$\lim_{j\to\infty} f_j = f \ \text{ in } \tau_{\mathscr{D}'} \iff \lim_{j\to\infty}\langle f_j,\phi\rangle = \langle f,\phi\rangle \ \text{ in } \mathbb{R} \ \text{ for each } \ \phi \in \mathscr{D}(X,\rho). \qquad (4.553)$$

The space of distributions on a quasimetric space is sequentially complete, in the sense made precise in the theorem below.

Theorem 4.85. *Suppose that* $(X,\mathbf{q})$ *is a quasimetric space and that* $\rho \in \mathbf{q}$. *If a sequence* $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'(X,\rho)$ *has the property that*

$$\lim_{j\to\infty} \langle f_j, \phi\rangle \text{ exists in } \mathbb{R} \text{ for each } \phi \in \mathscr{D}(X,\rho), \tag{4.554}$$

then the functional that associates to each test function $\phi \in \mathscr{D}(X,\rho)$ the number defined as the limit in (4.554) is a distribution $f \in \mathscr{D}'(X,\rho)$ that satisfies the following properties.

(1) $\lim_{j\to\infty} f_j = f$ *in* $\tau_{\mathscr{D}'}$.

(2) For every $n \in \mathbb{N}$ there exist $C \in (0,+\infty)$ and $\beta \in \big(0, [\log_2 C_\rho]^{-1}\big)$ such that

$$\big|\langle f_j, \phi\rangle\big| \leq C\big(\|\phi\|_\infty + \|\phi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big) \text{ for all } \phi \in \mathscr{D}_n(X,\rho) \text{ and all } j \in \mathbb{N}. \tag{4.555}$$

(3) $\lim_{j\to\infty} \langle f_j, \phi_j\rangle = \langle f, \phi\rangle$ *for every sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}(X,\rho)$ converging in $\tau_{\mathscr{D}}$ to a limit $\phi \in \mathscr{D}(X,\rho)$.*

Proof. This is essentially a consequence of the Banach–Steinhaus principle of uniform boundedness (cf. [61, Theorems 2.1.8, pp. 38–39] for details in the standard Euclidean setting). □

As in the classical Euclidean setting, any nonnegative distribution on a locally compact quasimetric space is given by integration against a Borel regular measure. More precisely, the following theorem holds.

Theorem 4.86. *Assume that $(X, \mathbf{q})$ is a quasimetric space for which $\tau_{\mathbf{q}}$, the topology canonically induced by the quasimetric space structure $\mathbf{q}$ on X, is locally compact. Fix $\rho \in \mathbf{q}$. Then for any distribution $f \in \mathscr{D}'(X,\rho)$ with the property that*

$$\langle f, \phi\rangle \geq 0 \quad \text{for every } \ \phi \in \mathscr{D}(X,\rho) \ \text{ such that } \ \phi \geq 0 \tag{4.556}$$

there exists a (unique) locally finite Borel measure μ on the locally compact topological space $(X, \tau_{\mathbf{q}})$ such that

$$\langle f, \phi\rangle = \int_X \phi(x)\,\mathrm{d}\mu(x) \quad \text{for every } \ \phi \in \mathscr{D}(X,\rho) \tag{4.557}$$

and that satisfies the following regularity conditions:

$$\mu(E) = \inf\{\mu(U) : \ E \subseteq U \subseteq X, \ U \text{ open}\} \quad \text{for every Borel set } E \subseteq X \tag{4.558}$$

and

$$\mu(E) = \sup\{\mu(K) : \ K \subseteq E, \ K \text{ compact subset of } X\} \tag{4.559}$$

whenever $E \subseteq X$ is either open or a Borel set of finite measure.

Proof. Denote by $\mathscr{C}^0_c(X)$ the space of continuous, real-valued functions of compact support in X. We claim that for each $\varphi \in \mathscr{C}^0_c(X)$ there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}(X,\rho)$ with the property that all functions ϕ_j vanish outside a common compact subset K of X and $\phi_j \to \varphi$ uniformly on X as $j \to +\infty$. When $C_\rho > 1$, this follows from Theorems 4.15 and 4.12 (used with $\beta := [\log_2 C_\rho]^{-1}$). The case $C_\rho = 1$, i.e., when ρ is an ultrametric, is handled similarly, with the help of Remark 4.8.

To proceed, assume for a moment that for each compact set $K \subseteq X$ there exists a finite constant $C_K \geq 0$ with the property that

$$|\langle f, \phi\rangle| \leq C_K \|\phi\|_\infty \quad \text{for every} \quad \phi \in \mathscr{D}(X,\rho) \quad \text{vanishing on} \quad X \setminus K. \tag{4.560}$$

Since this entails $|\langle f, \phi_j\rangle - \langle f, \phi_k\rangle| \leq C_K \|\phi_j - \phi_k\|_\infty$ for every $j, k \in \mathbb{N}$, it follows that

$$\Lambda(\varphi) := \lim_{j\to\infty} \langle f, \phi_j\rangle \quad \text{exists in } \mathbb{R} \text{ for every} \quad \varphi \in \mathscr{C}_c^0(X). \tag{4.561}$$

Furthermore, by mixing sequences, we see that the value of this limit is independent of the choice of the approximating sequence $\{\phi_j\}_{j\in\mathbb{N}}$ with the specified properties. Moreover, since matters may be arranged so that each ϕ_j is nonnegative if $\varphi \geq 0$ to begin with, we ultimately deduce that Λ is a well-defined, nonnegative linear functional on $\mathscr{C}_c^0(X)$ with the property that $\Lambda(\varphi) = \langle f, \varphi\rangle$ whenever $\varphi \in \mathscr{D}(X,\rho)$. At this stage, the existence of a locally finite Borel measure μ satisfying (4.557)–(4.559) becomes a consequence of Riesz's representation theorem.

We are therefore left with verifying the claim made in (4.560). To this end, fix a compact set $K \subseteq X$ and choose some $\phi \in \mathscr{D}(X,\rho)$ that vanishes in $X \setminus K$. Based on Theorem 4.12 (cf. also Remark 4.8), it is possible to produce a nonnegative function $\chi \in \mathscr{D}(X,\rho)$ such that $\chi(x) = 1$ for every $x \in K$. Then, it follows that $\phi_\pm := \|\phi\|_\infty \chi \pm \phi \in \mathscr{D}(X,\rho)$ and $\phi_\pm \geq 0$. In concert with (4.556), this yields $0 \leq \langle f, \phi_\pm\rangle = \|\phi\|_\infty \langle f, \chi\rangle \pm \langle f, \phi\rangle$. Hence, (4.560) holds with $C_K := \langle f, \chi\rangle$, and this completes the proof of the theorem. □

4.9 Hardy Spaces on Ahlfors-Regular Quasimetric Spaces

Let $(X, \mathbf{q})$ be a quasimetric space, and suppose that μ is a measure on X with the property that the following Ahlfors-regularity condition holds:

$$\begin{gathered} \text{there exist } d \in (0,+\infty) \text{ and } \rho_o \in \mathbf{q} \text{ such that} \\ \text{all } \rho_o\text{-balls are } \mu\text{-measurable, and } \ \mu\big(B_{\rho_o}(x,r)\big) \approx r^d \ \text{ uniformly} \\ \text{for every } x \in X \text{ and every finite number } r \in (0, \operatorname{diam}_{\rho_o}(X)]. \end{gathered} \tag{4.562}$$

It is not difficult to check that the Ahlfors-regularity condition (4.562) implies that $(X, \mathbf{q})$ is geometrically doubling (in the sense of Definition 4.10). Consequently, if we equip X with the topology $\tau_{\mathbf{q}}$, naturally induced by the quasimetric space structure $\mathbf{q}$ on X, it follows from property (1) in Theorem 4.21 that

$$\mu \text{ is a Borel measure on } (X, \tau_{\mathbf{q}}). \tag{4.563}$$

In such a setting, if

$$\rho \in \mathbf{q} \quad \text{and} \quad 0 < \gamma < \left[\log_2 C_\rho\right]^{-1}, \tag{4.564}$$

then for each $x \in X$ we define the class $\mathcal{T}_\rho^\gamma(x)$ of (ρ,γ)-normalized bump functions supported near x according to

$$\mathcal{T}_\rho^\gamma(x) := \Big\{\psi \in \mathscr{D}(X,\rho) : \exists\, r > 0 \text{ such that } \psi = 0 \text{ on } X \setminus B_\rho(x,r) \text{ and } \|\psi\|_\infty + r^\gamma \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho)} \leq r^{-d}\Big\}. \tag{4.565}$$

Next, given a quasidistance ρ and a number γ as in (4.564), define the `grand maximal function` of a distribution $f \in \mathscr{D}'(X,\rho)$ by setting (with the duality paring understood as before)

$$f^*_{\rho,\gamma}(x) := \sup_{\psi \in \mathcal{T}_\rho^\gamma(x)} \big|\langle f, \psi\rangle\big|, \qquad \forall\, x \in X. \tag{4.566}$$

Lemma 4.87. *Assume $(X,\mathbf{q})$ is a quasimetric space and μ is a measure on X that satisfies (4.562). Also, suppose that the quasidistance ρ and the number γ are as in (4.564). Finally, recall the regularized version $\rho_\#$ of ρ as defined in Theorem 3.46. Then there exist two finite constants $C_0, C_1 > 0$, depending only on ρ and γ, with the property that for any $f \in \mathscr{D}'(X,\rho)$ one has*

$$C_0\, f^*_{\rho_\#,\gamma}(x) \leq f^*_{\rho,\gamma}(x) \leq C_1\, f^*_{\rho_\#,\gamma}(x) \ \textit{ for all } x \in X. \tag{4.567}$$

Furthermore, for each distribution $f \in \mathscr{D}'(X,\rho)$,

$$\textit{the function } \ f^*_{\rho_\#,\gamma} : (X,\tau_{\mathbf{q}}) \to [0,+\infty] \ \textit{ is lower semicontinuous.} \tag{4.568}$$

As a corollary of this and (4.563), for each distribution $f \in \mathscr{D}'(X,\rho)$ the function $f^*_{\rho_\#,\gamma}$ is μ-measurable.

Proof. As a preamble, let us note that if $\rho, \rho' \in \mathbf{q}$, then there exists $C \in (0,+\infty)$ with the property that for each $x \in X$ and each number $\gamma \in \big(0, \min\{[\log_2 C_\rho]^{-1}, [\log_2 C_{\rho'}]^{-1}\}\big)$ we have

$$\psi \in \mathcal{T}_\rho^\gamma(x) \Longrightarrow \psi/C \in \mathcal{T}_{\rho'}^\gamma(x). \tag{4.569}$$

In turn, this shows that

$$\begin{aligned}&\text{if } \rho, \rho' \in \mathbf{q} \text{ and } 0 < \gamma < \min\{[\log_2 C_\rho]^{-1}, [\log_2 C_{\rho'}]^{-1}\}, \text{ then}\\ &f^*_{\rho,\gamma}(x) \approx f^*_{\rho',\gamma}(x) \quad \text{uniformly in } x \in X \text{ and } f \in \mathscr{D}'(X,\rho) \cap \mathscr{D}'(X,\rho').\end{aligned} \tag{4.570}$$

From (3.533) in conclusion (11) of Theorem 3.46 it follows that the regularized version $\rho_\#$ of ρ satisfies $\rho_\# \in \mathbf{q}$ and $C_{\rho_\#} \leq C_\rho$. Granted this, (4.567) becomes a consequence of (4.570). Consider now the claim made in (4.568) for an arbitrary distribution $f \in \mathscr{D}'(X,\rho)$. Let $a \geq 0$, and define $\mathcal{O} := (f^*_{\rho_\#,\gamma})^{-1}\big((a,+\infty)\big) \subseteq X$. Thus, if we fix $x_o \in \mathcal{O}$, then there exists $\psi \in \mathcal{T}^\gamma_{\rho_\#}(x_o)$ such that $b := |\langle f, \psi\rangle| > a$. Assume that $r > 0$ is such that ψ is supported in $B_{\rho_\#}(x_o, r)$ and is normalized as in (4.565) relative to this number. Then the desired conclusion follows as soon as we show that

$$\text{there exists } \varepsilon = \varepsilon(\rho, d, \gamma, r, a, b) > 0 \text{ such that } B_{\rho_\#}(x_o, \varepsilon) \subseteq \mathcal{O}. \tag{4.571}$$

To proceed, fix an index $0 < \beta < \min\{1, [\log_2 C_\rho]^{-1}\}$ and use the Hölder condition (3.542) for $\rho_\#$ to conclude that there exists a constant $C = C(\rho,\beta) > 0$ with the property that

$$\varepsilon \in (0,r) \text{ and } x \in B_{\rho_\#}(x_o,\varepsilon) \Longrightarrow \begin{cases} B_{\rho_\#}(x_o,r) \subseteq B_{\rho_\#}(x,R), \\ \text{where } R := r + C r^{1-\beta}\varepsilon^\beta. \end{cases} \tag{4.572}$$

Let ε and x be as on the left-hand side of (4.572). Then definition (4.565) and the fact that $B_{\rho_\#}(x_o,r) \subseteq B_{\rho_\#}(x,R)$ with $R \geq r$ readily imply that $(r/R)^{d+\gamma}\psi \in \mathcal{T}^\gamma_{\rho_\#}(x)$. Hence, $f^*_{\rho_\#,\gamma}(x) \geq |\langle f, (r/R)^{d+\gamma}\psi\rangle| = (r/R)^{d+\gamma}b$. Given that $b > a$ and that R is related to r and ε as in (4.572), it follows that there exists $\varepsilon = \varepsilon(\rho,d,\gamma,r,a,b) > 0$ such that $f^*_{\rho_\#,\gamma}(x) > a$ whenever $x \in B_{\rho_\#}(x_o,\varepsilon)$. This justifies the claim made in (4.571) and completes the proof of the lemma. □

Lemma 4.88. *Let $(X,\mathbf{q})$ be a quasimetric space, and suppose that μ is a measure on X that satisfies (4.562). Also, assume that $\rho \in \mathbf{q}$, $p \in (0,1]$ and $\gamma \in \mathbb{R}$ are such that $0 < \gamma < [\log_2 C_\rho]^{-1}$. Finally, consider a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'(X,\rho)$ with the property that*

$$\forall\, \varepsilon > 0 \;\exists\, N_\varepsilon \in \mathbb{N} \textit{ such that } \|(f_j - f_k)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} < \varepsilon \textit{ whenever } j,k \geq N_\varepsilon. \tag{4.573}$$

Then there exists a (unique) distribution $f \in \mathscr{D}'(X,\rho)$ for which

$$\lim_{j\to\infty} f_j = f \;\; \textit{in} \;\; \mathscr{D}'(X,\rho) \;\; \textit{and} \;\; \lim_{j\to\infty} \|(f - f_j)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} = 0. \tag{4.574}$$

Proof. With the goal of eventually employing Theorem 4.85, in a first stage we propose to show that

$$\big\{\langle f_j, \phi\rangle\big\}_{j\in\mathbb{N}} \text{ is a Cauchy sequence in } \mathbb{R} \text{ for each fixed } \phi \in \mathscr{D}(X,\rho). \tag{4.575}$$

To see this, pick an arbitrary $\phi \in \mathscr{D}(X,\rho) \subseteq \dot{\mathscr{C}}^\gamma_c(X,\rho)$. In particular, there exist $x_o \in X$ and $r > 0$ such that ϕ vanishes in $X \setminus B_{\rho_\#}(x_o,r)$. Hence, we can select a finite constant $C_\phi > 0$ with the property that $\phi/C_\phi \in \mathcal{T}^\gamma_{\rho_\#}(x)$ for every point $x \in B_{\rho_\#}(x_o,r)$. Consequently, for each $j,k \in \mathbb{N}$ we may write

$$|\langle f_j - f_k, \phi\rangle| \leq C_\psi (f_j - f_k)^*_{\rho_\#,\gamma}(x), \qquad \forall\, x \in B_{\rho_\#}(x_o,r). \tag{4.576}$$

In turn, raising both sides of this inequality to the pth power and integrating in $x \in B_{\rho_\#}(x_o, r)$ with respect to μ yields

$$|\langle f_j - f_k, \phi\rangle|^p \mu\big(B_{\rho_\#}(x_o, r)\big) \le C_\psi^p \int_{B_{\rho_\#}(x_o,r)} (f_j - f_k)^*_{\rho_\#,\gamma}(x)^p \,\mathrm{d}\mu(x), \qquad (4.577)$$

and hence

$$|\langle f_j - f_k, \phi\rangle| \le C_{\psi,r} \|(f_j - f_k)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)}, \qquad \forall\, j,k \in \mathbb{N}. \qquad (4.578)$$

Now, (4.575) follows from this and (4.573). Thus, Theorem 4.85 applies and gives the existence of a distribution $f \in \mathscr{D}'(X,\rho)$ for which $\lim_{j\to\infty} f_j = f$ in $\mathscr{D}'(X,\rho)$.

We are therefore left with showing that $(f - f_j)^*_{\rho_\#,\gamma} \to 0$ in $L^p(X,\mu)$ as $j \to \infty$. To this end, pick an arbitrary $\varepsilon > 0$ and, based on (4.573), select $N_\varepsilon \in \mathbb{N}$ with the property that $\|(f_j - f_k)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} < \varepsilon$ if $j,k \ge N_\varepsilon$. By once again relying on (4.573), we may inductively construct a subsequence $\{f_{j_n}\}_{n\in\mathbb{N}}$ of the original sequence of distributions such that

$$\int_X (f_{j_n} - f_{j_{n+1}})^*_{\rho_\#,\gamma}(x)^p \,\mathrm{d}\mu(x) < 2^{-n}, \qquad \forall\, n \in \mathbb{N}. \qquad (4.579)$$

Finally, consider a natural number $i \ge N_\varepsilon$ and pick $\ell \in \mathbb{N}$ so that $j_\ell \ge N_\varepsilon$ and $2^{-\ell} < \varepsilon$. Since we have

$$f - f_i = f_{j_\ell} - f_i + \sum_{n=\ell}^{\infty} (f_{j_{n+1}} - f_{j_n}) \quad \text{in } \mathscr{D}'(X,\rho), \qquad (4.580)$$

it follows that for every $x \in X$

$$(f - f_i)^*_{\rho_\#,\gamma}(x) \le (f_{j_\ell} - f_i)^*_{\rho_\#,\gamma}(x) + \sum_{n=\ell}^{\infty} (f_{j_{n+1}} - f_{j_n})^*_{\rho_\#,\gamma}(x). \qquad (4.581)$$

Given that $0 < p \le 1$, this and (4.568) in Lemma 4.87 further imply that

$$\|(f - f_i)^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)} \le \|(f_{j_\ell} - f_i)^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)} + \sum_{n=\ell}^{\infty} \|(f_{j_{n+1}} - f_{j_n})^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)}. \qquad (4.582)$$

Finally, on account of (4.579) and the choices we have made on the parameters N_ε, i, ℓ, we obtain from (4.582) that $\|(f - f_i)^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)} \le 3\varepsilon$. With this in hand, the desired conclusion (i.e., the last condition in (4.574)) follows. □

In the setting described at the beginning of this section, consider next a real number

$$p \in \Big(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\Big], \qquad (4.583)$$

and observe that this membership amounts to demanding that $0 < p \leq 1$ together with the existence of some $\rho \in \mathbf{q}$ with the property that $d(1/p - 1) < [\log_2 C_\rho]^{-1}$. For each such index p and quasidistance ρ define the `Hardy space` $H^p(X, \rho, \mu)$ by setting

$$H^p(X,\rho,\mu) := \Big\{ f \in \mathscr{D}'(X,\rho) : \ \forall\, \gamma \in \mathbb{R} \text{ so that } d\big(\tfrac{1}{p} - 1\big) < \gamma < [\log_2 C_\rho]^{-1}, \text{ it follows that } f^*_{\rho_\#,\gamma} \in L^p(X,\mu) \Big\}. \tag{4.584}$$

A closely related version of this Hardy space is $\widetilde{H}^p(X, \rho, \mu)$, with p and ρ as before, defined as

$$\widetilde{H}^p(X,\rho,\mu) := \Big\{ f \in \mathscr{D}'(X,\rho) : \ \exists\, \gamma \in \mathbb{R} \text{ so that } d\big(\tfrac{1}{p} - 1\big) < \gamma < [\log_2 C_\rho]^{-1} \text{ and with the property that } f^*_{\rho_\#,\gamma} \in L^p(X,\mu) \Big\}. \tag{4.585}$$

From this discussion it follows that $H^p(X, \rho, \mu) \subseteq \widetilde{H}^p(X, \rho, \mu)$, and our goal is to prove that one actually has equality. This is done in Theorem 4.91 below and, as preamble, it requires that we discuss the notions of atom and atomic Hardy space.

Specifically, given an exponent $p \in (0, 1]$ and assuming that (4.562) holds, call a function $a \in L^\infty(X, \mu)$ a p-`atom`, provided there exist $x \in X$ and a real number $r > 0$ with the property that (with $d \in (0, +\infty)$ as in (4.562))

$$\operatorname{supp} a \subseteq B_{\rho_o}(x,r), \quad \|a\|_{L^\infty(X,\mu)} \leq r^{-d/p}, \quad \int_X a \, \mathrm{d}\mu = 0. \tag{4.586}$$

In the case when $\mu(X) < +\infty$, it is also agreed that the constant function given by $a(x) := \big[\mu(X)\big]^{-1/p}$, for each $x \in X$, is a p-atom. Then, for each $p \in (0, 1]$, the `atomic Hardy space` $H^p_{\rm at}(X) := H^p_{\rm at}(X, \mathbf{q}, \mu)$ is introduced as

$$H^p_{\rm at}(X,\mathbf{q},\mu) := \Big\{ f \in \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \text{ and } p\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \ \text{ in } \ \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})\big)^* \Big\}. \tag{4.587}$$

Also, consider the quasinorm $\|\cdot\|_{H^p_{\rm at}(X)}$ defined for each $f \in H^p_{\rm at}(X)$ by

$$\|f\|_{H^p_{\rm at}(X)} := \inf \left\{ \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{1/p} : \ f = \sum_{j\in\mathbb{N}} \lambda_j a_j \ \text{ as in (4.587)} \right\}. \tag{4.588}$$

It is worth noting that, in the preceding context, for any quasidistance $\rho \in \mathbf{q}$ with the property that $d(1/p-1) < [\log_2 C_\rho]^{-1}$ we have

$$H^p_{\rm at}(X,\mathbf{q},\mu) \subseteq \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\rho)\big)^* \hookrightarrow \mathscr{D}'(X,\rho); \tag{4.589}$$

hence, in such a scenario, the elements in $H^p_{\rm at}(X,\mathbf{q},\mu)$ may be naturally viewed as distributions on X.

Let us momentarily digress for the purpose of establishing that the atomic Hardy space $H^p_{\rm at}(X,\mathbf{q},\mu)$ from (4.587) becomes trivial if $p \in (0,1]$ is small enough. To state a result to this effect, recall (4.323) (cf. also (4.333) in this regard).

Proposition 4.89. *Assume that* $(X,\mathbf{q})$ *is a quasimetric space and that* μ *is a measure on* X *satisfying* (4.562). *Then, if* $d \in (0,+\infty)$ *is as in* (4.562), *it follows that*

$$0 < p < \frac{d}{d+\mathrm{ind}_0(X,\mathbf{q})} \quad\Longrightarrow\quad H^p_{\rm at}(X,\mathbf{q},\mu) = \begin{cases} \{0\} & \textit{if} \quad \mu(X)=+\infty,\\ \mathbb{R} & \textit{if} \quad \mu(X)<+\infty. \end{cases} \tag{4.590}$$

Proof. Fix $p \in \mathbb{R}$ with

$$0 < p < \frac{d}{d+\mathrm{ind}_0(X,\mathbf{q})}, \tag{4.591}$$

and observe that this condition is equivalent to $\mathrm{ind}_0(X,\mathbf{q}) < d(1/p-1)$. Then, by Theorem 4.59 and Definition 4.57, we have $\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q}) = \mathbb{R}$ as vector spaces. First suppose that $\mu(X)=+\infty$, and recall that in this situation $\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})$ is regarded as a normed space modulo constant functions. Hence,

$$H^p_{\rm at}(X,\mathbf{q},\mu) \subseteq \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})\big)^* = \{0\}, \tag{4.592}$$

which completes the proof of (4.590) in the case when $\mu(X)=+\infty$. Next, assume that $\mu(X)<+\infty$. In this scenario, the constant function $a(x) := \big[\mu(X)\big]^{-1/p}$, for $x \in X$, is by definition a p-atom and, hence, $H^p_{\rm at}(X,\mathbf{q},\mu) \neq \{0\}$. On the other hand, when $\mu(X)<+\infty$, we have $\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q}) = \mathbb{R}$ as normed vector spaces. Thus,

$$H^p_{\rm at}(X,\mathbf{q},\mu) \subseteq \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})\big)^* = \mathbb{R}^* = \mathbb{R}, \tag{4.593}$$

forcing $H^p_{\rm at}(X,\mathbf{q},\mu) = \mathbb{R}$ in this case. This completes the proof of the proposition. □

Returning to the topic of studying the relationship between $H^p_{\rm at}(X)$ and $H^p(X,\rho,\mu)$, one inclusion between these spaces is established in the next lemma.

Lemma 4.90. *Assume that* $(X, \mathbf{q})$ *is a quasimetric space and that* μ *is a measure on* X *satisfying* (4.562)*. Then for each* p *as in* (4.583) *and each quasidistance* $\rho \in \mathbf{q}$ *for which* $d(1/p-1) < [\log_2 C_\rho]^{-1}$ *there holds*

$$H^p_{\rm at}(X) \subseteq H^p(X, \rho, \mu). \tag{4.594}$$

Proof. Fix p as in (4.583) along with a quasidistance $\rho \in \mathbf{q}$ and some index $\gamma \in \mathbb{R}$ satisfying $d(1/p-1) < \gamma < [\log_2 C_\rho]^{-1}$. Also, suppose that a is a p-atom supported in $B_{\rho_o}(x_*, r_*)$. Our goal is to show that there exists a finite constant $C > 0$ that is independent of the atom a, with the property that

$$\|a^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} \leq C. \tag{4.595}$$

Then, in the last part of the proof, we will indicate how (4.594) may be derived from (4.595) and Lemma 4.88.

Note that, by replacing ρ with its regularized version $\rho_\#$ as defined in Theorem 3.46 [which is known to satisfy $C_{\rho_\#} \leq C_\rho$; cf. (3.533)], there is no loss of generality in assuming that $\rho = \rho_\#$. Granted this, it follows that ρ is continuous in each of its variables. In particular, all ρ-balls are open and, hence, μ-measurable by (4.563) (the relevance of this comment will become apparent shortly).

To prove (4.595), pick an arbitrary point $x \in X$ and suppose that $\psi \in \mathcal{T}^\gamma_\rho(x)$ is supported in $B_\rho(x, r)$ for some real number $r > 0$ and is normalized as in (4.565) relative to r. Then, since $\rho \approx \rho_o$ and $B_\rho(x, r)$ is μ-measurable, we have

$$\begin{aligned} \big|\langle a, \psi\rangle\big| &\leq \int_{B_\rho(x,r)} |a(y)||\psi(y)|\,\mathrm{d}\mu(y) \leq \|a\|_{L^\infty(X,\mu)}\|\psi\|_{L^\infty(X,\mu)}\,\mu\big(B_\rho(x,r)\big) \\ &\leq C r_*^{-d/p} r^{-d} \mu\big(B_{\rho_o}(x, Cr)\big) \leq C r_*^{-d/p} r^{-d} r^d = C r_*^{-d/p} \end{aligned} \tag{4.596}$$

for some finite constant $C = C(\rho, \rho_o, \mu) > 0$ independent of a and ψ. Consequently, for each $x \in X$ we obtain

$$a^*_{\rho,\gamma}(x) = \sup_{\psi \in \mathcal{T}^\gamma_\rho(x)} \big|\langle a, \psi\rangle\big| \leq C r_*^{-d/p}, \tag{4.597}$$

with $C > 0$ as before. If we now pick a sufficiently large constant $M > 1$ (which is allowed to depend only on ρ), then we may estimate

$$\int_{B_{\rho_o}(x_*, Mr_*)} |a^*_{\rho,\gamma}(x)|^p\,\mathrm{d}\mu(x) \leq C(r_*^{-d/p})^p \mu\big(B_{\rho_o}(x_*, Mr_*)\big) \leq C, \tag{4.598}$$

again, for some finite constant $C > 0$ that is independent of a.

To estimate the contribution away from the support of the atom, for each $k \in \mathbb{N}$ let us introduce $A_k := B_{\rho_o}(x_*, M^{k+1}r_*) \setminus B_{\rho_o}(x_*, M^k r_*)$. To proceed, pick an arbitrary point $x \in X \setminus B_{\rho_o}(x_*, Mr_*)$ and suppose that $\psi \in \mathcal{T}_\rho^\gamma(x)$ is supported in $B_\rho(x, r)$ for some real number $r > 0$ and is normalized as in (4.565) relative to r. Then there exists $k \in \mathbb{N}$ so that $x \in A_k$, and we claim that there exist two constants $c = c(\rho, \rho_o) > 0$ and $C = C(\rho, \rho_o) > 0$, independent of a, ψ, k, r, r_*, with the property that

$$B_{\rho_o}(x_*, r_*) \cap B_\rho(x, r) \neq \emptyset \Longrightarrow r > cM^{k-1}r_*. \tag{4.599}$$

To justify this claim, note that if there exists $y \in B_{\rho_o}(x_*, r_*) \cap B_\rho(x, r)$, then we may write (keeping in mind that $\rho \approx \rho_o$)

$$\begin{aligned} M^k r_* &\leq \rho_o(x, x_*) \leq C\rho(x, x_*) \leq C\big(\rho(x, y) + \rho(y, x_*)\big) \leq C\big(r + \rho(y, x_*)\big) \\ &\leq C\big(r + C'\rho_o(y, x_*)\big) \leq C''r + C'''r_*, \end{aligned} \tag{4.600}$$

where all constants involved depend only on the proportionality factors of ρ and ρ_o. Hence, by eventually increasing M (in a manner that depends only on ρ and ρ_o), we may deduce from (4.600) that $r > cM^{k-1}r_*$, where $c = c(\rho, \rho_o) > 0$. This proves (4.599).

Next, based on the size, support, and cancellation conditions for the atom a, as well as the Hölder bound for the bump function ψ, we may write (keeping in mind that $\rho(\cdot, x_*)$ is continuous, hence μ-measurable; cf. (4.563))

$$\begin{aligned} \big|\langle a, \psi\rangle\big| &= \Big|\int_X a(y)(\psi(y) - \psi(x_*))\,\mathrm{d}\mu(y)\Big| \\ &= \Big|\int_{B_{\rho_o}(x_*, r_*)} a(y)(\psi(y) - \psi(x_*))\,\mathrm{d}\mu(y)\Big| \\ &\leq \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho)}\|a\|_{L^\infty(X,\mu)} \int_{B_{\rho_o}(x_*, r_*)} \rho(y, x_*)^\gamma\,\mathrm{d}\mu(y) \\ &\leq C\|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho)}\|a\|_{L^\infty(X,\mu)} \int_{B_{\rho_o}(x_*, r_*)} \rho_o(y, x_*)^\gamma\,\mathrm{d}\mu(y) \\ &\leq Cr^{-d-\gamma}r_*^{-d/p}r_*^\gamma\,\mu\big(B_{\rho_o}(x_*, r_*)\big) \leq Cr^{-d-\gamma}r_*^{-d/p+\gamma+d}. \end{aligned} \tag{4.601}$$

In turn, (4.601), (4.599), and support considerations imply that, for every $k \in \mathbb{N}$, we have

$$a^*_{\rho,\gamma}(x) \leq CM^{(k-1)(-d-\gamma)}r_*^{-d/p} \quad \text{whenever} \;\; x \in A_k, \tag{4.602}$$

where C is a positive, finite constant independent of a and k. Having established this, we may then proceed to estimate

$$
\begin{aligned}
\int_{X\setminus B_{\rho_o}(x_*,Mr)} |a^*_{\rho,\gamma}(x)|^p\,\mathrm{d}\mu(x) &= \sum_{k\in\mathbb{N}} \int_{A_k} |a^*_{\rho,\gamma}(x)|^p\,\mathrm{d}\mu(x) \\
&\leq C \sum_{k\in\mathbb{N}} M^{(k-1)(-dp-\gamma p)} r_*^{-d}\,\mu\big(B_{\rho_o}(x_*, M^{k+1}r_*)\big) \\
&\leq C \sum_{k\in\mathbb{N}} M^{(k-1)(-dp-\gamma p)} r_*^{-d}(M^{k+1}r_*)^d \\
&= C \sum_{k\in\mathbb{N}} M^{-k(-d+\gamma p+dp)} < +\infty
\end{aligned}
\tag{4.603}
$$

since $M > 1$ and since $\gamma > d(1/p-1)$ entails $-d+\gamma p+dp > 0$. In concert, (4.598) and (4.603) imply (4.595).

To complete the proof of the lemma, it remains to indicate how (4.594) may be derived from (4.595) and Lemma 4.88. In concrete terms, consider $f \in H^p_{\mathrm{at}}(X)$. Hence, $f \in \big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\rho)\big)^*$, and there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ and a sequence of p-atoms $\{a_j\}_{j\in\mathbb{N}}$ with the property that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $\big(\dot{\mathscr{C}}^{d(1/p-1)}(X,\rho)\big)^*$. For each $n \in \mathbb{N}$ introduce $f_n := \sum_{j=1}^n \lambda_j a_j$ and notice that, on the one hand,

$$
\lim_{n\to\infty} f_n = f \quad \text{in} \quad \mathscr{D}'(X,\rho). \tag{4.604}
$$

On the other hand, thanks to (4.595), whenever $n, m \in \mathbb{N}$ are such that $m \geq n$, we have

$$
\|(f_n - f_m)^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)} \leq \sum_{j=n}^m |\lambda_j|^p \|(a_j)^*_{\rho_\#,\gamma}\|^p_{L^p(X,\mu)} \leq C \sum_{j=n}^m |\lambda_j|^p. \tag{4.605}
$$

Together, (4.604), (4.605), and Lemma 4.88 then show that

$$
\lim_{n\to\infty} \|(f-f_n)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} = 0. \tag{4.606}
$$

In turn, since (4.595) also gives that for each $n \in \mathbb{N}$ we have

$$
\|(f_n)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} \leq C\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p}, \tag{4.607}
$$

with $C \in (0,+\infty)$ independent of n, we deduce from (4.606) and (4.607) that

$$
\|f^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} \leq C\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p} \tag{4.608}
$$

for some finite constant $C > 0$ that is independent of f. Hence, $f^*_{\rho_\#,\gamma} \in L^p(X,\mu)$, which proves (4.594). □

To summarize, the analysis so far shows that $H^p_{\rm at}(X) \subseteq H^p(X,\rho,\mu) \subseteq \widetilde{H}^p(X,\rho,\mu)$. Thus, to prove that all these spaces coincide (i.e., may be identified with one another in a natural fashion), it suffices to check that the injection $H^p_{\rm at}(X) \hookrightarrow \widetilde{H}^p(X,\rho,\mu)$ suffices. We will do so using a power-rescaling argument to reduce matters to the special case of the so-called normal spaces considered in [79].

Theorem 4.91. *Assume that* $(X,\mathbf{q})$ *is a quasimetric space and* μ *is a measure on* X *satisfying* (4.562)*, and fix*

$$p \in \Big(\frac{d}{d+\min\{d\,,\,\operatorname{ind}(X,\mathbf{q})\}}\,,\,1\Big],\tag{4.609}$$

where $d \in (0,+\infty)$ *is as in* (4.562)*. Suppose that*

$$\rho \in \mathbf{q} \text{ is such that } d\big(\tfrac{1}{p}-1\big) < \min\big\{d, [\log_2 C_\rho]^{-1}\big\}\tag{4.610}$$

and, for every functional $f \in H^p_{\rm at}(X)$*, denote by* $\widetilde{f}$ *the distribution in* $\mathscr{D}'(X,\rho)$ *defined as the restriction of* f *to* $\mathscr{D}(X,\rho)$*. Then the assignment* $f \mapsto \widetilde{f}$ *induces a well-defined, injective linear mapping from* $H^p_{\rm at}(X)$ *onto the space* $\widetilde{H}^p(X,\rho,\mu)$*. Moreover, for each*

$$\rho \in \mathbf{q} \text{ and } \gamma \in \mathbb{R} \text{ with } d\big(\tfrac{1}{p}-1\big) < \gamma < \min\big\{d, [\log_2 C_\rho]^{-1}\big\}\tag{4.611}$$

there exist two finite constants $c_1, c_2 > 0$ *such that*

$$c_1\|f\|_{H^p_{\rm at}(X)} \le \|(\widetilde{f}\,)^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} \le c_2\|f\|_{H^p_{\rm at}(X)} \quad \text{for all } f \in H^p_{\rm at}(X).\tag{4.612}$$

Consequently, the spaces $H^p(X,\rho,\mu)$ *and* $\widetilde{H}^p(X,\rho,\mu)$ *are naturally identified with* $H^p_{\rm at}(X)$*. In particular, they do not depend on the particular choice of the quasidistance* ρ *and index* γ *as in* (4.610)*, and their notation will be abbreviated to simply* $H^p(X)$ *and* $\widetilde{H}^p(X)$*. Hence,* $H^p(X) = \widetilde{H}^p(X) = H^p_{\rm at}(X)$*, and also*

$$\big(H^p(X)\big)^* = \dot{\mathscr{C}}^{d(1/p-1)}(X,\rho) \quad \text{if } \frac{d}{d+\min\{d,\operatorname{ind}(X,\mathbf{q})\}} < p < 1.\tag{4.613}$$

As a corollary, whenever (4.611) *holds, one can find a finite constant* $c = c(p,\rho,\gamma) > 0$ *such that for every distribution* $f \in \mathscr{D}'(X,\rho)$ *with the property that its grand maximal function* $f^*_{\rho_\#,\gamma}$ *belongs to* $L^p(X,\mu)$ *there exist a sequence of* p*-atoms* $\{a_j\}_{j\in\mathbb{N}}$ *on* X *and a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ *for which*

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } \mathscr{D}'(X,\rho)\tag{4.614}$$

and

$$\Big(\sum_{j\in\mathbb{N}}|\lambda_j|^p\Big)^{1/p} \le c\|f^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)}. \tag{4.615}$$

Finally, whenever (4.611) *holds, one can find a finite constant* $c' = c'(p,\rho,\gamma) > 0$ *such that, given a distribution* $f \in \mathscr{D}'(X,\rho)$*, a sequence of* p*-atoms* $\{a_j\}_{j\in\mathbb{N}}$*, and a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ *such that* (4.614) *holds, then*

$$\|f^*_{\rho_\#,\gamma}\|_{L^p(X,\mu)} \le c'\Big(\sum_{j\in\mathbb{N}}|\lambda_j|^p\Big)^{1/p}. \tag{4.616}$$

Proof. Assume that ρ and γ are as in (4.611), and note that these conditions imply that

$$p \in \Big(\frac{1}{1+d^{-1}[\log_2 C_{\rho,d}]^{-1}}\,,\,1\Big], \quad \text{where } \ C_{\rho,d} := \max\{C_\rho, 2^{1/d}\}. \tag{4.617}$$

Since $C_{\rho,d} \ge C_\rho$ and $d > 0$, it follows that ρ^d is a quasidistance on X satisfying

$$\rho(x,y)^d \le (C_{\rho,d})^d\ \max\{\rho(x,z)^d, \rho(y,z)^d\}, \quad \forall\, x,y,z \in X. \tag{4.618}$$

Consider now the triplet $(X,(\rho^d)_\#,\mu)$, where $(\rho^d)_\#$ is the canonical regularization of the dth power of the original quasidistance ρ, in the sense of (3.528). From conclusion (12) in Theorem 3.46 we know that $(\rho^d)_\# \approx \rho^d$ and that $(\rho^d)_\#$ satisfies a Hölder regularity condition as in (3.542) for any $\beta \in (0,\alpha]$, where

$$\alpha := [\log_2(C_{\rho,d})^d]^{-1} \in (0,1]. \tag{4.619}$$

This expression of α is derived from (3.514) and (3.515), used in the context of (4.618), while the fact that $\alpha \le 1$ is ensured by our choice of $C_{\rho,d}$ in (4.617).

Furthermore, since by conclusion (12) in Theorem 3.46 the function $(\rho^d)_\#$ is continuous in each of its variables on the topological space $(X,\tau_{\mathbf{q}})$, it follows that for every $x \in X$ and $r > 0$ the set $B_{(\rho^d)_\#}(x,r)$ is open in $\tau_{\mathbf{q}}$, hence μ-measurable by (4.563). In turn, this and condition (4.562) then allow us to write

$$\mu\big(B_{(\rho^d)_\#}(x,r)\big) \approx \mu\big(B_{\rho_o^d}(x,r)\big) = \mu\big(B_{\rho_o}(x,r^{1/d})\big) \approx (r^{1/d})^d = r \tag{4.620}$$

uniformly for $x \in X$ and $0 < r \le \operatorname{diam}_{\rho_o}(X)$. This analysis then shows that $(X,(\rho^d)_\#,\mu)$ becomes a normal space of order $\alpha := [\log_2(C_{\rho,d})^d]^{-1} \in (0,1]$, as defined in [80, p. 272] (we wish to stress that it is the condition $\alpha \le 1$ that allows us to derive the analog of the estimate [80, (1.9), p. 272] for $(\rho^d)_\#$ from the aforementioned Hölder regularity condition satisfied by this function). Also, there exist $C_1, C_2 \in (0,+\infty)$ with the property that for each $x \in X$ we have

$$\psi \in \mathcal{T}_\rho^\gamma(x) \Longrightarrow \psi/C_1 \in \mathcal{T}_{(\rho^d)_\#}^{\gamma/d}(x),$$
$$\psi \in \mathcal{T}_{(\rho^d)_\#}^{\gamma/d}(x) \Longrightarrow \psi/C_2 \in \mathcal{T}_\rho^\gamma(x). \tag{4.621}$$

Last, observe that our conditions on p and γ imply that

$$0 < \gamma/d < \alpha \quad \text{and} \quad \frac{1}{1+(\gamma/d)} < p \leq 1. \tag{4.622}$$

Then all claims in the statement of the theorem, with the exception of (4.613), follow from [80, Theorem 5.9, p. 306] (cf. also [80, Theorem 4.13, p. 299]) used for $(X, (\rho^d)_\#, \mu)$, and after rewriting the corresponding conclusions in terms of the quasidistance ρ.

Finally, the duality formula (4.613) is a consequence of the identification of $H^p_{\rm at}(X)$ with $H^p(X)$ proved earlier, and of [35, Theorem B, p. 593]. □

We close this section by noting that if $(X, \mathbf{q})$ is a quasimetric space, μ is a measure on X satisfying (4.562), and $\rho \in \mathbf{q}$ and $\gamma \in \mathbb{R}$ are such that

$$0 \leq d\Big(\tfrac{1}{p} - 1\Big) < \gamma < \min\big\{d, [\log_2 C_\rho]^{-1}\big\}, \tag{4.623}$$

then Theorem 4.91 shows that

$$H^p_{\rm at}(X, \mathbf{q}, \mu) = \Big\{f \in \mathscr{D}'(X, \rho) :\ f^*_{\rho\#,\gamma} \in L^p(X, \mu)\Big\}. \tag{4.624}$$

4.10 Approximation to the Identity on Ahlfors-Regular Quasimetric Spaces

Here we are concerned with the smoothness (measured on Hölder scales) of approximations to the identity on Ahlfors-regular quasimetric spaces. Our main result in this section, Theorem 4.93, extends similar results established in [37, p. 40], [39, p. 16], [56, pp. 10–11], [80, Lemma 3.15, pp. 285–286] to the optimal range of smoothness. To get started, we record the following definition.

Definition 4.92. Assume that $(X, \mathbf{q})$ is a quasimetric space, and suppose that k_X is a finite integer if $\operatorname{diam} X < +\infty$ and $k_X := -\infty$ if $\operatorname{diam} X = +\infty$ (for example, pick $k_X \in \mathbb{Z} \cup \{-\infty\}$ such that $\operatorname{diam} X \leq 2^{-k_X} \leq 2 \cdot \operatorname{diam} X$). Also, let μ be a measure on X satisfying the Ahlfors-regularity condition stated in (4.562). In this context, call a family $\{\mathcal{S}_k\}_{k\in\mathbb{Z}, k\geq k_X}$ of integral operators

$$\mathcal{S}_k f(x) := \int_X S_k(x, y) f(y)\, d\mu(y), \qquad x \in X, \tag{4.625}$$

with integral kernels $S_k : X \times X \to \mathbb{R}$, an `approximation to the identity of order` $\varepsilon > 0$ provided there exist $\rho \in \mathbf{q}$ and $C \in (0, +\infty)$ such that, for every $k \in \mathbb{Z}$ with $k \geq k_X$, the following properties hold (recall that $d > 0$ is as in (4.562)):

(i) $0 \leq S_k(x, y) \leq C2^{kd}$ for all $x, y \in X$, and $S_k(x, y) = 0$ if $\rho(x, y) \geq C2^{-k}$.
(ii) $|S_k(x, y) - S_k(x', y)| \leq C2^{k(d+\varepsilon)}\rho(x, x')^{\varepsilon}$ for every $x, x', y \in X$.
(iii) $\big|[S_k(x, y) - S_k(x', y)] - [S_k(x, y') - S_k(x', y')]\big| \leq C2^{k(d+2\varepsilon)}\rho(x, x')^{\varepsilon}\rho(y, y')^{\varepsilon}$ for every $x, x', y, y' \in X$.
(iv) $S_k(x, y) = S_k(y, x)$ for every $x, y \in X$, and $\int_X S_k(x, y)\, d\mu(y) = 1$ for every $x \in X$.

Clearly, if the operators $\{\mathcal{S}_k\}_{k \in \mathbb{Z}, k \geq k_X}$ form an approximation to the identity of certain order $\varepsilon > 0$, then their integral kernels continue to satisfy (i)–(iv) with ρ replaced by any other quasidistance $\rho' \in \mathbf{q}$.

Theorem 4.93. *Let $(X, \mathbf{q})$ be a quasimetric space, and assume that μ is a measure on X that satisfies the d-dimensional Ahlfors-regularity condition stated in (4.562) for some $d \in (0, +\infty)$. Then for any number ε_o such that*

$$0 < \varepsilon_o < \min\{d + 1, \operatorname{ind}(X, \mathbf{q})\} \tag{4.626}$$

there exists a family $\{\mathcal{S}_k\}_{k \in \mathbb{Z}, k \geq k_X}$ of integral operators on X constituting an approximation to the identity (in the sense of Definition 4.92) of any order $\varepsilon \in (0, \varepsilon_o]$.

Furthermore, given $p \in [1, \infty]$ and $f \in L^p(X, \mu)$, it follows that any approximation to the identity $\{\mathcal{S}_k\}_{k \in \mathbb{Z}, k \geq k_X}$, of any positive order ε, satisfies

$$\sup_{k \in \mathbb{Z}, k \geq k_X} \|\mathcal{S}_k\|_{L^p(X,\mu) \to L^p(X,\mu)} < +\infty, \tag{4.627}$$

$$\|\mathcal{S}_k f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q})} \leq C2^{k(\varepsilon + d/p)}\|f\|_{L^p(X,\mu)}, \quad \forall\, k \in \mathbb{Z}, \quad k \geq k_X, \tag{4.628}$$

$$\sup_{k \in \mathbb{Z}, k \geq k_X} \|\mathcal{S}_k\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q}) \to \dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q})} < +\infty, \tag{4.629}$$

$$\sup_{k \in \mathbb{Z}, k \geq k_X} \|\mathcal{S}_k\|_{\mathscr{C}^{\varepsilon}(X,\mathbf{q}) \to \mathscr{C}^{\varepsilon}(X,\mathbf{q})} < +\infty, \tag{4.630}$$

$$\lim\nolimits_{k \to +\infty} \mathcal{S}_k g = g \quad \textit{in } \dot{\mathscr{C}}^{\alpha}(X, \mathbf{q}) \textit{ for any } g \in \dot{\mathscr{C}}^{\varepsilon}(X, \mathbf{q}) \textit{ and } \alpha \in (0, \varepsilon), \tag{4.631}$$

$$\lim\nolimits_{k \to +\infty} \mathcal{S}_k g = g \quad \textit{in } \mathscr{C}^{\alpha}(X, \mathbf{q}) \textit{ for any } g \in \mathscr{C}^{\varepsilon}(X, \mathbf{q}) \textit{ and } \alpha \in (0, \varepsilon). \tag{4.632}$$

In addition, if the measure μ is actually Borel regular on $(X, \tau_{\mathbf{q}})$ and $1 \leq p < \infty$, then

$$\lim_{k \to +\infty} \mathcal{S}_k f = f \quad \textit{in } L^p(X, \mu), \tag{4.633}$$

whereas if diam $X = +\infty$ *and* $1 < p < \infty$*, then any approximation to the identity satisfies*

$$\lim_{k\to-\infty} \mathcal{S}_k f = 0 \quad \textit{in } L^p(X,\mu). \tag{4.634}$$

Proof. We revisit an approach originally due to Coifman (see the discussion on [37, pp. 16–17 and p. 40]) with the goal of monitoring the maximal amount of Hölder regularity for the integral kernels in the setting we are considering. We will first treat the case when diam $X = +\infty$, in which scenario $k_X = -\infty$. To get started, fix some number $\varepsilon_o \in \big(0, \min\{d+1, \operatorname{ind}(X,\mathbf{q})\}\big)$ and select $\rho \in \mathbf{q}$ with the property that $0 < \varepsilon_o < [\log_2 C_\rho]^{-1}$ [cf. (4.255)]. Also, fix an arbitrary number $\varepsilon \in (0, \varepsilon_o]$. Next, let $\rho_\#$ be the regularized version of ρ as described in conclusion (11) of Theorem 3.46 and recall from (3.530) that $\rho_\# \in \mathbf{q}$. In addition, from conclusion (12) of Theorem 3.46 (cf. (3.542)) we have

$$\big|\rho_\#(x,y) - \rho_\#(x,z)\big| \leq \frac{1}{\varepsilon} \max\big\{\rho_\#(x,y)^{1-\varepsilon}, \rho_\#(x,z)^{1-\varepsilon}\big\}\big[\rho_\#(y,z)\big]^{\varepsilon} \tag{4.635}$$

whenever $x, y, z \in X$ (with the understanding that $x \notin \{y,z\}$ when $\varepsilon > 1$). The idea now is to consider a nonnegative function $h \in \mathscr{C}^1(\mathbb{R})$ with the property that $h \equiv 1$ on $[-1/2, 1/2]$ and $h \equiv 0$ on $\mathbb{R} \setminus (-2,2)$ and, for each $k \in \mathbb{Z}$, let T_k be the integral operator on (X,μ) with integral kernel $2^{kd} h\big(2^k \rho_\#(x,y)\big)$ for $x, y \in X$. Based on properties of the function h and the Ahlfors-regularity condition for μ, it is straightforward to check that there exists a finite constant $C_o \geq 1$ such that

$$C_o^{-1} \leq (T_k 1)(x) \leq C_o \qquad \text{for each } x \in X \text{ and each } k \in \mathbb{Z}. \tag{4.636}$$

Keeping this in mind, for each $k \in \mathbb{Z}$ it is then meaningful to define

$$S_k(x,y) := \frac{2^{2kd}}{(T_k 1)(x)(T_k 1)(y)} \int_X \frac{h\big(2^k \rho_\#(x,z)\big) h\big(2^k \rho_\#(z,y)\big)}{T_k\Big(\frac{1}{T_k 1}\Big)(z)}\, d\mu(z) \tag{4.637}$$

for each $x, y \in X$. Also, for $\rho_o \in \mathbf{q}$ as in (4.562) we have

$$\begin{aligned} 0 \leq \int_X \frac{h\big(2^k \rho_\#(x,z)\big) h\big(2^k \rho_\#(z,y)\big)}{T_k\Big(\frac{1}{T_k 1}\Big)(z)}\, d\mu(z) &\leq C \int_X h\big(2^k \rho_\#(x,z)\big)\, d\mu(z) \\ &\leq C\mu\big(B_{\rho_o}(x, C2^{-k})\big) \leq C2^{-kd}, \end{aligned} \tag{4.638}$$

by the choice of h, the Ahlfors-regularity condition for μ, and (4.636). With this in hand, the properties listed in (i) are direct consequences of (4.637), (4.636), and (4.638). In turn, it is easy to check that (i) implies (ii) in the case when $y \in X$ and $x, x' \in X$ satisfy $\rho_\#(x,x') \geq c_o 2^{-k}$ for some fixed $c_o \in (0,+\infty)$. Hence, as far as property (ii) is concerned, there remains to check the case when $y \in X$ and $x, x' \in X$ satisfy $\rho_{\#}(x, x') < c_o 2^{-k}$ for some fixed $c_o \in (0, +\infty)$. To this end, write

$$S_k(x, y) - S_k(x', y) = I + II, \tag{4.639}$$

where I, II are given respectively by

$$\frac{2^{2kd}}{(T_k 1)(y)}\Big(\frac{1}{(T_k 1)(x)} - \frac{1}{(T_k 1)(x')}\Big)\int_X \frac{h\big(2^k\rho_{\#}(x,z)\big)h\big(2^k\rho_{\#}(z,y)\big)}{T_k\Big(\frac{1}{T_k 1}\Big)(z)}\,\mathrm{d}\mu(z) \tag{4.640}$$

and

$$\frac{2^{2kd}}{(T_k 1)(y)(T_k 1)(x')}\int_X \frac{\big[h\big(2^k\rho_{\#}(x,z)\big) - h\big(2^k\rho_{\#}(x',z)\big)\big]h\big(2^k\rho_{\#}(z,y)\big)}{T_k\Big(\frac{1}{T_k 1}\Big)(z)}\,\mathrm{d}\mu(z). \tag{4.641}$$

Going further, for some constant $C \in (0, +\infty)$ sufficiently large relative to c_o, we estimate

$$\begin{aligned}\Big|\frac{1}{(T_k 1)(x)} - \frac{1}{(T_k 1)(x')}\Big| &\le C\big|(T_k 1)(x) - (T_k 1)(x')\big| \\ &\le C 2^{kd}\int_X \big|h\big(2^k\rho_{\#}(x,y)\big) - h\big(2^k\rho_{\#}(x',y)\big)\big|\,\mathrm{d}\mu(y) \\ &= C 2^{kd}\int_D \big|h\big(2^k\rho_{\#}(x,y)\big) - h\big(2^k\rho_{\#}(x',y)\big)\big|\,\mathrm{d}\mu(y),\end{aligned} \tag{4.642}$$

where $D := \{y \in X : 2^k\rho_{\#}(x,y) < 2 \text{ or } 2^k\rho_{\#}(x',y) < 2\}$, by the support condition on h. In particular, given that we are assuming $\rho_{\#}(x,x') < c_o 2^{-k}$, it follows that

$$D \subseteq B_{\rho_{\#}}(x, C2^{-k}) \cap B_{\rho_{\#}}(x', C2^{-k}) \tag{4.643}$$

for some finite, large constant $C > 0$. Consequently, using the mean value theorem and (4.635), the last expression in (4.642) may be further bounded by

$$\begin{aligned}&C\varepsilon^{-1}2^{k(d+1)}\|h'\|_{L^\infty(\mathbb{R})}\big[\rho_{\#}(x,x')\big]^{\varepsilon}\int_D \max\big\{\rho_{\#}(x,y)^{1-\varepsilon}, \rho_{\#}(x',y)^{1-\varepsilon}\big\}\,\mathrm{d}\mu(y) \\ &\quad\le C2^{k(d+1)}\big[\rho(x,x')\big]^{\varepsilon}\int_D \big\{\rho_{\#}(x,y)^{1-\varepsilon} + \rho_{\#}(x',y)^{1-\varepsilon}\big\}\,\mathrm{d}\mu(y)\end{aligned}$$

$$\le C2^{k(d+1)}\big[\rho_\#(x,x')\big]^{\varepsilon}\times$$
$$\times\Big\{\int_{B_{\rho_\#}(x,C2^{-k})}\rho_\#(x,y)^{1-\varepsilon}\,\mathrm{d}\mu(y)+\int_{B_{\rho_\#}(x',C2^{-k})}\rho_\#(x',y)^{1-\varepsilon}\,\mathrm{d}\mu(y)\Big\},\tag{4.644}$$

by (4.643). On the other hand, if $\rho_o\in\mathbf{q}$ is as in (4.562), then

$$\int_{B_{\rho_\#}(x,C2^{-k})}\rho_\#(x,y)^{1-\varepsilon}\,\mathrm{d}\mu(y)\le C\int_{B_{\rho_o}(x,C2^{-k})}\rho_o(x,y)^{1-\varepsilon}\,\mathrm{d}\mu(y)$$
$$\le C\sum_{j=0}^{+\infty}\int_{2^{-j-1}(C2^{-k})\le\rho_o(x,y)<2^{-j}(C2^{-k})}\rho_o(x,y)^{1-\varepsilon}\,\mathrm{d}\mu(y)$$
$$\le C\sum_{j=0}^{+\infty}\big(2^{-j}(C2^{-k})\big)^{1-\varepsilon}\big(2^{-j}(C2^{-k})\big)^{d}=C(2^{-k})^{1-\varepsilon+d}\tag{4.645}$$

for some $C\in(0,+\infty)$, given that $\varepsilon<d+1$. In concert, (4.642)–(4.645) give that

$$\Big|\frac{1}{(T_k1)(x)}-\frac{1}{(T_k1)(x')}\Big|\le C2^{k(d+1)}\big[\rho(x,x')\big]^{\varepsilon}(2^{-k})^{1-\varepsilon+d}=C2^{k\varepsilon}\big[\rho(x,x')\big]^{\varepsilon}.\tag{4.646}$$

Hence, all together, from (4.640), (4.636), (4.638), and (4.646) we deduce that

$$|I|\le C2^{k(d+\varepsilon)}\big[\rho(x,x')\big]^{\varepsilon},\tag{4.647}$$

which is of the right order. Moreover, based on the same ingredients, we may also show that $|II|\le C2^{k(d+\varepsilon)}\big[\rho(x,x')\big]^{\varepsilon}$, completing the proof of (ii) in the statement of the theorem.

Moving on, the estimate in part (iii) of the statement of the theorem is justified by first observing that if $k\in\mathbb{Z}$, then for every $x,x',y,y'\in X$ we have

$$[S_k(x,y)-S_k(x',y)]-[S_k(x,y')-S_k(x',y')]$$
$$=\int_X\Big[\frac{2^{kd}}{(T_k1)(x)}h\big(2^k\rho_\#(x,z)\big)-\frac{2^{kd}}{(T_k1)(x')}h\big(2^k\rho_\#(x',z)\big)\Big]\times$$
$$\times\left[\frac{2^{kd}h\big(2^k\rho_\#(z,y)\big)}{(T_k1)(y)T_k\Big(\frac{1}{T_k1}\Big)(z)}-\frac{2^{kd}h\big(2^k\rho_\#(z,y')\big)}{(T_k1)(y')T_k\Big(\frac{1}{T_k1}\Big)(z)}\right]\mathrm{d}\mu(z)\tag{4.648}$$

and then estimating the two expressions in the square brackets using the same type of ideas as in the first part of the proof. Finally, the algebraic identities in part (iv)

of the statement of the theorem are seen directly from (4.637) and Fubini's theorem. This concludes the proof of the fact that the family of integral operators (4.625) with kernels as in (4.637) is an approximation to the identity of order ε.

We will now turn to the proof of (4.627). With this goal in mind, given $p \in [1, \infty]$ and $f \in L^p(X, \mu)$, the properties listed in (i) give that for each $k \in \mathbb{Z}$ and $x \in X$,

$$|\mathcal{S}_k f(x)| \leq C \fint_{B_{\rho_\#}(x, C2^{-k})} |f| \, d\mu$$
$$\leq C \sup_{r>0} \Big(\fint_{B_{\rho_\#}(x,r)} |f| \, d\mu \Big) =: (\mathcal{M}_{\rho_\#} f)(x), \tag{4.649}$$

i.e., $|\mathcal{S}_k f|$ is pointwise dominated by a fixed multiple (which is independent of k) of $\mathcal{M}_{\rho_\#} f$, the Hardy–Littlewood maximal operator of f (constructed in relation to $\rho_\#$). Then (4.627) follows from this and the boundedness of $\mathcal{M}_{\rho_\#}$ on $L^p(X, \mu)$ in the case when $p > 1$. If $p = 1$, then we may directly estimate, based on Fubini's theorem and properties (i), (iii) from Definition 4.92,

$$\|\mathcal{S}_k f\|_{L^1(X,\mu)} \leq \int_X \Big(\int_X S_k(x, y) |f(y)| \, d\mu(y) \Big) d\mu(x)$$
$$= \int_X \Big(\int_X S_k(x, y) \, d\mu(x) \Big) |f(y)| \, d\mu(y) = \|f\|_{L^1(X,\mu)}. \tag{4.650}$$

Note that the fact that $|\mathcal{S}_k f(x)| < +\infty$ for μ-a.e. $x \in X$ is implicit in this estimate. Incidentally, this also shows (via interpolation between $p = 1$ and $p = \infty$) that the supremum in (4.627) is dominated by a constant independent of $p \in [1, \infty]$.

Consider now estimate (4.628). To set the stage, fix $k \in \mathbb{Z}$ satisfying $k \geq k_X$, along with $x, x' \in X$, and note that $S_k(x, y) = 0$ for each $y \in X \setminus B_{\rho_\#}(x, C2^{-k})$ and that $S_k(x', y) = 0$ for each $y \in X \setminus B_{\rho_\#}(x', C2^{-k})$. Assume first that $\rho_\#(x, x') < C2^{-k}$. In this scenario, $B_{\rho_\#}(x', C2^{-k}) \subseteq B_{\rho_\#}(x, C_\rho C2^{-k})$; hence if $p' \in [1, \infty]$ is such that $1/p + 1/p' = 1$, then by this observation, properties (i) and (ii) in Definition 4.92, and Hölder's inequality we may estimate

$$|\mathcal{S}_k f(x) - \mathcal{S}_k f(x')|$$
$$= \Big| \int_{B_{\rho_\#}(x, C_\rho C2^{-k})} S_k(x, y) f(y) \, d\mu(y) - \int_{B_{\rho_\#}(x, C_\rho C2^{-k})} S_k(x', y) f(y) \, d\mu(y) \Big|$$
$$\leq \int_{B_{\rho_\#}(x, C_\rho C2^{-k})} |S_k(x, y) - S_k(x', y)| |f(y)| \, d\mu(y)$$
$$\leq C 2^{k(d+\varepsilon)} \rho(x, x')^\varepsilon \int_{B_{\rho_\#}(x, C_\rho C2^{-k})} |f(y)| \, d\mu(y)$$

$$\leq C2^{k(d+\varepsilon)}\rho(x,x')^{\varepsilon}\Big(\int_X |f(y)|^p\,\mathrm{d}\mu(y)\Big)^{1/p}\mu\big(B_{\rho_\#}(x,C_\rho C2^{-k})\big)^{1/p'}$$
$$\leq C2^{k(d+\varepsilon)}2^{-kd/p'}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)}. \tag{4.651}$$

Hence,

$$|\mathcal{S}_k f(x)-\mathcal{S}_k f(x')| \leq C2^{k(\varepsilon+d/p)}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)} \quad \text{if} \;\; \rho_\#(x,x') < C2^{-k}. \tag{4.652}$$

Let us now consider the situation when $\rho_\#(x,x') \geq C2^{-k}$. Granted this, we may write

$$\frac{|\mathcal{S}_k f(x)-\mathcal{S}_k f(x')|}{\rho(x,x')^{\varepsilon}} \leq C2^{k\varepsilon}\big(|\mathcal{S}_k f(x)|+|\mathcal{S}_k f(x')|\big)$$
$$\leq C2^{k\varepsilon}\Big(⨍_{B_{\rho_\#}(x,C2^{-k})}|f|\,\mathrm{d}\mu + ⨍_{B_{\rho_\#}(x',C2^{-k})}|f|\,\mathrm{d}\mu\Big)$$
$$\leq C2^{k\varepsilon}\cdot 2^{kd/p}\|f\|_{L^p(X,\mu)} \tag{4.653}$$

by the first line in (4.649) and Hölder's inequality. Thus,

$$|\mathcal{S}_k f(x)-\mathcal{S}_k f(x')| \leq C2^{k(\varepsilon+d/p)}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)} \quad \text{if} \;\; \rho_\#(x,x') \geq C2^{-k}, \tag{4.654}$$

and (4.628) now follows from (4.652) and (4.654).

As regards (4.629), pick some $k \in \mathbb{Z}$ with $k \geq k_X$, fix two arbitrary points $x, x' \in X$, and select an arbitrary function $f \in \dot{\mathscr{C}}^{\varepsilon}(X,\rho)$. When $\rho(x,x') \leq C2^{-k}$, proceeding as in the first part of (4.651) and keeping in mind property (iv) from Definition 4.92, we obtain

$$|\mathcal{S}_k f(x)-\mathcal{S}_k f(x')|$$
$$= \Big|\int_{B_{\rho_\#}(x,C_\rho C2^{-k})} S_k(x,y)f(y)\,\mathrm{d}\mu(y) - \int_{B_{\rho_\#}(x,C_\rho C2^{-k})} S_k(x',y)f(y)\,\mathrm{d}\mu(y)\Big|$$
$$= \Big|\int_{B_{\rho_\#}(x,C_\rho C2^{-k})} \big[S_k(x,y)-S_k(x',y)\big](f(y)-f(x))\,\mathrm{d}\mu(y)\Big|$$
$$\leq \int_{B_{\rho_\#}(x,C_\rho C2^{-k})} |S_k(x,y)-S_k(x',y)||f(y)-f(x)|\,\mathrm{d}\mu(y)$$
$$\leq C2^{k(d+\varepsilon)}\rho(x,x')^{\varepsilon}2^{-kd}2^{-k\varepsilon}\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)} = C\rho(x,x')^{\varepsilon}\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}, \tag{4.655}$$

where we have also used properties (i) and (ii) from Definition 4.92. Furthermore,

$$|\mathcal{S}_k f(x)-\mathcal{S}_k f(x')|\leq\big|\big(\mathcal{S}_k f(x)-f(x)\big)-\big(\mathcal{S}_k f(x')-f(x')\big)\big|+|f(x)-f(x')|$$
$$\leq\big|\mathcal{S}_k f(x)-f(x)\big|+\big|\mathcal{S}_k f(x')-f(x')\big|+\rho(x,x')^{\varepsilon}\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\tag{4.656}$$

and when $\rho(x,x')\geq C2^{-k}$, we have, thanks to property (i) in Definition 4.92,

$$\big|\mathcal{S}_k f(x)-f(x)\big|=\Big|\int_{B_{\rho_\#}(x,C_\rho C2^{-k})}S_k(x,y)(f(y)-f(y))\,d\mu(y)\Big|$$
$$\leq C2^{kd}2^{-kd}2^{-k\varepsilon}\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}\leq C\rho(x,x')^{\varepsilon}\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\tag{4.657}$$

with a similar estimate for $\big|\mathcal{S}_k f(x')-f(x')\big|$. All together, this analysis proves that $\big\|\mathcal{S}_k f\big\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}\leq C\|f\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}$ for some finite constant $C>0$, independent of $k\in\mathbb{Z}$ with $k\geq k_X$. Hence, (4.629) follows. In turn, (4.630) is a consequence of (4.629) and (4.627) with $p=\infty$.

Turning our attention to (4.631), assume that $\alpha\in(0,\varepsilon)$, and fix an arbitrary function $g\in\dot{\mathscr{C}}^{\varepsilon}(X,\rho)$, along with $x,x'\in X$ and $k\in\mathbb{Z}$ with $k\geq k_X$. When $\rho(x,x')\leq C2^{-k}$, from (4.656) and (4.657) we have

$$\big|\big(\mathcal{S}_k g-g\big)(x)-\big(\mathcal{S}_k g-g\big)(x')|\leq C\rho(x,x')^{\varepsilon}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}$$
$$\leq C\rho(x,x')^{\alpha}\,2^{-k(\varepsilon-\alpha)}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\tag{4.658}$$

whereas when $\rho(x,x')\geq C2^{-k}$, from (4.657) we have

$$\big|\big(\mathcal{S}_k g-g\big)(x)-\big(\mathcal{S}_k g-g\big)(x')|\leq C2^{-k\varepsilon}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}$$
$$\leq C\rho(x,x')^{\alpha}\,2^{-k(\varepsilon-\alpha)}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}.\tag{4.659}$$

Combining (4.658) and (4.659) we therefore arrive at the conclusion that

$$\big\|\mathcal{S}_k g-g\big\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)}\leq C2^{-k(\varepsilon-\alpha)}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\tag{4.660}$$

which readily yields (4.631). In fact, since, much as in (4.657),

$$\sup_{x\in X}\big|\big(\mathcal{S}_k g-g\big)(x)\big|\leq C2^{-k\varepsilon}\|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\tag{4.661}$$

formula (4.632) subsequently follows from (4.661) and (4.631).

We next establish (4.634) while continuing to assume that diam $X=+\infty$. For starters, given $p\in[1,\infty)$ and $f\in L^p(X,\mu)$, the properties listed in (i) give that for each $k\in\mathbb{Z}$ and $x\in X$,

$$|\mathcal{S}_k f(x)| \le C \fint_{B_{\rho_\#}(x,C2^{-k})} |f|\,\mathrm{d}\mu \le C\Big(\fint_{B_{\rho_\#}(x,C2^{-k})} |f|^p\,\mathrm{d}\mu\Big)^{1/p}$$
$$\le C2^{kd/p}\Big(\int_X |f|^p\,\mathrm{d}\mu\Big)^{1/p} \to 0 \ \text{ as } k \to -\infty. \tag{4.662}$$

Since by (4.649) we have $|\mathcal{S}_k f| \le C\mathcal{M}_{\rho_\#} f$ pointwise on X for some $C \in (0,+\infty)$ independent of $k \in \mathbb{Z}$ and since $\mathcal{M}_{\rho_\#} f \in L^p(X,\mu)$ if $1 < p < \infty$, (4.634) follows with the help of Lebesgue's Dominated Convergence Theorem.

Moving on, in the situation when $\operatorname{diam} X < +\infty$, we proceed analogously, and properties (i)–(iv) in Definition 4.92, as well as (4.627), are established in a similar fashion (given that now $k \ge k_X \in \mathbb{Z}$ forces k to stay away from $-\infty$). It remains to prove that if μ is actually Borel regular on $(X,\tau_{\mathbf{q}})$, then (4.633) holds whenever $f \in L^p(X,\mu)$, $1 \le p < \infty$. To justify this in the case when $1 < p < \infty$, based on (i) and (iv) we may write

$$\int_X |\mathcal{S}_k f - f|^p\,\mathrm{d}\mu \le C\int_X \Big[\fint_{B_{\rho_\#}(x,C2^{-k})} |f(y)-f(x)|\,\mathrm{d}\mu(y)\Big]^p \mathrm{d}\mu(x). \tag{4.663}$$

With this in hand, the boundedness of $\mathcal{M}_{\rho_\#}$ on $L^p(X,\mu)$, together with Lebesgue's differentiation theorem (which, through the use of the density result from Theorem 4.13, presupposes that μ is a locally finite Borel regular measure) then yields (4.633).

When $p = 1$, given any $\delta > 0$, we may invoke Theorem 4.13 (recall that the measure μ is both locally finite and Borel regular on $(X,\tau_{\mathbf{q}})$) to obtain a function $g \in \dot{\mathscr{C}}^{\beta}_c(X,\mathbf{q})$, where β is a finite number with the property that $0 < \beta \le \big[\log_2 C_\rho\big]^{-1}$, satisfying

$$\|f-g\|_{L^1(X,\mu)} < \delta. \tag{4.664}$$

Then, for some finite constant $C > 0$ independent of k, g, and $x \in X$, we have (much as in (4.663))

$$|(\mathcal{S}_k g)(x) - g(x)| \le C\fint_{B_{\rho_\#}(x,C2^{-k})} |g(y)-g(x)|\,\mathrm{d}\mu(y) \le C\|g\|_{\dot{\mathscr{C}}^{\beta}_c(X,\mathbf{q})}2^{-k\beta}, \tag{4.665}$$

which shows that, on the one hand, $\|\mathcal{S}_k g - g\|_{L^\infty(X,\mu)} \le C\|g\|_{\dot{\mathscr{C}}^{\beta}_c(X,\mathbf{q})}2^{-k\beta}$, while, on the other hand, given that g vanishes outside of a ρ-bounded subset of X, it follows from property (i) in Definition 4.92 that there exists a bounded subset B of X outside of which $\mathcal{S}_k g$ vanishes for all $k \in \mathbb{N}$. From this analysis, and the fact that μ is locally finite, we may therefore conclude that

$$\lim_{k\to+\infty} \|\mathcal{S}_k g - g\|_{L^1(X,\mu)} = 0. \tag{4.666}$$

Since, thanks to (4.664) and (4.627) (with $p = 1$),

$$\begin{aligned}\|\mathcal{S}_k f - f\|_{L^1(X,\mu)} &\leq \|\mathcal{S}_k(f-g)\|_{L^1(X,\mu)} + \|\mathcal{S}_k g - g\|_{L^1(X,\mu)} + \|g - f\|_{L^1(X,\mu)} \\ &\leq C\delta + \|\mathcal{S}_k g - g\|_{L^1(X,\mu)}, \end{aligned} \tag{4.667}$$

it follows from (4.666) that $\lim_{k\to+\infty} \|\mathcal{S}_k f - f\|_{L^1(X,\mu)} = 0$, as desired. This completes the justification of (4.633) and completes the proof of Theorem 4.93. □

Remark 4.94. In the context of Theorem 4.93, if the Borel measure μ is not necessarily Borel regular, then the same proof that we saw previously yields, in place of (4.633), that for each $p \in [1, \infty]$ one has

$$\begin{aligned}&\lim_{k\to+\infty} \mathcal{S}_k f = f \quad \text{in } L^p(X,\mu) \text{ for each function } f \\ &\text{belonging to the closure of } \dot{\mathscr{C}}^\varepsilon(X,\mathbf{q}) \text{ in } L^p(X,\mu).\end{aligned} \tag{4.668}$$

4.11 Bi-Lipschitz Euclidean Embeddings of Quasimetric Spaces

Before presenting an application dealing with bi-Lipschitz Euclidean embeddings of quasimetric spaces, we review a few definitions and results.

Definition 4.95. Assume that (X, ρ) is a quasimetric space.

(1) Given $s \in [0, +\infty]$, call (X, ρ) s-`homogeneous` if there exists a finite constant $c \geq 0$ with the property that whenever $Y \subseteq X$ is such that there exist $b \geq a > 0$ satisfying $a \leq \rho(x, y) \leq b$ for all $x, y \in Y$ with $x \neq y$, then the cardinality of Y is $\leq c\left(\frac{b}{a}\right)^s$.
(2) The `Assouad dimension` of (X, ρ) is

$$\dim_A(X,\rho) := \inf\big\{s \in [0,+\infty] : (X,\rho) \text{ is } s\text{-homogeneous}\big\}. \tag{4.669}$$

Given a quasimetric space (X, ρ), it follows that $\dim_A(X, \rho) < +\infty$ if and only if there exist $C \geq 1$ and $s \in [0, +\infty)$ such that, for every $x \in X$ and for any real numbers R, r with $\operatorname{diam}_\rho(X) \geq R \geq r > 0$, the ρ-ball $B_\rho(x, R)$ can be covered by at most $C\left(\frac{R}{r}\right)^s$ ρ-balls of radii $\leq r$. The latter property is equivalent to the geometrically doubling condition (4.44), hence

$$\begin{aligned}&\text{a quasimetric space is geometrically doubling} \\ &\text{if and only if it has finite Assouad dimension.}\end{aligned} \tag{4.670}$$

In turn, since any space of homogeneous type is geometrically doubling, (4.670) gives that

$$(X, \rho, \mu) \text{ space of homogeneous type} \implies \dim_A(X, \rho) < +\infty. \tag{4.671}$$

In this vein, it is important to note that while there exist quasimetric spaces with finite Assouad dimension that do not carry a doubling measure, the following result, generalizing earlier work in the context of metric spaces, holds.

Theorem 4.96. *Any complete quasimetric space of finite Assouad dimension* (X, ρ) *carries a doubling measure. More precisely, for each exponent* $\alpha \in \big(\dim_A(X, \rho), +\infty\big)$ *there exists a Borel measure* μ *on* (X, τ_ρ) *with the property that, for some constant* $C \in (0, +\infty)$,

$$\begin{gathered} 0 < \mu\big(B_\rho(x, R)^{\text{circ}}\big) \leq C\Big(\frac{R}{r}\Big)^{\alpha} \mu\big(B_\rho(x, r)^{\text{circ}}\big) < +\infty \\ \textit{for all} \ \ x \in X \ \ \textit{and} \ \ 0 < r \leq R < +\infty, \end{gathered} \tag{4.672}$$

where, generally speaking, E^{circ} *denotes the interior, in the topology* τ_ρ*, of a set* $E \subseteq X$.

The completeness assumption is indispensable in the context of the foregoing theorem. For example, $(\mathbb{Q}, |\cdot - \cdot|)$ is a geometrically doubling quasimetric space that does not carry a doubling measure (since, in fact, no uniformly perfect, geometrically doubling metric space of zero Hausdorff dimension does; cf. the discussion in [59, p. 102]). The case of compact metric spaces has been dealt with by Vol'berg and Konyagin in [127], while the setting of complete metric spaces has been handled by Luukkainen and Saksman in [78]; see also the exposition in [59, Theorems 13.3, 13.5, pp. 103–104]. In the proof of the foregoing theorem, as well as for other subsequent considerations, the following lemma will be useful.

Lemma 4.97. *Let* (X, ρ) *be a quasimetric space. Then the following statements are true.*

(i) *For every* $\alpha \in (0, +\infty)$ *there holds* $\dim_A(X, \rho^\alpha) = \frac{1}{\alpha} \dim_A(X, \rho)$.
(ii) *If* ρ' *is a quasidistance on* X *such that* $\rho' \approx \rho$*, then* $\dim_A(X, \rho) = \dim_A(X, \rho')$.
(iii) *For every set* $\widetilde{X} \subseteq X$ *there holds* $\dim_A(\widetilde{X}, \rho) \leq \dim_A(X, \rho)$*. Moreover, if* $\widetilde{X}$ *is a dense subset of* X*, then actually* $\dim_A(\widetilde{X}, \rho) = \dim_A(X, \rho)$.
(iv) *The Assouad dimension is invariant under bi-Lipschitz surjections, in the sense that if* (X_j, ρ_j)*,* $j = 0, 1$*, are two quasimetric spaces and if* $\Phi : (X_0, \rho_0) \to (X_1, \rho_1)$ *is a bi-Lipschitz surjection, then* $\dim_A(X_0, \rho_0) = \dim_A(X_1, \rho_1)$.
(v) *For every* $n \in \mathbb{N}$ *one has* $\dim_A(\mathbb{R}^n, |\cdot - \cdot|) = n$*, where* $|\cdot|$ *denotes the standard Euclidean norm in* $\mathbb{R}^n$.

Proof. All the previously listed properties can be verified directly from definitions (cf. also [10, Proposition 2, p. 733], as well as [11, (a)–(c), p. 435] in the case of metric spaces). □

We are now prepared to give the proof of Theorem 4.96.

Proof of Theorem 4.96. Assume that (X, ρ) is a complete quasimetric space with the property that $\operatorname{diam}_A(X, \rho) < +\infty$, and fix a finite number $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$. From the discussion in conclusion (12) of Theorem 3.46 we know that $\big(X, (\rho_\#)^\beta\big)$ is a metric space. Given that $\big(X, \tau_{(\rho_\#)^\beta}\big) = (X, \tau_\rho)$ (again, see Theorem 3.46), and since we are assuming that (X, ρ) is complete, it follows that the metric space $\big(X, (\rho_\#)^\beta\big)$ is complete. In addition,

$$\dim_A\big(X, (\rho_\#)^\beta\big) = \frac{1}{\beta}\dim_A\big(X, \rho_\#\big) = \frac{1}{\beta}\dim_A\big(X, \rho\big) < +\infty, \tag{4.673}$$

by (i), (ii) in Lemma 4.97, and Theorem 3.46. Hence, if $\dim_A\big(X, \rho\big) < \alpha < +\infty$, then it follows from [59, Theorem 13.5, p. 104] that there exists a Borel measure μ on (X, τ_ρ) with the property that, for some finite constant $C > 0$,

$$0 < \mu\Big(B_{(\rho_\#)^\beta}(x, R)\Big) \le C\Big(\frac{R}{r}\Big)^{\alpha/\beta}\mu\Big(B_{(\rho_\#)^\beta}(x, r)\Big) < +\infty \tag{4.674}$$

for each $x \in X$ and $0 < r \le R < +\infty$. In turn, this implies that

$$0 < \mu\Big(B_{\rho_\#}(x, R)\Big) \le C\Big(\frac{R}{r}\Big)^{\alpha}\mu\Big(B_{\rho_\#}(x, r)\Big) < +\infty \tag{4.675}$$

for all $x \in X$ and $0 < r \le R < +\infty$. From (4.15) and the fact that the $\rho_\#$-balls are open in τ_ρ we also deduce that

$$B_{\rho_\#}(x, r) \subseteq B_\rho(x, r)^{\rm circ} \subseteq B_{\rho_\#}\big(x, C_\rho^2 r\big), \qquad \forall\, x \in X,\ \ \forall\, r \in (0, +\infty). \tag{4.676}$$

Now, (4.672) readily follows from (4.675) and (4.676). □

Moving on, thanks to property (ii) in Lemma 4.97, the following definition is meaningful.

Definition 4.98. For an arbitrary quasimetric space $(X, \mathbf{q})$ define its Assouad dimension as $\dim_A(X, \mathbf{q}) := \dim_A(X, \rho)$ for some (hence, any) $\rho \in \mathbf{q}$.

Assouad's embedding theorem (cf. [9, Proposition 1.30, p. I.29], [10, Remarque 2, p. 732], and [11, Proposition 2.6, p. 436]) asserts that each snowflaked version of a metric space of finite Assouad dimension admits a bi-Lipschitz embedding into some (finite-dimensional) Euclidean space. We may now state and prove an extension of this result to the setting of quasimetric spaces of finite Assouad dimension. Our theorem shows that the example of a subset of some $\mathbb{R}^n$ endowed with a power of the standard Euclidean distance is prototypical (i.e., always realizable, up to a bi-Lipschitz homeomorphism) in the class of all quasimetric spaces of finite Assouad dimension.

Theorem 4.99. *Let $(X, \mathbf{q})$ be a quasimetric space. Then the following three conditions are equivalent:*

(1) For any $\beta \in \mathbb{R}$ *satisfying* $0 < \beta < \text{ind}\,(X, \mathbf{q})$ *there exist a number* $n \in \mathbb{N}$*, a set* $\widetilde{X} \subseteq \mathbb{R}^n$*, a quasidistance* $\rho \in \mathbf{q}$*, and a bi-Lipschitz homeomorphism*

$$\Phi : (X, \rho) \longrightarrow (\widetilde{X}, |\cdot - \cdot|^{1/\beta}), \tag{4.677}$$

where $|\cdot|$ *denotes the standard Euclidean norm in* $\mathbb{R}^n$*.*

(2) The same embedding property as in (1) *but for just some number* $\beta \in \mathbb{R}$ *satisfying* $0 < \beta < \text{ind}\,(X, \mathbf{q})$.

(3) $(X, \mathbf{q})$ *is of finite Assouad dimension, i.e.,* $\dim_A(X, \mathbf{q}) < +\infty$.

Proof. That (1) $\Rightarrow$ (2) is obvious. As regards the implication (2) $\Rightarrow$ (3), note that for the given β, ρ, and $\widetilde{X}$ we may write

$$\begin{aligned}\dim_A(X, \rho) &= \dim_A(\widetilde{X}, |\cdot - \cdot|^{1/\beta}) = \beta \dim_A(\widetilde{X}, |\cdot - \cdot|) \\ &\leq \beta \dim_A(\mathbb{R}^n, |\cdot - \cdot|) = \beta n < +\infty, \end{aligned} \tag{4.678}$$

by (iv), (i), (iii), and (v) in Lemma 4.97. The desired conclusion follows.

To justify the implication (3) $\Rightarrow$ (1), given $\beta \in \big(0, \text{ind}\,(X, \mathbf{q})\big)$, consider a quasidistance $\rho \in \mathbf{q}$ with the property that $0 < \beta < [\log_2 C_\rho]^{-1}$ and select $\varepsilon \in (0, 1)$ such that $\beta/\varepsilon < [\log_2 C_\rho]^{-1}$. Then, if $\rho_\#$ is the regularized version of ρ as in (12) of Theorem 3.46, then we may conclude that $(X, (\rho_\#)^{\beta/\varepsilon})$ is a metric space. Furthermore, thanks to the properties (i) and (ii) listed in Lemma 4.97, we have $\dim_A(X, (\rho_\#)^{\beta/\varepsilon}) < +\infty$. Hence, Assouad's embedding theorem cited earlier ensures the existence of some $n \in \mathbb{N}$ and of a bi-Lipschitz function from the ε-snowflaked version of $(X, (\rho_\#)^{\beta/\varepsilon})$ into $(\mathbb{R}^n, |\cdot - \cdot|)$. Thus, Φ satisfies

$$|\Phi(x) - \Phi(y)| \approx [\rho(x, y)]^\beta \qquad \text{uniformly for } x, y \in X, \tag{4.679}$$

so that, in particular, H maps X homeomorphically onto its image. Consequently, taking $\widetilde{X} := \Phi(X) \subseteq \mathbb{R}^n$, the implication (3) $\Rightarrow$ (1) follows. □

Remark 4.100. (i) In general, it is not true that any metric space admits a bi-Lipschitz embedding into some finite-dimensional Euclidean space endowed with its standard Euclidean distance. A counterexample is offered by the Heisenberg group $\mathbb{H}_n$ equipped with the Carnot metric from (3.307). This follows from the work in [94] (cf. also the discussion in [59, p. 99]).

(ii) Given two quasimetric spaces (X, ρ), (Y, ϱ), it is natural to refer to a function $\Phi : X \to Y$ with the property that there exists $\beta \in (0, +\infty)$ such that

$$\varrho\big(\Phi(x), \Phi(y)\big) \approx \rho(x, y)^\beta \qquad \text{uniformly in } x, y \in X, \tag{4.680}$$

as being β-`bi-Hölder`. With this piece of terminology, (4.679) expresses the fact that Φ is a β-bi-Hölder embedding of (X, ρ) into $(\mathbb{R}^n, |\cdot - \cdot|)$. Thus, at least heuristically, any quasimetric space of finite Assouad dimension may be regarded as a Hölder manifold.

Theorem 4.99 extends [123, Theorem 1.192, p. 114], where a smaller range for β has been treated (more specifically, $0 < \beta < \alpha$, where α is as in (1.2)). As is apparent from the work in [123] (and the references therein), the significance of such an embedding result stems from the fact that this allows one to transport classes of smoothness and their properties from Euclidean spaces onto the quasimetric space, much as local coordinate charts are used to define smoothness spaces on a manifold. For example, Theorem 4.99 readily yields the following corollary.

Corollary 4.101. *Let $(X, \mathbf{q})$ be a complete quasimetric space with the property that $\dim_A(X, \mathbf{q}) < +\infty$. Then the topology $\tau_{\mathbf{q}}$ is both locally compact and sigma-compact.*

Proof. Pick $\beta \in \big(0, \mathrm{ind}\,(X, \mathbf{q})\big)$. From Theorem 4.99 and the current assumptions it follows that there exist a number $n \in \mathbb{N}$, a set $\widetilde{X} \subseteq \mathbb{R}^n$, a quasidistance $\rho \in \mathbf{q}$, and a bi-Lipschitz homeomorphism $\Phi : (X, \rho) \to (\widetilde{X}, |\cdot - \cdot|^{1/\beta})$. In particular, the fact that $(X, \mathbf{q})$ is complete forces $\widetilde{X}$ to be closed in $\mathbb{R}^n$. Hence, $(\widetilde{X}, \tau_{|\cdot-\cdot|^{1/\beta}}) = (\widetilde{X}, \tau_{|\cdot-\cdot|})$ is a topological space that is both locally compact and sigma-compact, and the desired conclusion follows by noting that these properties may be transported back to $(X, \mathbf{q})$ via the homeomorphism Φ. □

Another concrete example, addressed in Theorem 4.102 below, deals with the issue of interpolation of Hölder spaces in the context of quasimetric spaces of finite Assouad dimension.

Theorem 4.102. *Let $(X, \mathbf{q})$ be a quasimetric space of finite Assouad dimension, and let $E \subseteq X$ be an arbitrary nonempty set. Then*

$$\big\{\dot{\mathscr{C}}^{\alpha}(E)\big\}_{0<\alpha<\mathrm{ind}\,(X,\mathbf{q})} \quad \textit{is a real interpolation scale,} \tag{4.681}$$

in the sense that whenever $0 < \alpha_0, \alpha_1 < \mathrm{ind}\,(X, \mathbf{q})$, and $\theta \in (0, 1)$, one has

$$\big(\dot{\mathscr{C}}^{\alpha_0}(E)\,,\, \dot{\mathscr{C}}^{\alpha_1}(E)\big)_{\theta,\infty} = \dot{\mathscr{C}}^{\alpha}(E), \qquad \alpha := (1-\theta)\alpha_0 + \theta\alpha_1. \tag{4.682}$$

Furthermore, a similar result is valid for the inhomogeneous Hölder scale.

Proof. Thanks to the existence of the linear, bounded, universal extension operator $\mathscr{E}$ from Theorem 4.11, it follows that the scale $\big\{\dot{\mathscr{C}}^{\alpha}(E)\big\}_{0<\alpha<\mathrm{ind}\,(X,\mathbf{q})}$ is a retract of the scale $\big\{\dot{\mathscr{C}}^{\alpha}(X)\big\}_{0<\alpha<\mathrm{ind}\,(X,\mathbf{q})}$. As such, abstract interpolation results (cf. [16, Exercise 18(a), p. 81]) show that it suffices to establish (4.682) when $E := X$. In such a scenario, fix two exponents $0 < \alpha_0, \alpha_1 < \mathrm{ind}\,(X, \mathbf{q})$ and pick a quasidistance $\rho \in \mathbf{q}$ and a number β with the property that $\max\{\alpha_0, \alpha_1\} < \beta < [\log_2 C_\rho]^{-1}$. Next, consider the bi-Lipschitz homeomorphism $\Phi : (X, \rho) \to (\Phi(X), |\cdot - \cdot|^{1/\beta})$ from (4.677) in Theorem 4.99. Then the operator $T : \dot{\mathscr{C}}^{\alpha}(X) \to \dot{\mathscr{C}}^{\alpha/\beta}(\Phi(X))$, given by $Tf := f \circ \Phi^{-1}$, acts isomorphically for each $\alpha \in (0, \beta)$. Indeed, if $f \in \dot{\mathscr{C}}^{\alpha}(X)$, then for each $x, y \in \Phi(X)$

$$|Tf(x) - Tf(y)| = |f(\Phi^{-1}(x)) - f(\Phi^{-1}(y))| \leq \|f\|_{\dot{\mathscr{C}}^{\alpha}(X)} \rho\big(\Phi^{-1}(x), \Phi^{-1}(y)\big)^{\alpha}$$
$$\approx \|f\|_{\dot{\mathscr{C}}^{\alpha}(X)} |x-y|^{\alpha/\beta}, \tag{4.683}$$

which shows that $Tf \in \dot{\mathscr{C}}^{\alpha/\beta}(\Phi(X))$ and $\|Tf\|_{\dot{\mathscr{C}}^{\alpha/\beta}(\Phi(X))} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(X)}$. In a similar manner, one can then check that the inverse of T, namely, $T^{-1}f = f \circ \Phi$, is well defined and bounded from $\dot{\mathscr{C}}^{\alpha/\beta}(\Phi(X))$ into $\dot{\mathscr{C}}^{\alpha}(X)$.

At this stage, given the goal we have in mind, matters have been reduced to proving (4.682) when $E \subseteq \mathbb{R}^n$ and $\alpha_0, \alpha_1 \in (0,1)$. Going further, much as in the first part of the proof (based on the universal extension operator $\mathscr{E}$ and abstract retract results), there is no loss of generality in assuming that $E = \mathbb{R}^n$. However, in this latter context the interpolation result in question is well known (see, e.g., [16, Theorem 6.4.5(1), p. 152], [119, Sect. 2.7.2, p. 201]; this also follows from [96], or [60, (4.18), p. 193] and the reiteration theorem for the real method of interpolation from [16, Theorem 3.5.3, p. 50]).

Finally, the case of the inhomogeneous Hölder scale is treated similarly. □

Theorem 4.99 may also be used to construct examples of subsets K of Euclidean spaces that are homeomorphic to a unit cube and such that the upper smoothness index of K (viewed as metric space when equipped with the Euclidean distance) is larger than any a priori given bound. The relevance of such examples is substantiated in [54]. What follows is the precise statement of the result sketched above.

Corollary 4.103. *Let $(X, \mathbf{q})$ be a quasimetric space of finite Assouad dimension, and assume that $\gamma \in (0,+\infty)$ has the property that there exists $\rho \in \mathbf{q}$ such that $\log_2 C_\rho < \gamma$. Then there exist $n \in \mathbb{N}$, $Y \subseteq \mathbb{R}^n$, a finite constant $C \geq 0$, and a function $f : Y \to X$, which is a homeomorphism and satisfies*

$$\rho(f(x), f(y)) \leq C|x-y|^{\gamma}, \qquad \forall\, x, y \in Y. \tag{4.684}$$

In particular, for every $m \in \mathbb{N}$ and $\gamma \in (0,+\infty)$ there exist $n \in \mathbb{N}$ and a compact set $K \subseteq \mathbb{R}^n$ with the property that K is homeomorphic to $[0,1]^m$ and there are nonconstant, real-valued Hölder functions of order γ defined on K. Consequently, $\mathrm{ind}_0(K, |\cdot - \cdot|) \geq \gamma$*; hence, in particular,* $\mathrm{Ind}\,(K, |\cdot - \cdot|) \geq \gamma$.

Proof. The claim in the first part of the statement of the corollary follows from Theorem 4.99 by taking $f := \Phi^{-1}$, $\beta := 1/\gamma$ and $Y := \widetilde{X}$. The second claim is a consequence of this result, specialized to the case when $(X, \mathbf{q}) := ([0,1]^m, |\cdot - \cdot|)$ and $K := \Phi([0,1]^m)$. Finally, the fact that $\mathrm{ind}_0(K, |\cdot - \cdot|) \geq \gamma$ is seen with the help of Corollary 4.60. □

Assouad has conjectured that any metric space of finite Assouad dimension may be bi-Lipschitzly embedded into some $\mathbb{R}^n$ [i.e., one may take $\beta = 1$ in (4.677)]. However, the Heisenberg group, with its Carnot metric, is a counterexample to this conjecture (see the discussion in [59, p. 99], as well as [110]), and in [59, p. 99] J. Heinonen raises the following question:

Open problem. *Characterize the metric spaces that can be embedded bi-Lipschitzly into some Euclidean space.*

An inspection of the proof of the implication (2) $\Rightarrow$ (3) in Theorem 4.99 shows that a necessary condition for a quasimetric space (X, ρ) to be embedded bi-Lipschitzly into some Euclidean space is that $\dim_A(X, \rho) < +\infty$. Moreover, it is implicit in Assouad's work (cf. [10, Remarque 2 on p. 732, and Proposition 3(g) on p. 733]) that any ultrametric space (i.e., ρ is a metric for which $C_\rho = 1$) of finite Assouad dimension may be embedded bi-Lipschitzly into some Euclidean space. See also [77, Proposition 3.3, p. 186] for an explicit argument. Below, we extend this result to a larger class of quasimetric spaces.

Theorem 4.104. *If* $(X, \mathbf{q})$ *is a quasimetric space of finite Assouad dimension satisfying* $\operatorname{ind}(X, \mathbf{q}) > 1$*, then there exists* $n \in \mathbb{N}$ *with the property that* $(X, \mathbf{q})$ *may be embedded bi-Lipschitzly into* $\mathbb{R}^n$*. Conversely, if a quasimetric space* $(X, \mathbf{q})$ *is embedded bi-Lipschitzly into some finite-dimensional Euclidean space, then* $\operatorname{ind}(X, \mathbf{q}) \geq 1$.

As a corollary, if (X, ρ) *is a quasimetric space of finite Assouad dimension that has the property that*

$$\sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x, y)}{\max\{\rho(x, z), \rho(y, z)\}} < 2, \tag{4.685}$$

then (X, ρ) *may be embedded bi-Lipschitzly into some finite-dimensional Euclidean space. Hence, in particular, any ultrametric space may be embedded bi-Lipschitzly into some finite-dimensional Euclidean space.*

Proof. The claim in the first part of the statement of the theorem follows from Theorem 4.99 after observing that condition $\operatorname{ind}(X, \mathbf{q}) > 1$ permits us to select $\beta = 1$ in (4.677). The second claim is an immediate consequence of the last part of Corollary 4.41. Finally, the remaining claims are straightforward corollaries of the result proved in the first part (cf. also (4.260) in this regard). □

We conclude with two comments related to the nature of Theorem 4.104. First, we wish to note that if (X, ρ) is a quasimetric space with the property that (4.685) holds, then there exists a metric d on X such that $d \approx \rho$. This is a consequence of conclusion (12) in Theorem 3.46. Second, the manner in which Assouad's embedding theorem recalled previously is contained in Theorem 4.104 is made transparent by the observation that if (X, d) is a metric space and $\rho := d^\varepsilon$ for some $\varepsilon \in (0, 1)$, then $C_\rho \leq 2^\varepsilon < 2$, so (4.685) holds for the snowflaked version (X, d^ε) of (X, d).

4.12 Quasimetric Version of Kuratowski's and Fréchet's Embedding Theorems

Given an arbitrary, nonempty set X, denote by $L^\infty(X)$ the space of all bounded functions $f : X \to \mathbb{R}$. When equipped with the norm

$$\|f\|_{L^\infty(X)} := \sup_{x\in X} |f(x)|, \tag{4.686}$$

this may be canonically identified with the Lebesgue space $L^\infty(X, \mu_c)$, where μ_c is the counting measure on X and, as such, $(L^\infty(X), \|\cdot\|_{L^\infty(X)})$ becomes a Banach space. As is customary, corresponding to the case when $X = \mathbb{N}$, we will write $\ell^\infty(\mathbb{N})$ in place of $L^\infty(\mathbb{N})$.

The classical Kuratowski's embedding theorem (cf. [75]) asserts that any metric space (X, d) embeds isometrically into $L^\infty(X)$, and our theorem below extends this result to the setting of quasimetric space.

Theorem 4.105 (The quasimetric space version of Kuratowski's embedding theorem). *Assume that $(X, \mathbf{q})$ is a quasimetric space and that the real number β is such that there exists a quasidistance $\rho \in \mathbf{q}$ with the property that $0 < \beta \le [\log_2 C_\rho]^{-1}$. Then $(X, \mathbf{q})$ admits a bi-Lipschitz embedding into the quasi-Banach space $\big(L^\infty(X), \|\cdot\|^{1/\beta}_{L^\infty(X)}\big)$.*

In particular, any quasimetric space $(X, \mathbf{q})$ for which $\operatorname{ind}(X, \mathbf{q}) > 1$ *may be bi-Lipschitzly embedded into the Banach space $\big(L^\infty(X), \|\cdot\|_{L^\infty(X)}\big)$.*

Proof. Let $(X, \mathbf{q})$ be a quasimetric space, and suppose that the quasidistance $\rho \in \mathbf{q}$ and the real number β are such that $0 < \beta \le [\log_2 C_\rho]^{-1}$. Also, denote by $\rho_\#$ the regularized version of ρ (as in Theorem 3.46), and fix a point $x_o \in X$. Then, for each $x \in X$, consider the function

$$\phi^x : X \to \mathbb{R}, \qquad \phi^x(z) := \big[\rho_\#(x, z)\big]^\beta - \big[\rho_\#(z, x_o)\big]^\beta, \quad \forall\, z \in X. \tag{4.687}$$

Note that, by (3.537) and (3.538),

$$|\phi^x(y) - \phi^y(y)| = \big[\rho_\#(x, y)\big]^\beta \ge (C_\rho)^{-2\beta} \rho(x, y)^\beta, \qquad \forall\, x, y \in X, \tag{4.688}$$

as well as

$$\begin{aligned} |\phi^x(z) - \phi^y(z)| &= \Big|\big[\rho_\#(x, z)\big]^\beta - \big[\rho_\#(y, z)\big]^\beta\Big| \\ &\le \big[\rho_\#(x, y)\big]^\beta \le \rho(x, y)^\beta, \qquad \forall\, x, y, z \in X. \end{aligned} \tag{4.689}$$

Hence, if we define

$$\Phi : X \to L^\infty(X), \qquad \Phi(x) := \phi^x, \quad \forall\, x \in X, \tag{4.690}$$

then from (4.688) and (4.689) we may deduce that

$$C_\rho^{-2}\rho(x,y) \le \|\Phi(x)-\Phi(y)\|^{1/\beta}_{L^\infty(X)} \le \rho(x,y), \qquad \forall\, x,y\in X. \tag{4.691}$$

Then the claim in the first part of the statement of the theorem follows from this, whereas the last claim is a consequence of what we have just proved and (4.255). □

A drawback of Theorem 4.105 is the fact that the embedding discussed there takes place in a space that depends on the original quasimetric space. A version of this theorem, building on a classical result of M. Fréchet in the context of metric spaces (cf. [48]), that corrects the aforementioned deficiency is presented next.

Theorem 4.106 (Quasimetric space version of Fréchet's embedding theorem). *Let $(X,\mathbf{q})$ be a quasimetric space with the property that the topological space $(X,\tau_{\mathbf{q}})$ is separable. Also, assume that the real number β is such that there exists a quasidistance $\rho\in\mathbf{q}$ for which $0<\beta\le[\log_2 C_\rho]^{-1}$.*

Then the quasimetric space $(X,\mathbf{q})$ admits a bi-Lipschitz embedding into the quasi-Banach space $\big(\ell^\infty(\mathbb{N}),\|\cdot\|^{1/\beta}_{\ell^\infty(\mathbb{N})}\big)$. More precisely, if $\rho_\#$ denotes the regularized version of the quasidistance ρ (as in Theorem 3.46), then there exists a mapping $\Psi: X\to\ell^\infty(\mathbb{N})$ that satisfies

$$\|\Psi(x)-\Psi(y)\|^{1/\beta}_{\ell^\infty(\mathbb{N})} = \rho_\#(x,y), \qquad \forall\, x,y\in X. \tag{4.692}$$

As a corollary, any separable quasimetric space $(X,\mathbf{q})$ for which $\operatorname{ind}(X,\mathbf{q})>1$ *may be embedded bi-Lipschitzly into the Banach space $\big(\ell^\infty(\mathbb{N}),\|\cdot\|_{\ell^\infty(\mathbb{N})}\big)$.*

Proof. Assume that the quasidistance $\rho\in\mathbf{q}$ and the real number β are as in the first part of the theorem, and select a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ that is dense in $(X,\tau_{\mathbf{q}})$. As before, we let $\rho_\#$ be the regularized version of the quasidistance ρ (defined in Theorem 3.46). To proceed, consider the mapping

$$\begin{aligned} &\Phi:\{x_j: j\in\mathbb{N}\}\longrightarrow \ell^\infty(\mathbb{N}) \text{ defined for each } j\in\mathbb{N} \text{ by} \\ &\Phi(x_j):=\Big\{\big[\rho_\#(x_j,x_k)\big]^\beta-\big[\rho_\#(x_k,x_1)\big]^\beta\Big\}_{k\in\mathbb{N}}. \end{aligned} \tag{4.693}$$

Based on the fact that the function defined in (3.537) is a distance, we may then estimate

$$\begin{aligned} \big[\rho_\#(x_i,x_j)\big]^\beta &= \big(\Phi(x_i)-\Phi(x_j)\big)_j \\ &\le \|\Phi(x_i)-\Phi(x_j)\|_{\ell^\infty(\mathbb{N})} \le \big[\rho_\#(x_i,x_j)\big]^\beta, \quad \forall\, i,j\in\mathbb{N}; \end{aligned} \tag{4.694}$$

hence, ultimately,

$$\big[\rho_{\#}(x_i, x_j)\big]^{\beta} = \|\Phi(x_i) - \Phi(x_j)\|_{\ell^\infty(\mathbb{N})}, \quad \forall\, i, j \in \mathbb{N}. \tag{4.695}$$

From the density of the sequence $\{x_j\}_{j\in\mathbb{N}}$ in $(X, \tau_{\mathbf{q}})$, the continuity of the function $\rho_{\#} : (X \times X, \tau_{\mathbf{q}} \times \tau_{\mathbf{q}}) \to [0, +\infty)$ (which is a consequence of results established in Theorem 3.46), and (4.695) it follows that the function Φ extends to a mapping $\Psi : X \to \ell^\infty(\mathbb{N})$ that satisfies (4.692). In concert with (3.538), this estimate then gives

$$C_\rho^{-2}\rho(x, y) \le \|\Psi(x) - \Psi(y)\|^{1/\beta}_{\ell^\infty(\mathbb{N})} \le \rho(x, y), \qquad \forall\, x, y \in X. \tag{4.696}$$

Hence, Ψ is a bi-Lipschitz embedding of $(X, \mathbf{q})$ into $\big(\ell^\infty(\mathbb{N}), \|\cdot\|^{1/\beta}_{\ell^\infty(\mathbb{N})}\big)$ which also satisfies (4.692). Then the last claim in the statement of the theorem follows from this and (4.255). □

4.13 Pompeiu–Hausdorff Quasidistance on Quasimetric Spaces

In this section we introduce and study a version of the Pompeiu–Hausdorff distance[2] in the context of quasimetric spaces. To set the stage, given a set X, a quasidistance $\rho \in \mathfrak{Q}(X)$, a subset A of X, and a number $r \in (0, +\infty)$, define

$$N_{\rho,r}(A) := \{x \in X : \operatorname{dist}_\rho(x, A) \le r\}. \tag{4.697}$$

For each $U, V \subseteq X$ then introduce the `Pompeiu-Hausdorff quasidistance` between U and V as

$$D_\rho(U, V) := D_{(X,\rho)}(U, V) := \inf\big\{r > 0 : U \subseteq N_{\rho,r}(V) \text{ and } V \subseteq N_{\rho,r}(U)\big\}. \tag{4.698}$$

Recall that 2^X denotes the set of all subsets of a given set X. Also, given a topological space (X, τ), define

[2]What we here call the Pompeiu–Hausdorff distance has typically been referred to in the literature as the Hausdorff distance. For historical accuracy, however, it is significant to note that D. Pompeiu was the first to introduce (a slight version of) this concept in his thesis (written under the supervision of H. Poincaré). Pompeiu's thesis appeared in print in [100], published in 1905, where Pompeiu calls this notion *écart (mutuel)* between two sets. Subsequently, in 1914, F. Hausdorff revisited this topic, and on p. 463 of his book [57] he correctly attributes the introduction of this notion to Pompeiu.

$$\mathrm{Cl}\,(X,\tau) := \{E \subseteq X : E \text{ is closed in } \tau\}, \tag{4.699}$$

$$\mathrm{Cp}\,(X,\tau) := \{E \subseteq X : E \text{ is compact in } \tau\}. \tag{4.700}$$

Finally, a quasimetric space (X, ρ) is said to be `complete` if any sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq X$ that is Cauchy with respect to the quasidistance ρ is convergent to a point in X.

Theorem 4.107. *For each fixed set X the following properties are valid.*

(1) As a real-valued, nonnegative function defined on $2^X \times 2^X$, the Pompeiu–Hausdorff quasidistance satisfies

$$D_{\lambda\rho} = \lambda D_\rho, \qquad \forall\, \rho \in \mathfrak{Q}(X), \qquad \forall\, \lambda \in (0,+\infty), \tag{4.701}$$

$$D_{\rho^\alpha} = \big[D_\rho\big]^\alpha, \qquad \forall\, \rho \in \mathfrak{Q}(X), \qquad \forall\, \alpha \in (0,+\infty), \tag{4.702}$$

$$D_{\rho'} \le D_\rho \qquad \forall\, \rho, \rho' \in \mathfrak{Q}(X) \;\; \textit{with} \;\; \rho' \le \rho \;\; \textit{on} \;\; X. \tag{4.703}$$

Also, for each $\rho \in \mathfrak{Q}(X)$, the function D_ρ is symmetric in the sense that

$$D_\rho(U,V) = D_\rho(V,U), \qquad \forall\, U, V \subseteq X. \tag{4.704}$$

In addition, $D_\rho \approx D_{\rho'}$ whenever $\rho, \rho' \in \mathfrak{Q}(X)$ are such that $\rho \approx \rho'$. More precisely,

$$c_0\, D_\rho \le D_{\rho'} \le c_1\, D_\rho \;\; \textit{if} \;\; \rho, \rho' \in \mathfrak{Q}(X) \;\; \textit{satisfy} \;\; c_0\rho \le \rho' \le c_1\rho \;\; \textit{on} \;\; X. \tag{4.705}$$

(2) If $\rho \in \mathfrak{Q}(X)$, $p \in (0,+\infty)$, and

$$C_{\rho,p} := \sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{\rho(x,y)}{([\rho(x,z)]^p + [\rho(z,y)]^p)^{1/p}}, \tag{4.706}$$

then

$$D_\rho(U,V) \le C_{\rho,p}\Big(\big[D_\rho(U,W)\big]^p + \big[D_\rho(W,V)\big]^p\Big)^{1/p}, \qquad \forall\, U,V,W \subseteq X. \tag{4.707}$$

Furthermore, formally corresponding to the case $p = +\infty$, there holds

$$D_\rho(U,V) \le C_\rho \max\Big\{D_\rho(U,W), D_\rho(W,V)\Big\}, \qquad \forall\, U,V,W \subseteq X, \tag{4.708}$$

where, as usual, C_ρ is as in (4.2).

(4) Assume that $\rho \in \mathfrak{Q}(X)$. Then for every $U, V \subseteq X$ one has

$$D_\rho(U,V) = 0 \iff \overline{U} = \overline{V}, \tag{4.709}$$

where the closures are taken with respect to τ_ρ, the topology canonically induced by ρ on X. Moreover, one has

$$C_\rho^{-1} D_\rho(U,V) \le D_\rho(\overline{U},\overline{V}) \le C_\rho D_\rho(U,V) \quad \textit{for every } \ U,V \subseteq X. \tag{4.710}$$

(5) Let $\rho \in \mathfrak{Q}(X)$ be an arbitrary, fixed quasidistance, and recall (4.699). Then

$$D_\rho \in \mathfrak{Q}\big(\mathrm{Cl}\,(X,\tau_\rho)\big), \tag{4.711}$$

i.e., the Pompeiu–Hausdorff quasidistance is a genuine quasidistance when restricted to the collection of all closed subsets of (X,τ_ρ). Moreover,

$$\begin{gathered}\textit{if the quasimetric space } (X,\rho) \textit{ is complete, then}\\ \big(\mathrm{Cl}\,(X,\tau_\rho), D_\rho\big) \textit{ is a complete quasimetric space,}\end{gathered} \tag{4.712}$$

and

$$\begin{gathered}\textit{if the topological space } (X,\tau_\rho) \textit{ is compact, then}\\ \textit{the topological space } \big(\mathrm{Cp}\,(X,\tau_\rho), \tau_{D_\rho}\big) \textit{ is compact.}\end{gathered} \tag{4.713}$$

(6) Assume that the quasidistance $\rho \in \mathfrak{Q}(X)$ and the real number β are such that $0 < \beta \le [\log_2 C_\rho]^{-1}$. For every $U,V \subseteq X$ define

$$D_{\rho,\beta}(U,V) := \inf\Big\{\Big(\sum_{i=1}^{N}\big[D_\rho(A_i,A_{i+1})\big]^\beta\Big)^{\frac{1}{\beta}} : N \in \mathbb{N},\ (A_i)_{1\le i\le N+1} \subseteq 2^X \ \textit{such that}\ \ A_1 = U,\ A_{N+1} = V\Big\}. \tag{4.714}$$

Then the function $D_{\rho,\beta} : 2^X \times 2^X \to [0,+\infty)$ has the following properties:

$$D_{\rho,\beta} \textit{ is symmetric,} \tag{4.715}$$

$$D_{\rho,\beta} \textit{ restricted to } \mathrm{Cl}\,(X,\tau_\rho) \times \mathrm{Cl}\,(X,\tau_\rho) \textit{ belongs to } \mathfrak{Q}\big(\mathrm{Cl}\,(X,\tau_\rho)\big), \tag{4.716}$$

$$D_{\rho,\beta} \approx D_\rho \textit{ in the precise sense that } 2^{-2/\beta} D_\rho \le D_{\rho,\beta} \le D_\rho, \tag{4.717}$$

$$D_{\rho,\beta}(U,V) \le \Big(\big[D_{\rho,\beta}(U,W)\big]^\beta + \big[D_{\rho,\beta}(W,V)\big]^\beta\Big)^{\frac{1}{\beta}}, \quad \forall\, U,V,W \subseteq X, \tag{4.718}$$

$$D_{\rho,\beta}(U,V) \le 2^{1/\beta} \max\big\{D_{\rho,\beta}(U,W)\,,\ D_{\rho,\beta}(W,V)\big\}, \quad \forall\, U,V,W \subseteq X. \tag{4.719}$$

Moreover, for each $\gamma \in (0,\beta]$, the following Hölder-type regularity condition of order γ holds:

$$\begin{aligned} &\big|D_{\rho,\beta}(U,V) - D_{\rho,\beta}(U,W)\big| \\ &\qquad\qquad \leq \tfrac{1}{\gamma}\,\max\Big\{\big[D_{\rho,\beta}(U,V)\big]^{1-\gamma}, \big[D_{\rho,\beta}(U,W)\big]^{1-\gamma}\Big\}\big[D_{\rho,\beta}(V,W)\big]^{\gamma} \end{aligned} \tag{4.720}$$

whenever $U, V, W \subseteq X$, with the understanding that if $\gamma > 1$, then one also imposes the condition that $\overline{U} \neq \overline{V}$ and $\overline{U} \neq \overline{W}$, where the closures are taken in (X, τ_ρ).

Proof. If $\rho, \rho' \in \mathfrak{Q}(X)$, $r, r_0, r_1, \alpha, \lambda > 0$, $p \in (0, +\infty)$ and $U, V \subseteq X$, then the following properties may be verified based on definitions:

$$\begin{aligned} &N_{\rho^\alpha,r}(U) = N_{\rho,r^{1/\alpha}}(U), \quad N_{\rho,r}(U) \subseteq N_{\rho,r}(V) \ \text{ if } \ U \subseteq V; \\ &N_{\lambda\rho,r}(U) = N_{\rho,r/\lambda}(U), \quad N_{\rho,r}(U) \subseteq N_{\rho',r}(U) \ \text{ if } \ \rho' \leq \rho; \\ &\overline{N_{\rho,r}(U)} \subseteq N_{\rho,C_\rho r}(U) \ \text{ and } \ N_{\rho,r}(\overline{U}) \subseteq N_{\rho,C_\rho r}(U); \\ &N_{\rho,r_0}\big(N_{\rho,r_1}(U)\big) \subseteq N_{\rho,r}(U) \ \text{ if } \ r := C_{\rho,p}(r_0^p + r_1^p)^{1/p} \ \text{ or } \ r := C_\rho \max\{r_0, r_1\}. \end{aligned} \tag{4.721}$$

Furthermore, if $\rho \in \mathfrak{Q}(X)$ and $\rho_\#$ denotes its regularization (in the sense of Theorem 3.46), then

$$\bigcap_{r>0} N_{\rho_\#,r}(U) = \overline{U} \quad \text{and} \quad N_{\rho_\#,r}(U) = N_{\rho_\#,r}(\overline{U}), \qquad \forall\, U \subseteq X, \tag{4.722}$$

where the closure is taken in τ_ρ. This is a consequence of (4.697) and property (3) in Theorem 4.17, which shows that the function $\mathrm{dist}_{\rho_\#}(\cdot, U) : (X, \tau_\rho) \to [0, +\infty)$ is continuous and that $\mathrm{dist}_{\rho_\#}(\cdot, U) = \mathrm{dist}_{\rho_\#}(\cdot, \overline{U})$. Collectively, these properties then readily yield the claims made in parts (1)–(4), as well as (4.711), in the statement of the theorem. In turn, from (4.708), (4.711), and Theorem 3.46 we deduce that

$$\begin{aligned} &\big(X, (\rho_\#)^\beta\big) \ \text{ is a metric space, } \tau_{(\rho_\#)^\beta} = \tau_\rho, \ \text{ and} \\ &\big(\mathrm{Cl}\,(X, \tau_\rho), D_{(\rho_\#)^\beta}\big) \ \text{ is a metric space whenever} \\ &\rho \in \mathfrak{Q}(X) \text{ and } \beta \in \mathbb{R} \text{ satisfy } 0 < \beta \leq [\log_2 C_\rho]^{-1}. \end{aligned} \tag{4.723}$$

Then (4.712) and (4.713) follow from (4.723) and similar properties known for metric spaces (cf., e.g, [27, Proposition 7.3.7 and Theorem 7.3.8]). Finally, the claims made in part (6) are consequences of Theorem 3.46 and properties established earlier in the proof. ☐

4.14 Gromov–Pompeiu–Hausdorff Distance Between Quasimetric Spaces

The aim of this section is to adapt the notion of the Gromov–Pompeiu–Hausdorff distance between metric spaces to the more general context of quasimetric spaces. We begin by establishing a couple of definitions.

Definition 4.108. Given two quasimetric spaces (X_0, ρ_0) and (X_1, ρ_1), along with a function $f : X_0 \to X_1$ and a number $\gamma \in (0, +\infty)$, define the `distortion of order` γ of f with respect to ρ_0, ρ_1 as

$$\mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) := \sup_{x,y\in X_0} \big|\big[\rho_1(f(x), f(y))\big]^{\gamma} - \big[\rho_0(x, y)\big]^{\gamma}\big|. \tag{4.724}$$

If $\gamma = 1$, then it is agreed that the abbreviation $\mathrm{Dis}_{\rho_0,\rho_1}(f)$ may be used in place of $\mathrm{Dis}^{1}_{\rho_0,\rho_1}(f)$.

Definition 4.109. Given a nonempty set Y, a quasimetric space (X, ρ), and two functions $f, g : Y \to X$, define the `deviation` of f from g, with respect to the quasidistance ρ, as

$$\mathrm{Dev}_{\rho}(f, g) := \sup_{x\in Y} \rho(f(x), g(x)). \tag{4.725}$$

Given an arbitrary set X, we denote by id_X the identity mapping of X into itself.

Lemma 4.110. *(1) Let (X_0, ρ_0), (X_1, ρ_1), and (X_2, ρ_2) be three quasimetric spaces, and assume that $f : X_0 \to X_1$ and $g : X_1 \to X_2$ are given functions. Then for every number $\gamma \in (0, +\infty)$ there holds*

$$\mathrm{Dis}^{\gamma}_{\rho_0,\rho_2}(g \circ f) \le \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(g). \tag{4.726}$$

(2) Suppose that (X_0, ρ_0) and (X_1, ρ_1) are two quasimetric spaces and that $f : X_0 \to X_1$ and $g : X_1 \to X_0$ are two given functions. Then for every number $\gamma \in (0, +\infty)$ one has

$$\big[\mathrm{Dev}_{\rho_0}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma} \le \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) \tag{4.727}$$

and

$$\big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma} \le \big[\mathrm{Dev}_{\rho_0}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_0}(g). \tag{4.728}$$

(3) Let X_0 be a nonempty set and (X_1, ρ_1) and (X_2, ρ_2) two quasimetric spaces, and assume that $h : X_0 \to X_1$, $f : X_1 \to X_2$, $g : X_1 \to X_1$, and $k : X_0 \to X_2$ are given functions. Then for every $\gamma \in (0, +\infty)$ one has

$$\big[\mathrm{Dev}_{\rho_2}(k, f\circ g\circ h)\big]^{\gamma}$$
$$\leq (C_{\rho_2})^{\gamma}\max\big\{\big[\mathrm{Dev}_{\rho_2}(k, f\circ h)\big]^{\gamma}, \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, g)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(f)\big\}. \tag{4.729}$$

(4) *Assume that* (X_0, d_0) *and* (X_1, d_1) *are two metric spaces, and let* $f : X_0 \to X_1$ *and* $g : X_1 \to X_0$ *be two given functions. Then*

$$\mathrm{Dis}_{d_1,d_0}(g) \leq \mathrm{Dis}_{d_0,d_1}(f) + 2\,\mathrm{Dev}_{d_1}(\mathrm{id}_{X_1}, f\circ g), \tag{4.730}$$

$$\mathrm{Dis}_{d_0,d_1}(f) \leq \mathrm{Dis}_{d_1,d_0}(g) + 2\,\mathrm{Dev}_{d_0}(\mathrm{id}_{X_0}, g\circ f). \tag{4.731}$$

Proof. To deal with the claim made in part (1), fix $\gamma \in (0, +\infty)$. Using Definition 4.724 we may then write

$$\begin{aligned}
\mathrm{Dis}^{\gamma}_{\rho_0,\rho_2}(g\circ f) &= \sup_{x,y\in X_0}\big|\big[\rho_2(g(f(x)), g(f(y)))\big]^{\gamma} - \big[\rho_0(x,y)\big]^{\gamma}\big| \\
&\leq \sup_{x,y\in X_0}\big\{\big|\big[\rho_2(g(f(x)), g(f(y)))\big]^{\gamma} - \big[\rho_1(f(x), f(y))\big]^{\gamma}\big| \\
&\qquad + \big|\big[\rho_1(f(x), f(y))\big]^{\gamma} - \big[\rho_0(x,y)\big]^{\gamma}\big|\big\} \\
&\leq \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(g),
\end{aligned} \tag{4.732}$$

as desired. Consider next the claim made in part (2). For each $x \in X_0$ we have

$$\begin{aligned}
\big[\rho_0(x, g(f(x)))\big]^{\gamma} &\leq \big|\big[\rho_0(x, g(f(x)))\big]^{\gamma} - \big[\rho_1(f(x), f(g(f(x))))\big]^{\gamma}\big| \\
&\qquad + \big[\rho_1(f(x), f(g(f(x))))\big]^{\gamma} \\
&\leq \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) + \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, f\circ g)\big]^{\gamma}.
\end{aligned} \tag{4.733}$$

Taking the supremum over all $x \in X_0$ then yields (4.727). Finally, (4.728) is proved in a similar manner.

To deal with the claim made in part (3), making use of Definition 4.109 and the quasiultrametric property of ρ_2, we have

$$\begin{aligned}
\big[\mathrm{Dev}_{\rho_2}(k, f\circ g\circ h)\big]^{\gamma} &= \sup_{x\in X_0}\big[\rho_2(k(x), f(g(h(x))))\big]^{\gamma} \\
&\leq \sup_{x\in X_0}\Big\{(C_{\rho_2})^{\gamma}\max\big\{\big[\rho_2(k(x), f(h(x)))\big]^{\gamma}, \big[\rho_2(f(h(x)), f(g(h(x))))\big]^{\gamma}\big\}\Big\} \\
&\leq \sup_{x\in X_0}\Big\{(C_{\rho_2})^{\gamma}\max\big\{\big[\mathrm{Dev}_{\rho_2}(k, f\circ h)\big]^{\gamma}, \big[\rho_2(f(h(x)), f(g(h(x))))\big]^{\gamma}\big\}\Big\}.
\end{aligned} \tag{4.734}$$

In addition, for each $x \in X_0$, we have

$$\begin{aligned}\big[\rho_2(f(h(x)), f(g(h(x))))\big]^{\gamma} &\leq \big[\rho_1(h(x), g(h(x)))\big]^{\gamma} \\ &\quad + \Big|\big[\rho_2(f(h(x)), f(g(h(x))))\big]^{\gamma} - \big[\rho_1(h(x), g(h(x)))\big]^{\gamma}\Big| \\ &\leq \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, g)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(f). \end{aligned} \tag{4.735}$$

Now (3) follows by combining (4.734) and (4.735).

As regards the claims made in part (4), for each $x, y \in X_1$ we may, on the one hand, write

$$\begin{aligned}\big|d_0(g(x), g(y)) - d_1(x, y)\big| &\leq \big|d_1(f(g(x)), f(g(y))) - d_0(g(x), g(y))\big| \\ &\quad + \big|d_1(f(g(x)), f(g(y))) - d_1(x, y)\big| \\ &\leq \mathrm{Dis}_{d_0,d_1}(f) + \big|d_1(f(g(x)), f(g(y))) - d_1(x, y)\big|. \end{aligned} \tag{4.736}$$

On the other hand, since d_1 is a genuine distance, based on the triangle inequality, we may estimate

$$\begin{aligned}&\big|d_1(f(g(x)), f(g(y))) - d_1(x, y)\big| \\ &\quad \leq \big|d_1(f(g(x)), f(g(y))) - d_1(f(g(x)), y)\big| + \big|d_1(f(g(x)), y) - d_1(x, y)\big| \\ &\quad \leq d_1(f(g(y)), y) + d_1(f(g(x)), x) \\ &\quad \leq 2\,\mathrm{Dev}_{d_1}(\mathrm{id}_{X_1}, f \circ g). \end{aligned} \tag{4.737}$$

At this stage, a combination of (4.736) and (4.737) yields (4.730), after taking the supremum over all $x, y \in X_1$. Finally, (4.731) is proved in a similar manner, and this completes the proof of the lemma. □

Definition 4.111. Given two quasimetric spaces (X_0, ρ_0) and (X_1, ρ_1), along with two functions $f : X_0 \to X_1$, $g : X_1 \to X_0$, an exponent $\alpha \in (0, +\infty]$, and a number $\gamma \in (0, +\infty)$, set

$$\begin{aligned}[f, g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} := \max\Big\{&\mathrm{Dis}^{\gamma}_{(\rho_0)_\alpha,(\rho_1)_\alpha}(f), \mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\rho_0)_\alpha}(g), \big[\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma}, \\ &\big[\mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma}\Big\}^{1/\gamma}, \end{aligned} \tag{4.738}$$

where $(\rho_0)_\alpha$ and $(\rho_1)_\alpha$ are the α-subadditive regularizations of the quasidistances ρ_0, ρ_1 constructed as in Theorem 3.46. Then define

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) := \inf\Big\{[f, g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} : \ f : X_0 \to X_1,\ \ g : X_1 \to X_0\Big\}. \tag{4.739}$$

To proceed, recall the definition made in (4.16).

Proposition 4.112. *Fix two numbers* $\alpha, \gamma \in (0, +\infty)$ *and assume that* (X_0, ρ_0) *and* (X_1, ρ_1) *are quasimetric spaces. Then* $d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)$ *is a well-defined number belonging to* $[0, +\infty]$*, and the following symmetry property holds:*

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) = d_{\alpha,\gamma}\big((X_1, \rho_1), (X_0, \rho_0)\big). \tag{4.740}$$

Furthermore,

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) \le \max\big\{\mathrm{diam}_{\rho_0}(X_0), \mathrm{diam}_{\rho_1}(X_1)\big\}. \tag{4.741}$$

Proof. The claims in the first part of the statement are clear from Definition 4.111. As for (4.741), if $x_j \in X_j$, $j = 0, 1$, are two fixed points, then consider the functions $f : X_0 \to X_1$, $f(x) := x_1$ for every $x \in X_0$, and $g : X_1 \to X_0$, $g(x) := x_0$ for every $x \in X_1$. It follows then from (4.724), (4.725), and (3.530) that

$$\begin{aligned}
\mathrm{Dis}^{\gamma}_{(\rho_0)_\alpha,(\rho_1)_\alpha}(f) &= \Big[\sup_{x,y\in X_0} (\rho_0)_\alpha(x, y)\Big]^{\gamma} = \big[\mathrm{diam}_{(\rho_0)_\alpha}(X_0)\big]^{\gamma} \le \big[\mathrm{diam}_{\rho_0}(X_0)\big]^{\gamma},\\
\mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\rho_0)_\alpha}(g) &= \Big[\sup_{x,y\in X_1} (\rho_1)_\alpha(x, y)\Big]^{\gamma} = \big[\mathrm{diam}_{(\rho_1)_\alpha}(X_1)\big]^{\gamma} \le \big[\mathrm{diam}_{\rho_1}(X_1)\big]^{\gamma},\\
\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, g \circ f) &= \sup_{x\in X_0} (\rho_0)_\alpha(x, x_0) \le \sup_{x\in X_0} \rho_0(x, x_0) \le \mathrm{diam}_{\rho_0}(X_0),\\
\mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, f \circ g) &= \sup_{x\in X_1} (\rho_1)_\alpha(x, x_1) \le \sup_{x\in X_1} \rho_1(x, x_1) \le \mathrm{diam}_{\rho_1}(X_1).
\end{aligned} \tag{4.742}$$

Now (4.741) follows from (4.738), (4.739), and (4.742). □

Proposition 4.113. *Let* (X_0, ρ_0)*,* (X_1, ρ_1)*, and* (X_2, ρ_2) *be three quasimetric spaces, and fix two numbers* $\alpha, \gamma \in (0, +\infty)$*. Then*

$$\begin{aligned}
&d_{\alpha,\gamma}\big((X_0, \rho_0), (X_2, \rho_2)\big) \\
&\quad\le 2^{1/\alpha}\Big[\big[d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)\big]^{\gamma} + \big[d_{\alpha,\gamma}\big((X_1, \rho_1), (X_2, \rho_2)\big)\big]^{\gamma}\Big]^{1/\gamma}.
\end{aligned} \tag{4.743}$$

In particular, there holds

$$\begin{aligned}
&d_{\alpha,\gamma}\big((X_0, \rho_0), (X_2, \rho_2)\big) \\
&\quad\le 2^{1/\gamma+1/\alpha} \max\Big\{d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big), d_{\alpha,\gamma}\big((X_1, \rho_1), (X_2, \rho_2)\big)\Big\}.
\end{aligned} \tag{4.744}$$

Proof. Fix $r_{01} > d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)$ and $r_{12} > d_{\alpha,\gamma}\big((X_1, \rho_1), (X_2, \rho_2)\big)$. Hence, there exist functions $f_{01} : X_0 \to X_1$, $f_{10} : X_1 \to X_0$, $f_{12} : X_1 \to X_2$, and $f_{21} : X_2 \to X_1$ such that

$$[f_{01}, f_{10}]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} < r_{01} \quad\text{and}\quad [f_{12}, f_{21}]^{\alpha,\gamma}_{(X_1,\rho_1),(X_2,\rho_2)} < r_{12}. \tag{4.745}$$

Consider the functions $f := f_{12} \circ f_{01} : X_0 \to X_2$ and $g := f_{10} \circ f_{21} : X_2 \to X_0$. Then, applying (4.726) in concert with (4.745), we obtain

$$\mathrm{Dis}^{\gamma}_{(\rho_0)_\alpha,(\rho_2)_\alpha}(f) \leq \mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\rho_2)_\alpha}(f_{12}) + \mathrm{Dis}^{\gamma}_{(\rho_0)_\alpha,(\rho_1)_\alpha}(f_{01}) < r_{01}^{\gamma} + r_{12}^{\gamma}, \tag{4.746}$$

$$\mathrm{Dis}^{\gamma}_{(\rho_2)_\alpha,(\rho_0)_\alpha}(g) \leq \mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\rho_0)_\alpha}(f_{10}) + \mathrm{Dis}^{\gamma}_{(\rho_2)_\alpha,(\rho_1)_\alpha}(f_{21}) < r_{01}^{\gamma} + r_{12}^{\gamma}. \tag{4.747}$$

Next, we use part (3) of Lemma 4.110 and the fact that $C_{(\rho_0)_\alpha} \leq 2^{1/\alpha}$ (itself a consequence of (3.107)) to be able to write

$$\begin{aligned}\big[\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma} &\leq (C_{(\rho_0)_\alpha})^{\gamma} \max\Big\{\big[\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, f_{10} \circ f_{01})\big]^{\gamma},\\ &\qquad \big[\mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, f_{21} \circ f_{12})\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\rho_0)_\alpha}(f_{10})\Big\}\\ &\leq 2^{\gamma/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma}).\end{aligned} \tag{4.748}$$

Similarly, we obtain

$$\big[\mathrm{Dev}_{(\rho_2)_\alpha}(\mathrm{id}_{X_2}, f \circ g)\big]^{\gamma} \leq 2^{\gamma/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma}). \tag{4.749}$$

In concert, (4.746)–(4.749) and (4.738) and (4.739) imply that

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_2, \rho_2)\big) \leq 2^{1/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma})^{1/\gamma}. \tag{4.750}$$

Now letting $r_{01} \searrow d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)$ and $r_{12} \searrow d_{\alpha,\gamma}\big((X_1, \rho_1), (X_2, \rho_2)\big)$ in (4.750) we arrive at (4.743), from which (4.744) also follows. □

Proposition 4.114. *If the quasiquasimetric spaces (X_0, ρ_0) and (X_1, ρ_1) are bi-Lipschitzly homeomorphic with one another, then there exists $\widetilde{\rho}_0 \in \mathfrak{Q}(X_0)$ such that*

$$\widetilde{\rho}_0 \approx \rho_0 \quad \textit{and} \quad d_{\alpha,\gamma}\big((X_0, \widetilde{\rho}_0), (X_1, \rho_1)\big) = 0 \;\; \textit{for every} \;\; \alpha, \gamma \in (0, +\infty). \tag{4.751}$$

More specifically, suppose that $\phi : X_0 \to X_1$ is a bijection with the property that there exist $c', c'' \in (0, +\infty)$ such that

$$c'\rho_0(x, y) \leq \rho_1(\phi(x), \phi(y)) \leq c''\rho_0(x, y), \qquad \forall\, x, y \in X_0. \tag{4.752}$$

If one then defines

$$\widetilde{\rho}_0(x, y) := \rho_1(\phi(x), \phi(y)), \qquad \forall\, x, y \in X_0, \tag{4.753}$$

it follows that

$$\begin{aligned}&\widetilde{\rho}_0 \in \mathfrak{Q}(X_0), \quad c'\rho_0 \le \widetilde{\rho}_0 \le c''\rho_0, \quad \textit{and}\\ &d_{\alpha,\gamma}\big((X_0,\widetilde{\rho}_0),(X_1,\rho_1)\big) = 0, \quad \forall\, \alpha,\gamma \in (0,+\infty).\end{aligned} \tag{4.754}$$

Proof. Assume that the function $\phi : X_0 \to X_1$ is bijective and that there exist two constants $c', c'' \in (0,+\infty)$ with the property that (4.752) holds. Define $\widetilde{\rho}_0$ as in (4.753). Then, by design, $\widetilde{\rho}_0 \in \mathfrak{Q}(X_0)$ and $c'\rho_0 \le \widetilde{\rho}_0 \le c''\rho_0$ on $X_0 \times X_0$. Furthermore, from property (8) in Lemma 3.14 it follows that

$$(\widetilde{\rho}_0)_\alpha(x,y) = (\rho_1)_\alpha(\phi(x),\phi(y)), \qquad \forall\, x,y \in X_0, \tag{4.755}$$

and, hence,

$$\mathrm{Dis}^{\gamma}_{(\widetilde{\rho}_0)_\alpha,(\rho_1)_\alpha}(\phi) = 0 \quad \text{and} \quad \mathrm{Dis}^{\gamma}_{(\rho_1)_\alpha,(\widetilde{\rho}_0)_\alpha}(\phi^{-1}) = 0, \qquad \forall\, \alpha,\gamma \in (0,+\infty). \tag{4.756}$$

Since, clearly, for each $\alpha, \gamma \in (0,+\infty)$ we also have

$$\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, \phi^{-1}\circ\phi) = \mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, \phi\circ\phi^{-1}) = 0, \tag{4.757}$$

we deduce that $[\phi,\phi^{-1}]^{\alpha,\gamma}_{(X_0,\widetilde{\rho}_0),(X_1,\rho_1)} = 0$ for every $\alpha,\gamma \in (0,+\infty)$. Thus, ultimately, $d_{\alpha,\gamma}\big((X_0,\widetilde{\rho}_0),(X_1,\rho_1)\big) = 0$ for every $\alpha,\gamma \in (0,+\infty)$, completing the proof of (4.754). □

Definition 4.115. Define the `Gromov–Pompeiu–Hausdorff distance`

$$d_{\mathrm{GPH}}\big((X_0,d_0),(X_1,d_1)\big) \tag{4.758}$$

between any two given metric spaces (X_0,d_0) and (X_1,d_1) as the infimum of all numbers $r > 0$ with the property that there exist a metric space (X,d) and isometric embeddings $\phi_0 : (X_0,d_0) \to (X,d)$ and $\phi_1 : (X_1,d_1) \to (X,d)$ for which

$$D_{(X,d)}\big(\phi_0(X_0),\phi_1(X_1)\big) \le r, \tag{4.759}$$

where $D_{(X,d)}$ denotes the Pompeiu–Hausdorff distance in (X,d) [cf. (4.698)].

Consider two metric spaces (X_0,d_0), (X_1,d_1). Given a number $\varepsilon > 0$, a function $f : X_0 \to X_1$ is said to be an ε-`isometry` between (X_0,d_0) and (X_1,d_1) provided

$$\mathrm{Dis}_{d_0,d_1}(f) < \varepsilon \quad \text{and} \quad X_1 = \bigcup_{x\in X_0} B_{d_1}(f(x),\varepsilon). \tag{4.760}$$

Denote by ε-Iso$\big((X_0,d_0),(X_1,d_1)\big)$ the collection of all ε-isometries between (X_0,d_0) and (X_1,d_1). With this piece of terminology, Corollary 7.3.28 in [27] then shows that the following implications are valid:

$$d_{\mathrm{GPH}}\big((X_0,d_0),(X_1,d_1)\big) < \varepsilon \implies (2\varepsilon)\text{-Iso}\,\big((X_0,d_0),(X_1,d_1)\big) \neq \emptyset, \tag{4.761}$$

$$\varepsilon\text{-Iso}\,\big((X_0,d_0),(X_1,d_1)\big) \neq \emptyset \implies d_{\mathrm{GPH}}\big((X_0,d_0),(X_1,d_1)\big) < 2\varepsilon. \tag{4.762}$$

Proposition 4.116. *Assume that* (X_j,ρ_j), $j=0,1$, *are two quasimetric spaces, and suppose that* $\alpha\in(0,+\infty]$ *and the real number* γ *satisfy*

$$\gamma\in(0,\alpha] \quad \text{and} \quad 0<\alpha\leq[\log_2 C_{\rho_j}]^{-1}, \quad j=0,1. \tag{4.763}$$

Then $(X_j,(\rho_j)^\gamma_\alpha)$, $j=0,1$, *are metric spaces [where* $(\rho_j)^\gamma_\alpha$ *abbreviates* $((\rho_j)_\alpha)^\gamma$*], and*

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big) \leq 6^{1/\gamma}\Big[d_{\mathrm{GPH}}\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)\Big]^{1/\gamma}, \tag{4.764}$$

$$\Big[d_{\mathrm{GPH}}\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)\Big]^{1/\gamma} \leq 2^{1/\gamma} d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big). \tag{4.765}$$

Proof. That $(X_j,(\rho_j)^\gamma_\alpha)$, $j=0,1$, are metric spaces whenever (4.763) holds is a consequence of Theorem 3.46. To proceed, let $\varepsilon > d_{\mathrm{GPH}}\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)$. Make use of (4.761) and select $f\in(2\varepsilon)\text{-Iso}\,\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)$. In particular, this entails $X_1=\cup_{x\in X_0}B_{(\rho_1)^\gamma_\alpha}(f(x),2\varepsilon)$ and we may invoke the axiom of choice to construct $g: X_1\to X_0$ with the property that $[(\rho_1)_\alpha(f(g(x)),x)]^\gamma < 2\varepsilon$ for every $x\in X_1$. Thus, we have

$$\big[\mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, f\circ g)\big]^\gamma < 2\varepsilon, \tag{4.766}$$

$$\mathrm{Dis}^\gamma_{(\rho_0)_\alpha,(\rho_1)_\alpha}(f) = \mathrm{Dis}_{(\rho_0)^\gamma_\alpha,(\rho_1)^\gamma_\alpha}(f) < 2\varepsilon, \tag{4.767}$$

where the fact that $f\in(2\varepsilon)\text{-Iso}\,\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)$ was used in the last inequality. Based on (4.766) and (4.767) and parts (2) and (4) of Lemma 4.110, we then deduce that

$$\mathrm{Dis}^\gamma_{(\rho_1)_\alpha,(\rho_0)_\alpha}(g) = \mathrm{Dis}_{(\rho_1)^\gamma_\alpha,(\rho_0)^\gamma_\alpha}(g) < 6\,\varepsilon \quad \text{and} \quad \big[\mathrm{Dev}_{(\rho_0)_\alpha}(\mathrm{id}_{X_0}, g\circ f)\big]^\gamma < 4\varepsilon. \tag{4.768}$$

Collectively, (4.766)–(4.768) prove that

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big) \leq [f,g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} < (6\,\varepsilon)^{1/\gamma}. \tag{4.769}$$

After letting $\varepsilon \searrow d_{\mathrm{GPH}}\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big)$ this yields (4.764).

To justify (4.765), consider $\varepsilon > d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)$. Then there exist two functions, $f: X_0\to X_1$ and $g: X_1\to X_0$, with the property that

$[f,g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} < \varepsilon$. In particular,

$$\mathrm{Dis}_{(\rho_0)^\gamma_\alpha,(\rho_1)^\gamma_\alpha}(f) = \mathrm{Dis}^\gamma_{(\rho_0)_\alpha,(\rho_1)_\alpha}(f) < \varepsilon^\gamma, \tag{4.770}$$

$$\mathrm{Dev}_{(\rho_1)^\gamma_\alpha}(\mathrm{id}_{X_1}, f\circ g) = \big[\mathrm{Dev}_{(\rho_1)_\alpha}(\mathrm{id}_{X_1}, f\circ g)\big]^\gamma < \varepsilon^\gamma. \tag{4.771}$$

Upon observing that (4.771) entails

$$X_1 = \bigcup_{x\in X_0} B_{(\rho_1)^\gamma_\alpha}(f(x),\varepsilon^\gamma), \tag{4.772}$$

we may conclude from (4.770) and (4.772) that ε^γ-Iso$\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha)\big) \neq \emptyset$. With this in hand, (4.762) then gives $d_{\mathrm{GPH}}\big((X_0,(\rho_0)^\gamma_\alpha),(X_1,(\rho_1)^\gamma_\alpha\big) < 2\varepsilon^\gamma$. By letting $\varepsilon \searrow d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)$, estimate (4.765) follows. This completes the proof of the proposition. □

We are now ready to introduce a version of the Gromov–Pompeiu–Hausdorff distance that is suitably adapted to the context of quasimetric spaces.

Definition 4.117. Fix $\alpha \in (0,+\infty]$ and a number $\gamma \in (0,\alpha]$, and assume that the quasimetric spaces (X,ρ_X) and (Y,ρ_Y) have the property that

$$0 < \alpha \leq \min\Big\{[\log_2 C_{\rho_X}]^{-1}, [\log_2 C_{\rho_Y}]^{-1}\Big\}. \tag{4.773}$$

For every number $\beta \in (0,(1/\alpha + 1/\gamma)^{-1}]$ define

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) := \inf\Big\{\Big(\sum_{i=1}^{N}\big[d_{\alpha,\gamma}\big((Z_i,\rho_i),(Z_{i+1},\rho_{i+1})\big)\big]^\beta\Big)^{\frac{1}{\beta}}\Big\}, \tag{4.774}$$

where $d_{\alpha,\gamma}$ is as in Definition 4.111 and the infimum is taken over all numbers $N \in \mathbb{N}$ and all families $(Z_i,\rho_i)_{1\leq i\leq N+1}$ of quasimetric spaces with the property that

$$\begin{aligned}&(Z_0,\rho_0) = (X,\rho_X), \qquad (Z_{N+1},\rho_{N+1}) = (Y,\rho_Y),\\ &\text{and} \quad 0<\alpha \leq [\log_2 C_{\rho_i}]^{-1}, \quad \forall\, i \in \{1,\dots,N+1\}.\end{aligned} \tag{4.775}$$

The theorem below, summarizing some of the most basic properties of the function $\delta_{\alpha,\gamma,\beta}$ introduced in Definition 4.117, is the main result in this section.

Theorem 4.118. *Fix $\alpha \in (0,+\infty]$, $\gamma \in (0,\alpha]$ and assume that $\beta \in (0,(1/\alpha + 1/\gamma)^{-1}]$. Then the following properties are valid.*

(i) If (X,ρ_X) and (Y,ρ_Y) are quasimetric spaces satisfying $[\log_2 C_{\rho_X}]^{-1} \geq \alpha$ and $[\log_2 C_{\rho_Y}]^{-1} \geq \alpha$, then the following symmetry condition holds:

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) = \delta_{\alpha,\gamma,\beta}\big((Y,\rho_Y),(X,\rho_X)\big). \tag{4.776}$$

Moreover,

$$2^{-\frac{2}{\beta}} d_{\alpha,\gamma}\big((X,\rho_X),(Y,\rho_Y)\big) \leq \delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) \leq d_{\alpha,\gamma}\big((X,\rho_X),(Y,\rho_Y)\big); \tag{4.777}$$

hence, in particular,

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) \leq \max\big\{\mathrm{diam}_{\rho_X}(X), \mathrm{diam}_{\rho_Y}(Y)\big\}. \tag{4.778}$$

In addition,

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) \leq 6^{1/\gamma}\Big[d_{\mathrm{GPH}}\big((X,(\rho_X)^{\gamma}_{\alpha}),(Y,(\rho_Y)^{\gamma}_{\alpha})\big)\Big]^{1/\gamma}, \tag{4.779}$$

$$\Big[d_{\mathrm{GPH}}\big((X,(\rho_X)^{\gamma}_{\alpha}),(Y,(\rho_Y)^{\gamma}_{\alpha})\big)\Big]^{1/\gamma} \leq 2^{1/\alpha+1/\beta+1/\gamma}\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big). \tag{4.780}$$

(ii) *The function* $(\delta_{\alpha,\gamma,\beta})^{\beta}$ *satisfies the triangle inequality in the sense that if* (X,ρ_X), (Y,ρ_Y), (Z,ρ_Z) *are three quasimetric spaces with the property that*

$$0 < \alpha \leq \min\Big\{[\log_2 C_{\rho_X}]^{-1}, [\log_2 C_{\rho_Y}]^{-1}, [\log_2 C_{\rho_Z}]^{-1}\Big\}, \tag{4.781}$$

then

$$\begin{aligned}&\big[\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big)\big]^{\beta}\\&\quad\leq \big[\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Z,\rho_Z)\big)\big]^{\beta} + \big[\delta_{\alpha,\gamma,\beta}\big((Z,\rho_Z),(Y,\rho_Y)\big)\big]^{\beta}.\end{aligned} \tag{4.782}$$

(iii) *Suppose that* (X,ρ_X) *and* (Y,ρ_Y) *are quasimetric spaces for which* $[\log_2 C_{\rho_X}]^{-1} \geq \alpha$ *and* $[\log_2 C_{\rho_Y}]^{-1} \geq \alpha$, *and such that the topological spaces* (X,τ_{ρ_X}) *and* (Y,τ_{ρ_Y}) *are compact. Then*

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) = 0 \Leftrightarrow (X,(\rho_X)_{\alpha}) \text{ and } (Y,(\rho_Y)_{\alpha}) \text{ are isometric.} \tag{4.783}$$

In particular, $\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) = 0$ *implies that* (X,ρ_X) *and* (Y,ρ_Y) *are bi-Lipschitzly homeomorphic.*

Conversely, given two bi-Lipschitzly homeomorphic quasimetric spaces (X,ρ_X) *and* (Y,ρ_Y) *that satisfy* $[\log_2 C_{\rho_X}]^{-1} \geq \alpha$ *and* $[\log_2 C_{\rho_Y}]^{-1} \geq \alpha$, *there exists a quasidistance* $\widetilde{\rho}_X \in \mathfrak{Q}(X)$ *such that* $\widetilde{\rho}_X \approx \rho_X$ *and* $\delta_{\alpha,\gamma,\beta}\big((X,\widetilde{\rho}_X),(Y,\rho_Y)\big) = 0.$

(iv) Assume that $\{(X_i, \rho_i)\}_{i\in I}$ *is an infinite family of quasimetric spaces satisfying the following conditions:*

$$\text{the topological space } (X_i, \tau_{\rho_i}) \text{ is compact for every } i \in I, \tag{4.784}$$

$$\inf_{i\in I} [\log_2 C_{\rho_i}]^{-1} \geq \alpha \quad \text{and} \quad \sup_{i\in I} \operatorname{diam}_{\rho_i}(X_i) < +\infty, \tag{4.785}$$

and

$$\begin{gathered}\forall\, \varepsilon > 0\ \exists\, N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall\, i \in I\ \exists\, A_i \subseteq X_i \text{ with} \\ \text{the property that } \#A_i \leq N \text{ and } \textstyle\bigcup_{x\in A_i} B_{\rho_i}(x, \varepsilon) = X_i.\end{gathered} \tag{4.786}$$

In addition, assume that α *is finite. Then for every countable set* $J \subseteq I$ *there exist a countable set* $\{j_k\}_{k\in\mathbb{N}} \subseteq J$ *and a quasimetric space* (X, ρ) *with the property that* (X, τ_ρ) *is compact,* $[\log_2 C_\rho]^{-1} \geq \alpha$*, and*

$$\lim_{k\to\infty} \delta_{\alpha,\alpha,\beta}\big((X_{j_k}, \rho_{j_k}), (X, \rho)\big) = 0. \tag{4.787}$$

Proof. Recall that estimate (4.744) holds and that, by hypothesis, the exponent β satisfies $0 < \beta \leq [\log_2 2^{1/\gamma+1/\alpha}]^{-1}$. Keeping these observations in mind, the symmetry property (4.776) is a consequence of (4.740) and (3.529) in part (11) of Theorem 3.46, whereas the double inequality in (4.777) follows from (3.129). Furthermore, (4.778) follows from (4.777) and (4.741). A combination of (4.777) and Proposition 4.116 proves (4.779) and (4.780). This concludes the proof of (*i*). Next, based on our earlier observations and (3.532) in part (11) of Theorem 3.46, we deduce that the claim in (*ii*) holds.

Moving on, the first claim in (*iii*) is a corollary of (4.780) and [27, Theorem 7.3.30]. Together, (4.777) and Proposition 4.114 then imply the second claim in (*iii*), completing the proof of this portion of the statement of the theorem.

There remains to deal with the claim made in part (*iv*). In this regard, the key observation is that if α is finite, then the current hypotheses imply that $\{(X_i, (\rho_i)^\alpha_\alpha)\}_{i\in I}$ is a uniformly totally bounded family of compact metric spaces, in the sense of [27, Definition 7.4.13]. As such, Gromov's compactness theorem (cf., e.g., [27, Theorem 7.4.15]) gives that for every countable set $J \subseteq I$ there exist a countable set $\{j_k\}_{k\in\mathbb{N}} \subseteq J$ and a metric space (X, d) with the property that (X, τ_ρ) is compact and

$$\lim_{k\to\infty} d_{\rm GPH}\big((X_{j_k}, ((\rho_{j_k})_\alpha)^\alpha), (X, d)\big) = 0. \tag{4.788}$$

Hence, if we now define $\rho := d^{1/\alpha}$ on $X \times X$, then (X, ρ) is a quasimetric space, and, since $\tau_{d^{1/\alpha}} = \tau_d$, the topological space (X, τ_ρ) is compact. Furthermore, we have that $C_\rho = C_{d^{1/\alpha}} = (C_d)^{1/\alpha} \leq 2^{1/\alpha}$ since $C_d \leq 2$ given that d is a distance. Consequently, $[\log_2 C_\rho]^{-1} \geq \alpha$. Going further, observe that

$$(\rho_\alpha)^\alpha = \big((d^{1/\alpha})_\alpha\big)^\alpha = d \tag{4.789}$$

by properties (5) and (9) in Lemma 3.14 since, as a distance, d is 1-subadditive. This shows that $(X, d) = (X, (\rho_\alpha)^\alpha)$. With this in hand, (4.787) follows by virtue of (4.789) and (4.779). This concludes the treatment of the claim made in part (iv) and completes the proof of the theorem. □

Chapter 5
Nonlocally Convex Functional Analysis

The goal in this chapter is to establish completeness and separability criteria for large classes of topological vector spaces that are typically nonlocally convex (such as Lebesgue-like spaces, Lorentz spaces, Orlicz spaces, mixed-normed spaces, tent spaces, and discrete Triebel–Lizorkin and Besov spaces) and to derive pointwise convergence results in the case of vector spaces of measurable functions.

The proofs of our results in this chapter make essential use of abstract capacitary estimates. By a capacity we will understand a nonnegative function $\mathscr{C}$ defined in some algebraic environment G equipped with some associative binary operation $*$ that is allowed to be only partially defined (i.e., its domain could, in principle, be just a subset of $G \times G$) and that is quasisubadditive. The latter property indicates that there exists a constant $c \in [0, +\infty)$ such that $\mathscr{C}(f * g) \leq c\big(\mathscr{C}(f) + \mathscr{C}(g)\big)$ whenever $f, g \in G$ have a meaningfully defined product $f * g \in G$.

This topic was the subject of extensive work in earlier chapters. Here we use this analysis and further expand upon it. For example, we will employ capacitary estimates in such settings as the case when $(G, *)$ is the underlying Abelian additive group of a given vector space X (in which scenario, $\mathscr{C}$ may be allowed to be a quasinorm on X), when $(G, *)$ consists of a sigma-algebra of sets $\mathfrak{M}$ equipped with the operation of taking unions, or, more generally, when G is a lattice $\mathscr{X}$, with $f * g$ taken to be $f \vee g := \sup\{f, g\}$ for each $f, g \in \mathscr{X}$ (in which case $\mathscr{C}$ may be thought of as a rough version of a measure). The presentation in this chapter follows closely [82].

5.1 Formulation of Results

Typically, completeness results are proved via ad hoc methods by reducing matters to the completeness of other, more standard spaces (a case in point is the treatment of tent spaces from [33]) or, when done abstractly, such considerations are largely limited to genuine Banach spaces (as is the case with the treatment in Chap. 15 of

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the classical monograph [130] of Zaanen, or the more timely presentation in Theorem 1.7, p. 6 in the monograph [15] by Bennett and Sharpley). More specifically, in [15, 74, 130] (as well as in many other works based on these references), the authors consider Köthe function spaces, i.e., having fixed a background measure space, spaces of the form $L^\rho := \{f \text{ measurable} : \|f\| := \rho(|f|) < +\infty\}$, where ρ is a mapping defined on $\mathcal{M}_+$, the collection of all nonnegative measurable functions, satisfying

$$\text{for each } f \in \mathcal{M}_+,\ \rho(f) \in [0, +\infty], \text{ and } \rho(f) = 0 \Leftrightarrow f = 0, \quad (5.1)$$

$$\rho(\lambda f) = \lambda \rho(f) \text{ for each } f \in \mathcal{M}_+ \text{ and each } \lambda \geq 0, \quad (5.2)$$

$$\rho(f + g) \leq \rho(f) + \rho(g) \text{ for each } f, g \in \mathcal{M}_+, \quad (5.3)$$

$$\rho(f) \leq \rho(g) \text{ whenever } f, g \in \mathcal{M}_+ \text{ satisfy } f \leq g \text{ a.e.} \quad (5.4)$$

In particular, the subadditivity property (5.3) precludes one from considering arbitrary quasinormed spaces. A notable exception is the treatment in Sect. 2.3 of the monograph [91] by Okada, Ricker, and Sánchez Pérez, where a completeness result is proved for quasinormed spaces, though the scope of this work is limited to order ideals of measurable functions associated with finite measure spaces,[1] a setting too restrictive for the applications we have in mind.

By way of contrast, here we adopt an abstract, general point of view, aimed at identifying the essential characteristics of a topological/functional analytic nature of a given vector space that ensure completeness and/or separability. Regarding the former issue, a sample result, itself a consequence of more general theorems proved later in this chapter, is formulated in Theorem 5.3 below. Before stating it, we first make a couple of definitions, the first of which describes a general recipe for constructing topologies on a given group and clarifies the notion of completeness.

Definition 5.1. Let $(X, +)$ be a group, and denote by 0 the neutral element in X and by $-f$ the inverse of $f \in X$. In this context, for a given function $\psi : X \to [0, +\infty]$ with the property that $\psi(0) = 0$, define the topology τ_ψ induced by ψ on X by demanding that $\mathcal{O} \subseteq X$ be open in τ_ψ if and only if for each $f \in \mathcal{O}$ there exists $r > 0$ such that $B_\psi(f, r) \subseteq \mathcal{O}$, where $B_\psi(f, r) := \{g \in X : \psi(f - g) < r\}$.

In such a setting, call a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq X$ `Cauchy` provided for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\psi(f_n - f_m) < \varepsilon$ whenever $n, m \in \mathbb{N}$ are such that $n, m \geq N_\varepsilon$. Also, call (X, τ_ψ) `complete` if any Cauchy sequence in X is convergent in τ_ψ to some element in X.

Our second definition introduces a severely weakened notion of measure.

Definition 5.2. Given a measurable space $(\Sigma, \mathfrak{M})$, call a function $\mu : \mathfrak{M} \to [0, +\infty]$ a `feeble measure` provided the collection of its null sets, i.e.,

[1] As is indicated in the last paragraph on [91, p. 18].

$$\mathcal{N}_\mu := \{A \in \mathfrak{M} : \mu(A) = 0\}, \tag{5.5}$$

contains $\emptyset$, is closed under countable union, and satisfies $A \in \mathcal{N}_\mu$ whenever $A \in \mathfrak{M}$ and there exists $B \in \mathcal{N}_\mu$ such that $A \subseteq B$.

Let $(\Sigma, \mathfrak{M})$ be a measurable space, and let μ be a feeble measure on $\mathfrak{M}$. As in the case of genuine measures, we will say that a property is valid μ-a.e. provided the property in question is valid, with the possible exception of a set in $\mathcal{N}_\mu$. Identifying functions coinciding μ-a.e. on Σ then becomes an equivalence relation, and we will denote by $\mathcal{M}(\Sigma, \mathfrak{M}, \mu)$ the collection of all equivalence classes[2] of scalar-valued, $\mathfrak{M}$-measurable functions on Σ. Finally, we define

$$\mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) := \{f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : f \geq 0 \quad \mu\text{-a.e. on } \Sigma\}. \tag{5.6}$$

We now proceed to state the abstract completeness criterion alluded to earlier (a more general result of this nature is discussed in Theorem 5.9).

Theorem 5.3. *Assume that $(\Sigma, \mathfrak{M})$ is a measurable space and that μ is a feeble measure on $\mathfrak{M}$. Suppose that the function*[3]

$$\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \longrightarrow [0, +\infty] \tag{5.7}$$

satisfies the following properties:

(1) (Quasisubadditivity) There exists a constant $C_0 \in [1, +\infty)$ with the property that

$$\|f + g\| \leq C_0 \max\{\|f\|, \|g\|\}, \quad \forall\, f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu); \tag{5.8}$$

(2) (Pseudohomogeneity) There exists a function $\varphi : (0, +\infty) \to (0, +\infty)$ satisfying

$$\|\lambda f\| \leq \varphi(\lambda)\|f\|, \quad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \quad \forall\, \lambda \in (0, +\infty), \tag{5.9}$$

and such that[4]

$$\sup_{\lambda > 0}\big[\varphi(\lambda)\varphi(\lambda^{-1})\big] < +\infty \quad \textit{and} \quad \lim_{\lambda \to 0^+} \varphi(\lambda) = 0; \tag{5.10}$$

(3) (Nondegeneracy) There holds

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[2]Even though we will work with equivalence classes of functions, we will follow the common practice of ignoring this aspect in the choice of our notation.

[3]While the notation $\|\cdot\|$ is employed, the function in question satisfies much weaker conditions than the axioms used to define a genuine norm.

[4]Any function of the form $\varphi(\lambda) := \lambda^p$, with $p \in (0, \infty)$ fixed, satisfies (5.10). Such an example arises naturally if, e.g., μ is a measure and $\|f\| := \int_\Sigma f^p \, d\mu$ for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ (note that $\|\cdot\|$ satisfies all hypotheses of Theorem 5.3).

$$\|f\| = 0 \iff f = 0, \quad \forall f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu); \tag{5.11}$$

(4) *(Quasimonotonicity) There exists a constant* $C_1 \in [1, +\infty)$ *such that for any functions* $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ *satisfying* $f \le g$ μ*-a.e. on* Σ *there holds* $\|f\| \le C_1 \|g\|$;

(5) *(Weak Fatou property) If* $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ *is a sequence of functions satisfying* $f_i \le f_{i+1}$ μ*-a.e. on* Σ *for each* $i \in \mathbb{N}$ *and* $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$*, then* $\|\sup_{i\in\mathbb{N}} f_i\| < +\infty$.

Finally, define

$$\mathcal{L} := \{f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \|f\|_{\mathcal{L}} := \|\,|f|\,\| < +\infty\}. \tag{5.12}$$

Then functions in $\mathcal{L}$ *are finite* μ*-a.e. on* Σ *and, with the topology* $\tau_{\|\cdot\|_{\mathcal{L}}}$ *considered in the sense of Definition 5.1 (relative to the additive group structure on* $\mathcal{L}$*),*

$$(\mathcal{L}, \tau_{\|\cdot\|_{\mathcal{L}}}) \text{ is a Hausdorff, complete, metrizable, topological vector space.} \tag{5.13}$$

The proof of Theorem 5.3 is presented in Sect. 5.4.1, after Theorem 5.9 has been established. The latter theorem constitutes our main completeness result in the setting of topological vector spaces, and Theorem 5.3 is essentially obtained as a fairly routine corollary of it. Significantly, Theorem 5.9 is formulated in the setting of partially ordered vector spaces, without any reference to a background measure space.

Of course, any genuine quasinorm $\|\cdot\|$ on the vector space $L^0(\Sigma, \mathfrak{M}, \mu)$, consisting of (classes of) functions from $\mathcal{M}(\Sigma, \mathfrak{M})$ that are finite μ-a.e., satisfies the axioms (1)–(3). In such a scenario, property (4) is a relaxation of the demand that $\|\cdot\|$ is `monotone`, a condition that reads

$$\forall f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \text{ with } f \le g \ \mu\text{-a.e. on } \Sigma \implies \|f\| \le \|g\|. \tag{5.14}$$

Also, the weak Fatou property mimics (in abstract, and with a weaker conclusion) the familiar Fatou's lemma in the standard setting of Lebesgue spaces. Indeed, in Proposition 5.17 we will prove that, given a feeble measure μ on a measurable space $(\Sigma, \mathfrak{M})$ and a function $\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ that is quasisubadditive and quasimonotone, the weak Fatou property stated in Theorem 5.3 is equivalent to the demand that

$$(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \text{ and } \liminf_{i\to\infty} \|f_i\| < +\infty \implies \Big\|\liminf_{i\to\infty} f_i\Big\| < +\infty. \tag{5.15}$$

Moreover, under the hypotheses of Theorem 5.3, a quantitative version of this implication holds; see (5.16) below. It is worth noting that under the stronger assumption that $\|\cdot\|$ is genuinely monotone, establishing the equivalence between condition (5.15) and the weak Fatou property stated in Theorem 5.3 is a trivial matter.[5] However, the proof of this equivalence is considerably more subtle under the mere quasimonotonicity assumption we presently make (see the discussion in the proof of Proposition 5.17 for details). The interested reader is referred to, e.g., [6, 15], [130, Sect. 65, pp. 446–449] for more on the role and significance of the Fatou property, albeit under considerably stronger background assumptions than ours. Finally, we wish to note that (5.12) is a general recipe according to which large classes of function spaces naturally arise in practice; see Sect. 5.2 for concrete examples of interest.

In addition to completeness, we are also interested in establishing abstract results pertaining to the pointwise behavior of sequences of functions that are convergent in topological vector spaces created according to formula (5.12). Specifically, we will prove the following theorem.

Theorem 5.4. *Retain the same hypotheses as in Theorem 5.3, and recall the vector space $\mathcal{L}$ from (5.12). Then any sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{L}$ that is convergent to some $f\in\mathcal{L}$ in the topology $\tau_{\|\cdot\|_{\mathcal{L}}}$ has a subsequence that converges to f pointwise μ-a.e. on Σ.*

In particular, the positive cone in $\mathcal{L}$ equipped with the partial order induced by the pointwise μ-a.e. inequality of functions, i.e., $\mathcal{L}^+ := \{f\in\mathcal{L} : f\geq 0\ \mu\text{-a.e. on }\Sigma\}$, is closed in $\big(\mathcal{L},\tau_{\|\cdot\|_{\mathcal{L}}}\big)$.

Theorem 5.4 is proved in the last part of Sect. 5.4.2 by making use of capacitary estimates and the fact that the weak Fatou property implies a quantitative version of itself. More precisely, in the context of Theorem 5.3,

$$\begin{aligned} &\exists\, C\in[0,+\infty)\ \text{ such that }\ \forall\,(f_i)_{i\in\mathbb{N}}\subseteq\mathcal{M}_+(\Sigma,\mathfrak{M},\mu)\\ &\qquad\text{one has }\ \Big\|\liminf_{i\to\infty} f_i\Big\|\leq C\liminf_{i\to\infty}\|f_i\|. \end{aligned}\tag{5.16}$$

This is proved in Corollary 5.21.

It is also possible to specify general conditions ensuring the continuous embedding of the vector space $\mathcal{L}$ from (5.12) equipped with the topology $\tau_{\|\cdot\|_{\mathcal{L}}}$ into the space of measurable, a.e. finite functions equipped with the topology of convergence in measure. To state a theorem to this effect, let us agree that $\mathbf{1}_E$ will denote the characteristic function of the set E. Also, given a sigma-finite measure[6] space $(\Sigma,\mathfrak{M},\mu)$, let $L^0(\Sigma,\mathfrak{M},\mu)$ stand for the vector space $\{f\in\mathcal{M}(\Sigma,\mathfrak{M},\mu) : |f|<+\infty\ \mu\text{-a.e. on }\Sigma\}$, and denote by τ_μ the topology on this space induced by convergence in measure on sets of finite measure.

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[5] Since $\liminf_{i\to\infty} f_i=\sup_{i\in\mathbb{N}} g_i$, where $g_i:=\inf_{j\geq i} f_j$ for any sequence $(f_i)_{i\in\mathbb{N}}$ of functions in $\mathcal{M}_+(\Sigma,\mathfrak{M},\mu)$, it follows that (5.15) implies the weak Fatou property. For the converse implication, note that $\sup_{i\in\mathbb{N}}\|g_i\| = \liminf_{i\to\infty}\|g_i\|$ whenever $(g_i)_{i\in\mathbb{N}}\subseteq\mathcal{M}_+(\Sigma,\mathfrak{M},\mu)$, $g_i\leq g_{i+1}$ μ-a.e. on Σ for each $i\in\mathbb{N}$, given that the sequence $\big\{\|g_i\|\big\}_{i\in\mathbb{N}}$ is monotone.

[6] Throughout the chapter, unless otherwise specified, by “measure” we will always understand a positive measure.

Theorem 5.5. *Suppose that* $(\Sigma, \mathfrak{M}, \mu)$ *is a measure space, and assume that a function* $\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ *satisfying properties* (1)–(4) *listed in Theorem 5.3 has been given. In addition, suppose that*

$$\begin{gathered}\exists\,(K_j)_{j\in\mathbb{N}} \subseteq \mathfrak{M} \text{ satisfying } \bigcup_{j=1}^{\infty} K_j = \Sigma \text{ and with the property that} \\ K_j \subseteq K_{j+1},\ \ \mu(K_j) < +\infty,\ \ \|\mathbf{1}_{K_j}\| < +\infty \ \text{ for each } \ j \in \mathbb{N}.\end{gathered} \tag{5.17}$$

Then, if $\mathcal{L}$ *is as in* (5.12), *it follows that*

$$\big(\mathcal{L}, \tau_{\|\cdot\|_{\mathcal{L}}}\big) \hookrightarrow \big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big) \quad \text{continuously.} \tag{5.18}$$

Theorem 5.5 is a direct corollary of Theorem 5.26, stated and proved in Sect. 5.6.

The last topic of interest for us in this chapter concerns the separability of the topological vector space from (5.13). In this regard, we will establish the following result (see Definition 5.20 for the notion of separable measure).

Theorem 5.6. *Retain the hypotheses of Theorem 5.5, and in addition assume that* $\|\cdot\|$ *is absolutely continuous, in the sense that for any given* $f \in \mathcal{L}$ *there holds*

$$\left.\begin{array}{r}\forall\,(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M} \ \text{ such that } \ \mathbf{1}_{A_n} \to 0 \\ \text{pointwise } \mu\text{-a.e. on } \Sigma \text{ as } n \to \infty\end{array}\right\} \Longrightarrow \lim_{n\to\infty} \big\| \, |f| \cdot \mathbf{1}_{A_n} \big\| = 0. \tag{5.19}$$

Then $\big(\mathcal{L}, \tau_{\|\cdot\|_{\mathcal{L}}}\big)$ *is a separable topological space whenever the measure* μ *is separable.*

Theorem 5.6 is readily implied by Theorem 5.28 discussed in Sect. 5.7.

In closing, we wish to note that in the case when $\big(\mathcal{L}, \|\cdot\|_{\mathcal{L}}\big)$ is a Banach function space, it follows from [15, Theorem 5.5, p. 27] that both the absolute continuity property stated in (5.19) and the separability of the measure μ are actually necessary conditions for the separability of the topological space $\big(\mathcal{L}, \tau_{\|\cdot\|_{\mathcal{L}}}\big)$.

5.2 Examples

It is instructive to illustrate the scope of Theorems 5.3 and 5.6 by considering a multitude of examples of interest and studying, in each case, the extent to which the conditions stipulated in the statement of these theorems are satisfied.

Example 1. Abstract Lebesgue spaces $L^p(\Sigma, \mathfrak{M}, \mu)$, $0 < p \le \infty$, *associated with a measure space* $(\Sigma, \mathfrak{M}, \mu)$. This is, of course, a toy case, and the goal is to illustrate the role and necessity of the assumptions made in our earlier theorems. Here, for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ we take $\|f\| := \left(\int_\Sigma f^p \, d\mu\right)^{1/p}$ if $p \in (0, \infty)$ and, corresponding to $p = \infty$, $\|f\| := \text{ess-sup} f$. Then, for each $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ and $p \in (0, \infty]$,

$$\begin{gathered} \|f + g\| \leq c_p\big(\|f\| + \|g\|\big) \leq 2c_p \max\{\|f\|, \|g\|\}, \\ \text{where} \ \ c_p := 2^{\max\{0,1/p-1\}} \in [1, +\infty), \end{gathered} \tag{5.20}$$

which shows that the quasinorm condition (5.8) is satisfied. Moreover, for each index $p \in (0, \infty]$ the classic Fatou lemma gives that[7]

$$\Big\|\sup_{i\in\mathbb{N}} f_i\Big\| \leq \sup_{i\in\mathbb{N}} \|f_i\| \tag{5.21}$$

whenever the functions $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ satisfy $f_i \leq f_{i+1}$ μ-a.e. on Σ for each $i \in \mathbb{N}$. Thus, properties (1)–(5) in the statement of Theorem 5.3 hold. As a consequence, we recover the familiar result that $L^p(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric (hence, quasi-Banach) space with the property that any convergent sequence from this space has a subsequence that converges (to its limit in L^p) in a pointwise μ-a.e. fashion. Furthermore, if μ is sigma-finite, then $L^p(\Sigma, \mathfrak{M}, \mu)$ embeds continuously into $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$. Finally, if $p \in (0, \infty)$ (which is the range for which (5.19) holds) and the measure μ is separable and sigma-finite, then the space $L^p(\Sigma, \mathfrak{M}, \mu)$ is separable. As is well known (cf., e.g., [15, Theorem 5.5, p. 27]), for each $p \in [1, \infty)$ the separability of the measure μ is actually a necessary condition for the separability of the Lebesgue space $L^p(\Sigma, \mathfrak{M}, \mu)$.

Example 2. Generalized Lebesgue spaces $L^\theta(\Sigma, \mathfrak{M}, \mu)$*, associated with a measure space* $(\Sigma, \mathfrak{M}, \mu)$. Let $\theta : \mathbb{R} \to [0, +\infty)$ be an even, lower-semicontinuous function that vanishes at, and only at, the origin. In addition, assume there exist $c_0, c_1 \in [1, +\infty)$ and $p \in (0, +\infty)$ with the property that

$$\theta(t_1) \leq c_0\theta(t_2), \quad \forall\, t_1, t_2 \in [0, +\infty) \ \text{ such that } \ t_1 \leq t_2, \tag{5.22}$$

$$\theta(st) \leq cs^p\theta(t), \quad \forall\, s \in [0, +\infty) \ \text{ and } \ \forall\, t \in (0, +\infty). \tag{5.23}$$

Define $\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ by setting

$$\|f\| := \int_\Sigma \theta(f(x))\,\mathrm{d}\mu(x), \quad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \tag{5.24}$$

and, consistent with (5.12), consider

$$L^\theta(\Sigma, \mathfrak{M}, \mu) := \big\{f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \|\,|f|\,\| < +\infty\big\}. \tag{5.25}$$

[7]In fact, (5.21) holds with equality, as the observant reader has undoubtedly noted.

Of course, for each fixed $p \in (0,\infty)$ the function $\theta(t) := |t|^p$ satisfies all conditions stipulated previously and, corresponding to this choice of θ, the space $L^\theta(\Sigma, \mathfrak{M}, \mu)$ coincides, as a topological vector space, with the classical Lebesgue space $L^p(\Sigma, \mathfrak{M}, \mu)$ (thus justifying the terminology adopted here).

To understand the nature of the space defined in (5.25), observe that for each function $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ we may write

$$\begin{aligned}
\|f+g\| &= \int_\Sigma \theta\big(f(x)+g(x)\big)\,\mathrm{d}\mu(x) \\
&= \int_{\{x\in\Sigma:\, f(x)\geq g(x)\}} \theta\big(f(x)+g(x)\big)\,\mathrm{d}\mu(x) \\
&\quad + \int_{\{x\in\Sigma:\, f(x)< g(x)\}} \theta\big(f(x)+g(x)\big)\,\mathrm{d}\mu(x) \\
&\leq c_0 \int_\Sigma \theta\big(2f(x)\big)\,\mathrm{d}\mu(x) + c_0 \int_\Sigma \theta\big(2g(x)\big)\,\mathrm{d}\mu(x) \\
&\leq c_0 c_1 2^p\big(\|f\| + \|g\|\big) \leq c_0 c_1 2^{p+1} \max\big\{\|f\|, \|g\|\big\},
\end{aligned} \tag{5.26}$$

which shows that $\|\cdot\|$ satisfies the quasisubadditivity condition (5.8) with constant $C_0 := c_0 c_1 2^{p+1} \in [1,+\infty)$. In addition, for each function $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ and each scalar $\lambda \in (0,+\infty)$,

$$\|\lambda f\| = \int_\Sigma \theta\big(\lambda f(x)\big)\,\mathrm{d}\mu(x) \leq c_1\lambda^p \int_\Sigma \theta\big(f(x)\big)\,\mathrm{d}\mu(x) = c_1\lambda^p\|f\|, \tag{5.27}$$

hence the pseudo-homogeneity condition (2) holds for $\varphi : (0,+\infty) \to (0,+\infty)$ given by $\varphi(\lambda) := c_1\lambda^p$ for each $\lambda \in (0,+\infty)$. Moreover, since θ vanishes only at the origin, it is clear that $\|\cdot\|$ is nondegenerate. Also, whenever two functions $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ are such that $f \leq g$ pointwise μ-a.e. on Σ,

$$\|f\| = \int_\Sigma \theta\big(f(x)\big)\,\mathrm{d}\mu(x) \leq c_0 \int_\Sigma \theta\big(g(x)\big)\,\mathrm{d}\mu(x) = c_0\|g\|, \tag{5.28}$$

from which we deduce that $\|\cdot\|$ is quasimonotone. Finally, in concert with the lower semicontinuity of θ, the classic Fatou lemma gives that if $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ is a sequence of functions satisfying $f_i \leq f_{i+1}$ μ-a.e. on Σ for each $i \in \mathbb{N}$, then

$$\begin{aligned}
\Big\|\sup_{i\in\mathbb{N}} f_i\Big\| &= \int_\Sigma \theta\Big(\sup_{i\in\mathbb{N}} f_i(x)\Big)\,\mathrm{d}\mu(x) = \int_\Sigma \theta\Big(\liminf_{i\to\infty} f_i(x)\Big)\,\mathrm{d}\mu(x) \\
&\leq \int_\Sigma \Big[\liminf_{i\to\infty} \theta\big(f_i(x)\big)\Big]\,\mathrm{d}\mu(x) \leq \liminf_{i\to\infty} \int_\Sigma \theta\big(f_i(x)\big)\,\mathrm{d}\mu(x) \\
&= \liminf_{i\to\infty} \|f_i\| \leq \sup_{i\in\mathbb{N}} \|f_i\|.
\end{aligned} \tag{5.29}$$

Thus, the weak Fatou property (5) is satisfied as well; hence, all conditions hypothesized in Theorem 5.3 hold. Consequently, $L^\theta(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric space with the property that any convergent sequence from this space has a subsequence that converges (to its limit in L^θ) in a pointwise μ-a.e. fashion. Furthermore, Lebesgue's dominated convergence theorem shows that (5.19) is satisfied in the current setting. As such, the space $L^\theta(\Sigma, \mathfrak{M}, \mu)$ is separable granted that the measure μ is separable and sigma-finite. Lastly, if the measure μ is sigma-finite, then (5.17) holds, which further implies that $L^\theta(\Sigma, \mathfrak{M}, \mu)$ embeds continuously into $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$ in this case.

Example 3. Variable exponent Lebesgue spaces $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ associated with a measure space $(\Sigma, \mathfrak{M}, \mu)$. Let $p : \Sigma \to (0, +\infty)$ be a measurable function,[8] called a variable exponent, with the property that

$$p^+ := \text{ess-sup}\, p < +\infty \quad \text{and} \quad p^- := \text{ess-inf}\, p > 0. \tag{5.30}$$

Define the Luxemburg "norm" $\|\cdot\| = \|\cdot\|_{L^{p(\cdot)}(\Sigma,\mathfrak{M},\mu)}$ by setting (with the convention that $\inf \emptyset := +\infty$)

$$\|f\| := \inf\Big\{\lambda > 0 : \int_\Sigma \big(f(x)/\lambda\big)^{p(x)}\, \mathrm{d}\mu(x) \le 1\Big\},\ \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu). \tag{5.31}$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is then constructed as in (5.12) for the choice of $\|\cdot\|$ as in (5.31). Since $\int_\Sigma \big(f(x)/\|f\|\big)^{p(x)}\, \mathrm{d}\mu(x) \le 1$ for each nonnegative function $f \in L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$, a straightforward computation gives

$$\|f + g\| \le \begin{cases} 2^{\frac{\max\{p^+,1\}}{p^-}}\big(\|f\| + \|g\|\big) & \text{if } p^- < 1, \\ \|f\| + \|g\| & \text{if } p^- \ge 1, \end{cases} \quad \forall\, f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu). \tag{5.32}$$

By design, we have $\|\lambda f\| = \lambda\|f\|$ for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ and $\lambda \in (0, +\infty)$. Also, if $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ is such that $\|f\| = 0$, then $\int_\Sigma \big(f(x)/\varepsilon\big)^{p(x)}\, \mathrm{d}\mu(x) \le 1$ for each $\varepsilon > 0$, which immediately gives that $f = 0$ μ-a.e. on Σ. Clearly, $\|\cdot\|$ is monotone, and the classical Fatou lemma proves that $\|\cdot\|$ satisfies the strong Fatou property (i.e., (5.21) holds). This shows that all hypotheses in Theorem 5.3 hold in the current setting. Consequently, $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric (hence, quasi-Banach) space with the property that any convergent sequence from this space has a subsequence that converges (to its limit in $L^{p(\cdot)}$) in a pointwise μ-a.e. fashion.

[8] Typically, in the literature it is assumed that $p \ge 1$ μ-a.e. on Σ, but such a restriction is artificial for us here.

Moreover, since for any $A \in \mathfrak{M}$ one has $\|\mathbf{1}_A\| \le [\mu(A)]^{1/p_A}$, where we have set $p_A := p^+$ if $\mu(A) \le 1$ and $p_A := p^-$ if $\mu(A) > 1$, it follows from Theorem 5.5 that $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ embeds continuously into $(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu)$ whenever the measure μ is sigma-finite. Finally, the absolute continuity condition (5.19) is automatically verified in this setting; hence, $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is separable if the measure μ is separable and sigma-finite.

Example 4. Mixed-normed spaces L^P, with $P = (p_1, \dots, p_n) \in (0, \infty]^n$, of Benedek–Panzone. Let $(\Sigma_i, \mathfrak{M}_i, \mu_i)$, $1 \le i \le n$, be measure spaces, set $\Sigma := \Sigma_1 \times \cdots \times \Sigma_n$ and $\mathfrak{M} := \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$, and define the product measure $\mu := \mu_1 \otimes \cdots \otimes \mu_n$ on Σ. Next, given $P = (p_1, \dots, p_n) \in (0, \infty]^n$, consider $\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ defined for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ according to the formula

$$\|f\| := \left(\int_{\Sigma_1} \left(\int_{\Sigma_2} \cdots \left(\int_{\Sigma_n} f(x_1, \dots, x_n)^{p_n} \, d\mu_n(x_n)\right)^{\frac{p_{n-1}}{p_n}} \cdots \right)^{\frac{p_1}{p_2}} d\mu_1(x_1)\right)^{\frac{1}{p_1}}, \tag{5.33}$$

understood with natural alterations when $p_i = \infty$ for some $i \in \{1, \dots, n\}$. In [14], the mixed-normed spaces $L^P = L^P(\Sigma, \mathfrak{M}, \mu)$, constructed according to the recipe described in (5.12) adapted to the current context, have been introduced and studied in the case when $1 \le p_i \le \infty$ for each $i \in \{1, \dots, n\}$. Here we wish to note that, for the full range of indices $P = (p_1, \dots, p_n) \in (0, \infty]^n$, repeated applications of (5.20) yield

$$\|f + g\| \le 2\Big(\prod_{i=1}^{n} c_{p_i}\Big) \max\{\|f\|, \|g\|\}, \quad \forall\, f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \tag{5.34}$$

hence (5.8) is satisfied. Also, with $\|\cdot\|$ as in (5.33), the strong Fatou property (5.21) holds (applying the classic Fatou lemma n times), and the remaining hypotheses in the statement of Theorem 5.3 are trivially satisfied. As a corollary, $L^P(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric space (hence, quasi-Banach) with the property that any convergent sequence from this space has a subsequence that converges (to its limit in L^P) in a pointwise μ-a.e. fashion. In addition, this space is continuously embedded into $(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu)$ whenever each measure μ_i, $1 \le i \le n$, is sigma-finite. Finally, if $P = (p_1, \dots, p_n) \in (0, \infty)^n$ and each measure μ_i is separable and sigma-finite, $1 \le i \le n$, then the space $L^P(\Sigma, \mathfrak{M}, \mu)$ is separable.

Example 5. Variable exponent mixed-normed spaces $L^{P(\cdot)}$ where, for some $n \in \mathbb{N}$, we have set $P(\cdot) = (p_1(\cdot), \dots, p_n(\cdot))$. Let $(\Sigma_i, \mathfrak{M}_i, \mu_i)$, $1 \le i \le n$, be measure spaces, set $\Sigma := \Sigma_1 \times \cdots \times \Sigma_n$ and $\mathfrak{M} := \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$, and define the product measure $\mu := \mu_1 \otimes \cdots \otimes \mu_n$ on Σ. In this setting, assume that for each $i \in \{1, \dots, n\}$ a $\mathfrak{M}_i$-measurable function $p_i : \Sigma_i \to (0, +\infty)$ has been given such that

$$p_i^+ := \text{ess-sup}\, p_i < +\infty \quad \text{and} \quad p_i^- := \text{ess-inf}\, p_i > 0. \tag{5.35}$$

Consider $\varrho : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ defined for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ according to the formula

$$\varrho(f) := \int_{\Sigma_1} \Big(\int_{\Sigma_2} \cdots \Big(\int_{\Sigma_n} f(x_1, \dots, x_n)^{p_n(x_n)} \, \mathrm{d}\mu_n(x_n) \Big)^{p_{n-1}(x_{n-1})} \cdots \Big)^{p_1(x_1)} \, \mathrm{d}\mu_1(x_1), \tag{5.36}$$

and define the Luxemburg "norm" $\| \cdot \| = \| \cdot \|_{L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)}$ by setting

$$\|f\| := \inf \big\{ \lambda > 0 : \varrho(f/\lambda) \le 1 \big\}, \quad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu). \tag{5.37}$$

Then the variable exponent mixed-norm space $L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is constructed for this choice of $\| \cdot \|$ as in (5.12). Arguments similar in spirit to those presented in Examples 2–3 show that $L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is a quasi-Banach space with the property that any convergent sequence from this space has a subsequence that converges [to its limit in $L^{P(\cdot)}$] in a pointwise μ-a.e. fashion. Furthermore, $L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ embeds continuously into $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$, provided each measure μ_i is sigma-finite, and is separable whenever each measure μ_i is sigma-finite and separable.

Example 6. Lorentz spaces $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$, $0 < p < \infty$, $0 < q \le \infty$, *associated with a measure space* $(\Sigma, \mathfrak{M}, \mu)$. Recall that if $0 < p < \infty$ and $0 < q \le \infty$, then the Lorentz quasinorm $\| \cdot \| = \| \cdot \|_{L^{p,q}(\Sigma, \mathfrak{M}, \mu)}$ is defined for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ by

$$\|f\| := \begin{cases} \Big(\int_0^\infty \lambda^q \mu\big(\{x \in \Sigma : f(x) > \lambda\}\big)^{q/p} \frac{d\lambda}{\lambda} \Big)^{1/q} & \text{if } q < \infty, \\ \sup_{\lambda > 0} \Big[\lambda\, \mu\,(\{x \in \Sigma : f(x) > \lambda\})^{1/p} \Big] & \text{if } q = \infty. \end{cases} \tag{5.38}$$

The Lorentz space $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$ is defined as in (5.12) when $\| \cdot \|$ is as in (5.38). Clearly, $\| \cdot \|_{\mathcal{L}}$ is then a monotone quasinorm. From [26, Theorem 1.9.9(c), p. 55] it follows that the strong Fatou property (5.21) holds for the quasinorm (5.38), and it is straightforward to check that all the other hypotheses in Theorem 5.3 are also satisfied. Moreover, the absolute continuity condition from (5.19) holds whenever $q < \infty$ (cf. [26, Theorem 1.9.9(d), p. 55]), hence Theorem 5.6 applies in this case. In summary, the Lorentz space $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric space with the property that any of its convergent sequences has a subsequence that converges (to its limit in $L^{p,q}$) in a pointwise μ-a.e. fashion and that embeds continuously into $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$ provided the measure μ is sigma-finite. Moreover, whenever $p, q \in (0, \infty)$ and the measure μ is separable and sigma-finite, then the Lorentz space $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$ is separable.

Let us also note here that similar considerations apply to the scale of Lorentz–Orlicz spaces (cf. [70, 85, 118]), as well as to the so-called Lorentz–Sharpley spaces. We omit the details.

Example 7. Capacitary spaces $L^{p,q}(\Sigma,\mathfrak{M},\mathscr{C})$, $0 < p < \infty$, $0 < q \leq \infty$, *associated with a measurable space* $(\Sigma,\mathfrak{M})$ *and a standard Fatou capacity* $\mathscr{C}$ *on* $\mathfrak{M}$. By a standard capacity on a sigma-algebra of sets $\mathfrak{M}$ we understand here a function $\mathscr{C} : \mathfrak{M} \to [0,+\infty]$ satisfying for any $A, B \in \mathfrak{M}$ the following conditions:

$$\mathscr{C}(\emptyset) = 0, \quad \mathscr{C}(A) \leq \mathscr{C}(B) \ \text{ if } \ A \subseteq B, \ \text{ and } \ \mathscr{C}(A \cup B) \leq c\big(\mathscr{C}(A) + \mathscr{C}(B)\big), \tag{5.39}$$

where $c \geq 1$ is a fixed, finite constant. A standard Fatou capacity is then a standard capacity that has the Fatou property, that is,

$$\begin{array}{c} \lim_{n\to\infty} \mathscr{C}(A_n) = \mathscr{C}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \ \text{ for every } (A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M} \\ \text{with the property that } \mathscr{C}(A_n \setminus A_{n+1}) = 0 \text{ for each } n \in \mathbb{N}. \end{array} \tag{5.40}$$

In particular, if "$\triangle$" stands for the set-theoretic difference operation, this and (5.39) imply that[9]

$$\mathscr{C}(A) = \mathscr{C}(B) \quad \text{whenever } \ A, B \in \mathfrak{M} \ \text{ are such that } \ \mathscr{C}(A \triangle B) = 0. \tag{5.41}$$

Next, given a standard Fatou capacity $\mathscr{C}$, it follows that $\mathscr{C}$ is a feeble measure. Following [29], we define $\|\cdot\| = \|\cdot\|_{L^{p,q}(\Sigma,\mathfrak{M},\mathscr{C})}$ analogously to (5.38), i.e., for each $\mathfrak{M}$-measurable function f on Σ that is nonnegative $\mathscr{C}$-a.e., set

$$\|f\| := \begin{cases} \Big(\int_0^\infty \lambda^q \mathscr{C}\big(\{x \in \Sigma : f(x) > \lambda\}\big)^{q/p} \frac{d\lambda}{\lambda}\Big)^{1/q} & \text{if } q < \infty, \\ \sup_{\lambda>0}\Big[\lambda\, \mathscr{C}\,(\{x \in \Sigma : f(x) > \lambda\})^{1/p}\Big] & \text{if } q = \infty. \end{cases} \tag{5.42}$$

Then, based on (5.41), it is clear that any two $\mathfrak{M}$-measurable functions that coincide $\mathscr{C}$-a.e. have identical quasinorms (i.e., (5.7) is well-defined). Keeping this in mind and relying on [29, Theorem 1(d)], it follows that all hypotheses made in Theorem 5.3 are verified in this setting. Consequently, the capacitary space $L^{p,q}(\Sigma,\mathfrak{M},\mathscr{C})$ is a complete quasimetric space with the property that any of its convergent sequences has a subsequence that converges [to its limit in $L^{p,q}(\Sigma,\mathfrak{M},\mathscr{C})$] in a pointwise μ-a.e. fashion. Moreover, additional properties for this space may be obtained by suitably strengthening the assumptions on $\mathscr{C}$ (so as to fit the hypotheses of Theorems 5.4–5.6).

Example 8. Orlicz spaces $L_\theta(\Sigma,\mathfrak{M},\mu)$, *associated with a measure space* $(\Sigma,\mathfrak{M},\mu)$. Consider an even, lower-semicontinuous function $\theta : \mathbb{R} \to [0,+\infty]$ that is not identically zero. In addition, assume that θ is nondecreasing on $[0,+\infty)$ and that there exist $c \in [1,+\infty)$ and $p \in (0,+\infty)$ with the property that

$$\theta(st) \leq c s^p \theta(t), \quad \forall\, s \in [0,1], \ \ \forall\, t \in (0,+\infty). \tag{5.43}$$

[9]Cf. the discussion preceding Theorem 1 on p. 98 in [29].

Parenthetically, we note that any Young function satisfies the preceding conditions. Let us also note that if $t_o \in (0, +\infty)$ is such that $\theta(t_o) > 0$, then $c^{-1}s^{-p}\theta(t_o) \leq \theta(t_o/s)$ for each $s \in (0, 1)$, which, in particular, implies that

$$\lim_{t\to+\infty} \theta(t) = +\infty. \tag{5.44}$$

In this setting, introduce the Luxemburg "norm" of any function $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ by setting

$$\|f\| := \inf \left\{ a > 0 : \int_\Sigma \theta(f(x)/a)\, d\mu(x) \leq 1 \right\} \in [0, +\infty], \tag{5.45}$$

with the convention that $\inf \emptyset := +\infty$. Then the Orlicz space $L_\theta(\Sigma, \mathfrak{M}, \mu)$ is defined as

$$L_\theta(\Sigma, \mathfrak{M}, \mu) := \big\{ f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \| |f| \| < +\infty \big\}. \tag{5.46}$$

It is then clear that the Orlicz space $L_\theta(\Sigma, \mathfrak{M}, \mu)$ fits within the framework of (5.12). Clearly, $\|\cdot\|$ is monotone. Also, that $\|\cdot\|$ is nondegenerate readily follows from (5.44) upon observing that, for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$, the following level set estimate holds:

$$\theta(\varepsilon/a)\, \mu\big(\{x \in \Sigma : f(x) > \varepsilon\}\big) \leq \int_\Sigma \theta\big(f(x)/a\big)\, d\mu(x), \qquad \forall\, a, \varepsilon \in (0, +\infty). \tag{5.47}$$

To verify the quasisubadditivity condition, assume that $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ are such that $0 < \min\{\|f\|, \|g\|\}$ and $\max\{\|f\|, \|g\|\} < +\infty$. Then

$$\begin{aligned}
&\int_\Sigma \theta\left(\frac{f(x) + g(x)}{c^{1/p}(\|f\| + \|g\|)}\right) d\mu(x) \\
&\quad = \int_\Sigma \theta\left(\frac{\|f\|}{\|f\| + \|g\|} \cdot \frac{c^{-1/p} f(x)}{\|f\|} + \frac{\|g\|}{\|f\| + \|g\|} \cdot \frac{c^{-1/p} g(x)}{\|g\|}\right) d\mu(x) \\
&\quad \leq \max\left\{ \int_\Sigma \theta\Big(\frac{c^{-1/p} f(x)}{\|f\|}\Big) d\mu(x)\, , \int_\Sigma \theta\Big(\frac{c^{-1/p} g(x)}{\|g\|}\Big) d\mu(x) \right\} \\
&\quad \leq \max\left\{ \int_\Sigma \theta\Big(\frac{f(x)}{\|f\|}\Big) d\mu(x)\, , \int_\Sigma \theta\Big(\frac{g(x)}{\|g\|}\Big) d\mu(x) \right\} \leq 1.
\end{aligned} \tag{5.48}$$

Here, the first inequality follows from the fact that $\theta(at_1 + (1 - a)t_2) \leq \max\{\theta(t_1), \theta(t_2)\}$ for $a \in [0, 1]$ and $t_1, t_2 \in (0, +\infty)$ (which, in turn, is implied by the monotonicity of θ), the second uses (5.43), while the last inequality is a consequence of the readily verified observation that $\int_\Sigma \theta(f/\|f\|)\, d\mu \leq 1$ for each

$f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ with $\|f\| \in (0, +\infty)$. Having proved (5.48), we then deduce from (5.45) that

$$\|f + g\| \le c^{1/p}\big(\|f\| + \|g\|\big) \le 2c^{1/p} \max\big\{\|f\|, \|g\|\big\}, \quad \forall\, f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \tag{5.49}$$

as desired. Furthermore, it is routine to check that

$$\forall\, (f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \implies \Big\|\liminf_{i\to\infty} f_i\Big\| \le \liminf_{i\to\infty} \|f_i\|, \tag{5.50}$$

which, as was already noted, implies the weak Fatou property hypothesized in Theorem 5.3. Hence, all conditions stipulated in this theorem are satisfied. As such, $L_\theta(\Sigma, \mathfrak{M}, \mu)$ is a complete quasimetric space with the property that any convergent sequence from this space has a subsequence that converges (to its limit in L_θ) in a pointwise μ-a.e. fashion. Moreover, it is elementary to check that (5.19) holds in the current situation and, hence, the space $L_\theta(\Sigma, \mathfrak{M}, \mu)$ is separable whenever the measure μ is separable and sigma-finite. Finally, if $\lim_{t\to 0^+} \theta(t) = 0$ and the measure μ is sigma-finite, then (5.17) holds; hence, $L_\theta(\Sigma, \mathfrak{M}, \mu)$ embeds continuously into $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$ in this situation.

Example 9. Tent spaces $T^{p,q}$, $0 < p, q \le \infty$ with $\min\{p, q\} < \infty$, of Coifman-Meyer-Stein. With $\mathscr{L}^n$ denoting the n-dimensional Lebesgue measure, consider the function $\|\cdot\| = \|\cdot\|_{T^{p,q}}$ mapping a Lebesgue measurable function $f : \mathbb{R}^{n+1}_+ \to [0, +\infty]$ to

$$\|f\| := \begin{cases} \Big(\int_{\mathbb{R}^n} (\mathscr{A}_q f)(x)^p \, d\mathscr{L}^n(x)\Big)^{1/p} & \text{if } \ p < \infty, \\ \text{ess-sup}\, \mathscr{C}_q f & \text{if } \ p = \infty \end{cases} \tag{5.51}$$

where, for each $x \in \mathbb{R}^n$, if $\Gamma(x)$ stands for the upright cone with vertex at x given by $\{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}$, then

$$(\mathscr{A}_q f)(x) := \begin{cases} \Big(\int_{\Gamma(x)} f(y, t)^q t^{-(n+1)} \, d\mathscr{L}^{n+1}(y, t)\Big)^{1/q} & \text{if } \ q < \infty, \\ \sup_{(y,t)\in\Gamma(x)} |f(y, t)|, & \text{if } \ q = \infty \end{cases} \tag{5.52}$$

denotes the area operator, while if $T(B) := B \times \big(0, \mathrm{rad}(B)\big) \subseteq \mathbb{R}^{n+1}_+$ stands for the Carleson box located above the n-dimensional ball B, then

$$(\mathscr{C}_q f)(x) := \sup_{\substack{B \ni x \\ B \text{ ball in } \mathbb{R}^n}} \Big(\frac{1}{\mathscr{L}^n(B)} \int_{T(B)} f(y, t)^q t^{-1} \, d\mathscr{L}^{n+1}(y, t)\Big)^{1/q} \tag{5.53}$$

denotes the Carleson operator. Then the tent space $T^{p,q} := T^{p,q}(\mathbb{R}^{n+1}_+)$ is defined as in (5.12) for the choice of the quasinorm $\|\cdot\|$ as in (5.51), provided $p \in (0,\infty]$ and $q \in (0,\infty)$. In this situation, analogously to the case discussed in Example 4, all hypotheses of Theorem 5.3 are readily verified. From this and Theorems 5.4–5.6 we may then conclude (keeping in mind that the Lebesgue measure is both separable and sigma-finite) that the tent space $T^{p,q}$ with $q < \infty$ is a complete quasimetric space such that any of its convergent sequences has a subsequence that converges (to its limit in $T^{p,q}$) in a pointwise μ-a.e. fashion and that embeds continuously into the space of measurable, a.e. finite functions on $\mathbb{R}^{n+1}_+$. In addition, if $p, q \in (0,\infty)$, then the tent space $T^{p,q}$ is separable.

The case $q = \infty$ requires some special attention due to the presence of "sup" in (5.52). In this scenario, the space $T^{p,\infty}$ is defined in [33] as[10]

$$T^{p,\infty} := \big\{f : \mathbb{R}^{n+1}_+ \to \mathbb{R} : \ f \ \text{ continuous, and } \ \|f\|_{T^{p,\infty}} < +\infty\big\}, \quad 0 < p < \infty. \tag{5.54}$$

While this space does not fall directly under the scope of Theorem 5.3 (given that it consists of continuous functions), a closely related version of it does, and this suffices as far as the completeness of $T^{p,\infty}$ is concerned. Specifically, given a Lebesgue measurable function f in $\mathbb{R}^{n+1}_+$, observe that the mapping

$$\mathbb{R}^n \ni x \mapsto \|f\|_{L^\infty(\Gamma(x),\, \mathscr{L}^{n+1})} \in [0, +\infty] \tag{5.55}$$

is lower semicontinuous. Thus, for such a function it is meaningful to consider

$$\|f\|_{\widetilde{T}^{p,\infty}} := \left(\int_{\mathbb{R}^n} \|f\|^p_{L^\infty(\Gamma(x),\, \mathscr{L}^{n+1})}\, d\mathscr{L}^n(x)\right)^{1/p}, \quad 0 < p < \infty, \tag{5.56}$$

and then define (with $\overline{\mathbb{R}}$ denoting the extended real-axis $[-\infty, +\infty]$)

$$\widetilde{T}^{p,\infty} := \big\{f : \mathbb{R}^{n+1}_+ \to \overline{\mathbb{R}} : \ f \text{ Lebesgue measurable,} \\ \text{and } \ \|f\|_{\widetilde{T}^{p,\infty}} < \infty\big\}, \quad 0 < p < \infty. \tag{5.57}$$

It may then be verified without difficulty that $\|\cdot\|_{\widetilde{T}^{p,\infty}}$ satisfies the hypotheses of Theorem 5.3; hence $\widetilde{T}^{p,\infty}$ is a complete quasinormed space. Also, it is clear from definitions that the Coifman–Meyer–Stein space $T^{p,\infty}$ is a linear subspace of our space $\widetilde{T}^{p,\infty}$. Hence, to conclude that $T^{p,\infty}$ is a complete quasinormed space,

[10] Strictly speaking, in [33] the authors also impose the condition that functions in $T^{p,\infty}$ possess nontangential limits a.e. on $\mathbb{R}^n \equiv \partial\mathbb{R}^{n+1}_+$, but this condition turns out to be superfluous.

it suffices to show that $T^{p,\infty}$ is closed in $\widetilde{T}^{p,\infty}$. This is where the pointwise-convergence result from Theorem 5.4 plays a key role, given that there exists $c \in (0, +\infty)$ such that for every $(y_0, t_0) \in \mathbb{R}^{n+1}_+$

$$\sup_{B((y_0,t_0),\,t_0/2)} |f| \leq \mathscr{L}^n\big(B(y_0, ct_0)\big)^{-1/p} \|f\|_{\widetilde{T}^{p,\infty}} \tag{5.58}$$

for every function f that is continuous on $\mathbb{R}^{n+1}_+$.

Example 10. Homogeneous Triebel–Lizorkin sequence spaces $\dot{f}^{p,q}_\alpha(\mathbb{R}^n)$, with $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$, of Frazier–Jawerth. Denote by $\mathcal{Q}_n$ the standard family of dyadic cubes in $\mathbb{R}^n$, i.e., $\mathcal{Q}_n := \big\{2^{-j}\big([0,1]^n + k\big) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\big\}$. For each $Q \in \mathcal{Q}_n$, we will abbreviate $|Q| := \mathcal{L}^n(Q)$. Following [47], we may now introduce the homogeneous Triebel–Lizorkin scale of sequence spaces by defining $\dot{f}^{p,q}_\alpha(\mathbb{R}^n)$, for $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q \leq \infty$, as the collection of all sequences $s = \{s_Q\}_{Q \in \mathcal{Q}_n}$ with elements from $\overline{\mathbb{R}}$ such that

$$\|s\|_{\dot{f}^{p,q}_\alpha(\mathbb{R}^n)} := \big\|\, |s|\, \big\| < +\infty, \tag{5.59}$$

where $|s| := \{|s_Q|\}_{Q \in \mathcal{Q}_n}$ and, for each sequence $s = \{s_Q\}_{Q \in \mathcal{Q}_n}$ of numbers from $[0, +\infty]$, we have set

$$\|s\| := \left\| \left(\sum_{Q \in \mathcal{Q}_n} \Big(|Q|^{-\frac{1}{2} - \frac{\alpha}{n}} s_Q \mathbf{1}_Q \Big)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}, \quad \text{if } 0 < p < \infty,\ 0 < q \leq \infty, \tag{5.60}$$

and, corresponding to the case when $p = \infty$ and $0 < q \leq \infty$,

$$\|s\| := \sup_{P \in \mathcal{Q}_n} \left(\frac{1}{|P|} \int_P \sum_{Q \in \mathcal{Q}_n :\, Q \subseteq P} \big(|Q|^{-\frac{1}{2} - \frac{\alpha}{n}} s_Q \mathbf{1}_Q(x)\big)^q \, \mathrm{d}\mathcal{L}^n(x) \right)^{\frac{1}{q}}. \tag{5.61}$$

This corresponds to the setup in Theorem 5.3 for the case when $\Sigma := \mathcal{Q}_n$, $\mathfrak{M} := 2^{\mathcal{Q}_n}$ (i.e., the collection of all subsets of $\mathcal{Q}_n$), and μ is the counting measure on $\mathfrak{M}$ since measurable functions may be canonically identified with numerical sequences indexed by $\mathcal{Q}_n$. Then the quasitriangle inequality for $\|\cdot\|$ is a direct consequence of the quasitriangle inequality for the quasinorms associated with the classic sequence space ℓ^q (indexed by $\mathcal{Q}_n$) and Lebesgue space $L^p(\mathbb{R}^n)$, as, in fact, is the homogeneity and nondegeneracy of $\|\cdot\|$. Also, the fact that $\|\cdot\|$ is monotone is obvious, while the weak Fatou property for $\|\cdot\|$ may be easily verified by (twice) making suitable use of the Lebesgue monotone convergence theorem. Thus, the hypotheses of Theorem 5.3 hold in this case. In fact, the hypotheses of Theorem 5.6 are also verified, as may be seen by applying (twice) the Lebesgue dominated convergence theorem, provided $\max\{p, q\} < \infty$. All in all, this shows that $\dot{f}_\alpha^{p,q}(\mathbb{R}^n)$ is a complete quasimetric space with the property that any convergent sequence from this space has a subsequence that converges (to its limit in $\dot{f}_\alpha^{p,q}$) in a pointwise fashion and is separable whenever $\max\{p,q\} < \infty$.

Of course, similar considerations apply to the inhomogeneous Triebel–Lizorkin sequence spaces $f_\alpha^{p,q}(\mathbb{R}^n)$ defined in [47, Sect. 12]. Moreover, results for the discrete Triebel–Lizorkin spaces directly translate into analogous results for the continuous Triebel–Lizorkin scale, $F_\alpha^{p,q}(\mathbb{R}^n)$, via wavelet transforms (more details on the latter issue may be found in [68, 109, 120, 121]).

Example 11. The homogeneous Besov sequence spaces $\dot{b}_\alpha^{p,q}(\mathbb{R}^n)$, $0 < p,q \leq \infty$, $\alpha \in \mathbb{R}$, *of Frazier–Jawerth.* Recall that $\mathcal{Q}_n$ stands for the standard family of dyadic cubes in $\mathbb{R}^n$, and denote by $\ell(Q)$ the side length of $Q \in \mathcal{Q}_n$. Then, following [46], the homogeneous Besov sequence space $\dot{b}_\alpha^{p,q}(\mathbb{R}^n)$, where $0 < p,q \leq \infty$ and $\alpha \in \mathbb{R}$, is defined as the collection of all numerical sequences $s = \{s_Q\}_{Q\in\mathcal{Q}_n}$ satisfying (with natural interpretations when $p = \infty$, or $q = \infty$)

$$\|s\|_{\dot{b}_\alpha^{p,q}(\mathbb{R}^n)} := \left(\sum_{j\in\mathbb{Z}}\Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ \ell(Q)=2^{-j}}}[\,|Q|^{-\alpha/n-1/2+1/p}|s_Q|\,]^p\Big)^{q/p}\right)^{1/q} < +\infty. \quad (5.62)$$

Then it is clear that the same type of analysis and conclusions as in Example 10 apply to this context.

Example 12. Function spaces on spaces of homogeneous type (in the sense of Coifman and Weiss). See [34, 35] for a systematic introduction to the topic of spaces of homogeneous type. Here we only wish to mention that the definition of the tent spaces of Coifman–Meyer–Stein in $\mathbb{R}^{n+1}_+$ has a natural counterpart in the setting of spaces of homogeneous type (a detailed analysis of which is found in [5]) and that the arguments used in Example 9 make minimalistic use of the structure of the ambient Euclidean space and, as such, carry over to the setting of tent spaces in spaces of homogeneous type. In fact, a variety of other function spaces, naturally arising in the context of spaces of homogeneous type, are amenable to the scope of the results in this chapter. For example, this is the case for the discrete Triebel–Lizorkin and Besov spaces on spaces of homogeneous type, as defined in [39, 55]. We omit the details.

5.3 Abstract Capacitary Estimates

The goal here is to prove a capacitary estimate involving the action of the capacity on an infinite product. Such a result is of basic importance for the subsequent considerations in this chapter. To set the stage, we first record a technical estimate in the lemma below.

Lemma 5.7. *Suppose that* $(G, *)$ *is a given semigroup (hence,* $G^{(2)} = G \times G$*) and that* $\mathscr{C} : G \to [0, +\infty]$ *is a function for which there exists a constant* $C_0 \in [1, +\infty)$ *with the property that the following quasisubadditivity condition holds:*

$$\mathscr{C}(f * g) \le C_0 \max\big\{\mathscr{C}(f), \mathscr{C}(g)\big\} \quad \text{for all} \quad (f, g) \in G^{(2)}. \tag{5.63}$$

In this context, for each $N \in \mathbb{N}$ *and each* $f \in G$ *set*

$$f^N := \underbrace{f * \cdots * f}_{N \text{ times}} \quad \text{and} \quad c_N := \sup_{\substack{f \in G \\ \mathscr{C}(f) \neq 0, +\infty}} \left(\frac{\mathscr{C}(f^N)}{\mathscr{C}(f)}\right). \tag{5.64}$$

Then

$$c_N \le C_0^2 N^{\log_2 C_0}, \quad \forall\, N \in \mathbb{N}. \tag{5.65}$$

Proof. Fix a finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ and consider some $f \in G$ such that $\mathscr{C}(f) \in (0, +\infty)$. Then, given $N \in \mathbb{N}$, invoke (3.322) used with $a_1 = \cdots = a_N = f$ (and $\psi = \mathscr{C}$). Such a choice yields the estimate $\mathscr{C}(f^N) \le C_0^2 N^{1/\beta} \mathscr{C}(f)$, which in turn forces

$$c_N = \sup_{\substack{f \in G \\ \mathscr{C}(f) \neq 0, +\infty}} \left(\frac{\mathscr{C}(f^N)}{\mathscr{C}(f)}\right) \le C_0^2 N^{1/\beta}. \tag{5.66}$$

Optimizing this last estimate with respect to $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ then yields (5.65), completing the proof of the lemma. □

Before stating the main result in this section, we clarify one additional piece of notation. Concretely, assume that $(G, *)$ is a semigroupoid. Given a number $N \in \mathbb{N}$ and an N-tuple J of positive integers, say $J = (j_1, \dots, j_N) \in \mathbb{N}^N$, then for every family $(f_j)_{j \in J} \in G^{(N)}$ abbreviate

$$\prod_{j \in J} f_j := \prod_{i=1}^{N} f_{j_i} := f_{j_1} * f_{j_2} * \cdots * f_{j_N} \in G. \tag{5.67}$$

Since the binary operation $*$ is associative, this is unambiguously defined.

Theorem 5.8. *Let* $(S, *)$ *be a semigroup and assume that* $\preceq$ *is a partial order relation on* S *satisfying axioms* $(A1)$ *and* $(A2)$ *below:*

(A1) For every sequence $(f_i)_{i \in \mathbb{N}} \subseteq S$

$$\text{the set} \quad \left\{\prod_{i=1}^{N} f_i \,:\, N \in \mathbb{N}\right\} \text{ has a least upper bound in} \quad (S, \preceq), \tag{5.68}$$

i.e., there exists $f \in S$ *such that* $\prod_{i=1}^{N} f_i \preceq f$ *for all* $N \in \mathbb{N}$ *and such that if* $g \in S$ *satisfies* $\prod_{i=1}^{N} f_i \preceq g$ *for all* $N \in \mathbb{N}$*, then* $f \preceq g$*. Hence,* f *is uniquely determined, and one denotes*

$$\sup_{N\in\mathbb{N}} \left(\prod_{i=1}^{N} f_i \right) := f. \tag{5.69}$$

(A2) For every $N, M \in \mathbb{N}$ *with* $M \leq N$ *and for each family* $(f_j)_{1\leq j\leq N} \subseteq S$ *the following holds:*

$$\prod_{i=1}^{M} f_{j_i} \preceq \prod_{j=1}^{N} f_j \quad \text{if } (j_i)_{1\leq i\leq M} \subseteq \mathbb{N} \text{ satisfy } 1 \leq j_1 < j_2 < \cdots < j_M \leq N. \tag{5.70}$$

Hence for each family $(f_j)_{j\in\mathbb{N}} \subseteq S$*, by (A1) and (A2),*

$$\prod_{i=1}^{\infty} f_i := \sup_{N\in\mathbb{N}} \Big(\prod_{i=1}^{N} f_i \Big) \in S \quad \text{is well defined.} \tag{5.71}$$

Let $\mathscr{C} : S \to [0, +\infty]$ *be such that the following conditions are satisfied:*

- *(i) (Quasisubadditivity) There exists a constant* $C_0 \in [1, +\infty)$ *with the property that*

$$\mathscr{C}(f * g) \leq C_0 \max\{\mathscr{C}(f), \mathscr{C}(g)\}, \quad \forall\, f, g \in S. \tag{5.72}$$

- *(ii) (Weak monotonicity) Whenever* $(f_i)_{i\in\mathbb{N}} \subseteq S$ *and* $f \in S$ *are such that* $f_i \preceq f$ *for each* $i \in \mathbb{N}$ *and* $\mathscr{C}(f) < +\infty$*, it follows that* $\sup_{i\in\mathbb{N}} \mathscr{C}(f_i) < +\infty$.
- *(iii) (Weak Riesz–Fischer property) For every sequence* $(f_i)_{i\in\mathbb{N}} \subseteq S$ *with the property that* $\sup_{N\in\mathbb{N}} \mathscr{C}\Big(\prod_{i=1}^{N} f_i\Big) < +\infty$ *there holds* $\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i\Big) < +\infty$.

Then, for every finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$*, there exists* $C \in [0, +\infty)$ *such that*

$$\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i\Big) \leq 2^{\max\{1/\beta - 1, 0\}}\, C_0^2 \left\{ \sum_{i=1}^{\infty} \mathscr{C}(f_i)^{\beta} \right\}^{\frac{1}{\beta}} + C \quad \text{for each } (f_i)_{i\in\mathbb{N}} \subseteq S. \tag{5.73}$$

In general, one cannot take $C = 0$ *in* (5.73)*. However, one does have*

$$\limsup_{N\to\infty} \left(\frac{\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i^{N}\Big)}{c_N} \right) \leq 2^{\max\{1/\beta - 1, 0\}} C_0^2 \left\{ \sum_{i=1}^{\infty} \mathscr{C}(f_i)^{\beta} \right\}^{\frac{1}{\beta}}, \quad \forall\, (f_i)_{i\in\mathbb{N}} \subseteq S, \tag{5.74}$$

provided [with $(c_N)_{N\in\mathbb{N}}$ retaining the same significance as in (5.64)]

$$\lim_{N\to\infty} c_N = +\infty. \tag{5.75}$$

Proof. Let $\mathscr{C}_\#$ be the regularization of the quasisubadditive function $\mathscr{C}$ according to the procedure described in Theorem 3.28 for the semigroup $(S, *)$. That is, if

$$\alpha := \frac{1}{\log_2 C_0} \in (0, +\infty], \tag{5.76}$$

then $\mathscr{C}_\# : S \to [0, +\infty]$ is the function that associates to each $f \in S$ the number

$$\mathscr{C}_\#(f) := \inf\left\{ \Big(\sum_{i=1}^{N} \mathscr{C}(f_i)^\alpha\Big)^{\frac{1}{\alpha}} : N \in \mathbb{N},\ f_1, \dots, f_N \in S,\ f = f_1 * \cdots * f_N \right\} \tag{5.77}$$

if $\alpha < +\infty$ and, corresponding to the case when $\alpha = +\infty$,

$$\mathscr{C}_\#(f) := \inf\Big\{ \max_{1\le i\le N} \mathscr{C}(f_i) : N \in \mathbb{N},\ f_1, \dots, f_N \in S,\ f = f_1 * \cdots * f_N \Big\}. \tag{5.78}$$

Then Theorem 3.28 ensures that

$$C_0^{-2}\mathscr{C} \le \mathscr{C}_\# \le \mathscr{C} \quad \text{on } S \tag{5.79}$$

and that for each finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$

$$\mathscr{C}_\#\Big(\prod_{i=1}^{N} f_i\Big)^\beta \le \sum_{i=1}^{N} \mathscr{C}_\#(f_i)^\beta, \quad \forall\, N \in \mathbb{N}, \quad \forall\, (f_i)_{1\le i\le N} \subseteq S. \tag{5.80}$$

We next claim that, given any finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$,

$$\exists\, C \in [0, +\infty) \text{ such that } \mathscr{C}_\#\Big(\prod_{i=1}^{\infty} f_i\Big)^\beta \le \sum_{i=1}^{\infty} \mathscr{C}_\#(f_i)^\beta + C, \quad \forall\, (f_i)_{i\in\mathbb{N}} \subseteq S. \tag{5.81}$$

Seeking a contradiction, assume that there exists a finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ such that (5.81) does not hold. Then, for each $k \in \mathbb{N}$ there exists a family $(f_{i,k})_{i\in\mathbb{N}} \subseteq S$ such that

$$\mathscr{C}_\#\Big(\prod_{i=1}^{\infty} f_{i,k}\Big)^\beta > \sum_{i=1}^{\infty} \mathscr{C}_\#(f_{i,k})^\beta + k. \tag{5.82}$$

In particular, the right-hand side of (5.82) is finite and, as such,

$$\mathscr{C}_\#(f_{i,k}) < +\infty, \qquad \forall\, i \in \mathbb{N} \quad \text{and} \quad \forall\, k \in \mathbb{N}, \tag{5.83}$$

and

$$\text{for each} \quad k \in \mathbb{N} \quad \text{there exists} \quad i_k \in \mathbb{N} \quad \text{such that} \quad \sum_{i=i_k}^{\infty} \mathscr{C}_{\#}(f_{i,k})^{\beta} < \frac{1}{k^2}. \tag{5.84}$$

Then, by (5.80) and (5.82),

$$\sum_{i=1}^{i_k-1} \mathscr{C}_{\#}(f_{i,k})^{\beta} + \mathscr{C}_{\#}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big)^{\beta} \geq \mathscr{C}_{\#}\Big(\prod_{i=1}^{\infty} f_{i,k}\Big)^{\beta} > \sum_{i=1}^{\infty} \mathscr{C}_{\#}(f_{i,k})^{\beta} + k. \tag{5.85}$$

Thus, using (5.85) and (5.83) we may write

$$\mathscr{C}_{\#}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big)^{\beta} > k, \qquad \forall\, k \in \mathbb{N}. \tag{5.86}$$

Going further, for each $N \in \mathbb{N}$ define $g_N \in S$ by setting

$$g_N := \prod_{k=1}^{N}\Big(\prod_{i=i_k}^{i_k+N} f_{i,k}\Big). \tag{5.87}$$

We then obtain

$$\begin{aligned} \mathscr{C}(g_N)^{\beta} &\leq C_0^{2\beta}\mathscr{C}_{\#}(g_N)^{\beta} \leq C_0^{2\beta} \sum_{k=1}^{N} \sum_{i=i_k}^{i_k+N} \mathscr{C}_{\#}(f_{i,k})^{\beta} \\ &\leq C_0^{2\beta} \sum_{k=1}^{\infty} \sum_{i=i_k}^{i_k+N} \mathscr{C}_{\#}(f_{i,k})^{\beta} \leq C_0^{2\beta} \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty. \end{aligned} \tag{5.88}$$

Indeed, the first two inequalities follow from (5.79) and (5.80), the fourth inequality is a consequence of (5.84), and the remaining ones are obvious. Hence, from (5.88) we may conclude that

$$\sup_{N \in \mathbb{N}} \mathscr{C}(g_N) < +\infty. \tag{5.89}$$

Also, in light of the hypothesized property $(A2)$ we have $g_i \preceq g_{i+1}$ for each $i \in \mathbb{N}$. Keeping this in mind and using $(A1)$, (5.89), and the weak Riesz–Fischer property, we deduce that

$$g := \sup_{N \in \mathbb{N}} g_N \in S \quad \text{is well-defined and} \quad \mathscr{C}(g) < +\infty. \tag{5.90}$$

Now, for each $k \in \mathbb{N}$ and each $N \in \mathbb{N}$ there holds $\prod_{i=i_k}^{i_k+N} f_{i,k} \preceq g$, by $(A2)$ and (5.87). Having observed this, $(A1)$ then ensures that

$$\prod_{i=i_k}^{\infty} f_{i,k} \preceq g \qquad \text{for each} \;\; k \in \mathbb{N}. \tag{5.91}$$

Let us also note that, using (5.79) and (5.86), we have

$$\mathscr{C}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big) \geq \mathscr{C}_{\#}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big) > k^{1/\beta} \qquad \text{for each } \ k \in \mathbb{N}. \tag{5.92}$$

Hence, on the one hand, $\sup_{k\in\mathbb{N}} \mathscr{C}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big) = +\infty$. On the other hand, from (5.91), the weak monotonicity property [cf. (ii) in the statement of the theorem] and the second part of (5.90) we see that $\sup_{k\in\mathbb{N}} \mathscr{C}\Big(\prod_{i=i_k}^{\infty} f_{i,k}\Big) < +\infty$. This contradiction completes the proof of (5.81).

Using (5.81) along with (5.79) twice, we obtain that for each family $(f_i)_{i\in\mathbb{N}} \subseteq S$ and every finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ there holds

$$\begin{aligned}\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i\Big)^{\beta} &\leq C_0^{2\beta}\mathscr{C}_{\#}\Big(\prod_{i=1}^{\infty} f_i\Big)^{\beta} \leq C_0^{2\beta}\sum_{i=1}^{\infty}\mathscr{C}_{\#}(f_i)^{\beta} + C_0^{2\beta}C \\ &\leq C_0^{2\beta}\sum_{i=1}^{\infty}\mathscr{C}(f_i)^{\beta} + C_0^{2\beta}C,\end{aligned} \tag{5.93}$$

where the constant $C \in [0, +\infty)$ is as in (5.81). Using the elementary fact that for any $a, b \in [0, +\infty)$ one has $(a+b)^{1/\beta} \leq 2^{\max\{1/\beta-1,0\}}(a^{1/\beta}+b^{1/\beta})$, and redenoting $2^{\max\{1/\beta-1,0\}}C_0^2C^{1/\beta} \in [0, +\infty)$ by C, estimate (5.93) readily implies (5.73), for each finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$.

Next, we will prove that, in general, one cannot take $C = 0$ in (5.73). In fact, it is possible to envision a scenario in which (5.73) holds for an arbitrarily specified constant. To see this, fix an arbitrary number $\widetilde{C} > 0$ and consider the case in which $S := 2^{\mathbb{N}}$ (i.e., S is the set of subsets of $\mathbb{N}$), the binary operation $*$ is the union of sets (hence, $(S, \cup)$ is a semigroup), and $\mathscr{C} : 2^{\mathbb{N}} \to [0, +\infty)$ is defined by

$$\mathscr{C}(A) := \begin{cases} 0 & \text{if } A \text{ is finite} \\ \widetilde{C} & \text{if } A \text{ is infinite,} \end{cases} \qquad \forall\, A \subseteq \mathbb{N}. \tag{5.94}$$

Then it is straightforward to see that properties $(A1)$–$(A2)$ hold and that $\mathscr{C}$ is quasisubadditive, more precisely that

$$\mathscr{C}(A \cup B) \leq \max\{\mathscr{C}(A), \mathscr{C}(B)\}, \qquad \forall\, A, B \subseteq \mathbb{N}. \tag{5.95}$$

In particular, property (i) from the statement of Theorem 5.8 is satisfied with $C_0{:=}1$. Also, $\mathscr{C}$ is monotone, i.e.,

$$\mathscr{C}(A) \leq \mathscr{C}(B), \qquad \forall\, A \subseteq B \subseteq \mathbb{N}. \tag{5.96}$$

It is immediate that (5.96) implies the weak-monotonicity property (ii) from the statement of the theorem. Since $\mathscr{C}$ takes finite values on S, the weak Riesz–Fischer property (iii) is trivially satisfied. Consider next the family $(f_i)_{i\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$ defined by $f_i := \{i\}$ for every $i \in \mathbb{N}$. Then, on the one hand,

$$\prod_{i=1}^{\infty} f_i = \bigcup_{i=1}^{\infty}\{i\} = \mathbb{N} \quad \text{and} \quad \mathscr{C}\Big(\prod_{i=1}^{\infty} f_i\Big) = \widetilde{C}. \tag{5.97}$$

On the other hand, $\mathscr{C}(f_i) = 0$ for all $i \in \mathbb{N}$. This shows that the smallest value of the constant C for which the inequality (5.73) holds is precisely $\widetilde{C}$ (incidentally, since $\widetilde{C} > 0$, this shows that in general one cannot take $C = 0$ in (5.73)).

Finally, we are left with proving (5.74) under the assumption (5.75). To do so, first observe from (5.73) that for every finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ there exists $C \in [0, +\infty)$ such that for every family $(f_i)_{i\in\mathbb{N}} \subseteq S$ there holds

$$\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i^N\Big) \leq 2^{\max\{1/\beta-1,0\}} C_0^2 \Big\{\sum_{i=1}^{\infty} \mathscr{C}(f_i^N)^{\beta}\Big\}^{1/\beta} + C, \qquad \forall\, N \in \mathbb{N}. \tag{5.98}$$

To proceed, we note that

$$\mathscr{C}(f^N) \leq c_N \mathscr{C}(f), \quad \forall\, f \in S \quad \text{and} \quad \forall\, N \in \mathbb{N}. \tag{5.99}$$

Indeed, (5.99) follows from the definition of c_N when $\mathscr{C}(f) \neq 0, +\infty$. However, if $\mathscr{C}(f) = +\infty$, then there is nothing to prove, and if $\mathscr{C}(f) = 0$, then, as a consequence of (3.322), it follows that $\mathscr{C}(f^N) = 0$ for all $N \in \mathbb{N}$, so once again (5.99) is trivially satisfied. Having established (5.99), we use this to further bound the right-hand side of (5.98). As such, we obtain that

$$\mathscr{C}\Big(\prod_{i=1}^{\infty} f_i^N\Big) \leq c_N \cdot 2^{\max\{1/\beta-1,0\}} C_0^2 \Big\{\sum_{i=1}^{\infty} \mathscr{C}(f_i)^{\beta}\Big\}^{1/\beta} + C, \quad \forall\, N \in \mathbb{N}. \tag{5.100}$$

Then (5.74) readily follows by dividing both sides of the inequality in (5.100) by c_N and by taking lim sup as $N \to \infty$. This completes the proof of Theorem 5.8. □

5.4 Abstract Completeness Results

5.4.1 Linear Background

In this subsection we will formulate and prove the main completeness result in the setting of a linear vector space (with additional structure), as described below.

Theorem 5.9. *Let $(X, \preceq)$ be a partially ordered vector space,*[11] *and denote by X^+ the positive cone in $(X, \preceq)$, i.e., $X^+ := \{f \in X : 0 \preceq f\}$, where 0 is the null vector in X. Also, consider a mapping*

$$\|\cdot\| : X \longrightarrow [0, +\infty), \tag{5.101}$$

satisfying the following properties:

(1) (Quasitriangle inequality) There exists a constant $C_0 \in [1, +\infty)$ such that

$$\|f + g\| \leq C_0 \max\{\|f\|, \|g\|\}, \qquad \forall\, f, g \in X. \tag{5.102}$$

(2) (Pseudohomogeneity) There exists a function $\varphi : (0, +\infty) \to (0, +\infty)$ satisfying

$$\|\lambda f\| \leq \varphi(\lambda)\|f\|, \qquad \forall\, f \in X, \quad \forall\, \lambda \in (0, +\infty), \tag{5.103}$$

as well as

$$\sup_{\lambda>0}\big[\varphi(\lambda)\varphi(\lambda^{-1})\big] < +\infty \quad \textit{and} \quad \lim_{\lambda\to 0^+} \varphi(\lambda) = 0. \tag{5.104}$$

(3) (Nondegeneracy) For each $f \in X$ there holds

$$\|f\| = 0 \iff f = 0. \tag{5.105}$$

(4) (Weak monotonicity) If a countable family of vectors $(f_i)_{i\in\mathbb{N}} \subseteq X^+$ has an upper bound in $(X, \preceq)$, then $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$.

(5) (Weak Fatou property) Any family $(f_i)_{i\in\mathbb{N}} \subseteq X^+$ satisfying

$$f_i \preceq f_{i+1}, \quad \forall\, i \in \mathbb{N}, \quad \textit{and} \quad \sup_{i\in\mathbb{N}} \|f_i\| < +\infty \tag{5.106}$$

has a least upper bound in $(X, \preceq)$.

(6) (X^+ spans X, with control) There exist two functions $\mathcal{P}_\pm : X \to X^+$ satisfying

$$\mathcal{P}_+(f) - \mathcal{P}_-(f) = f, \qquad \forall\, f \in X, \tag{5.107}$$

and there exists a numerical sequence $(a_i)_{i\in\mathbb{N}} \subseteq (0, +\infty)$ with the property that

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[11]That is, X is a vector space and $\preceq$ is a partial order relation on X with the property that if $f, g \in X$ are such that $f \preceq g$, then $f + h \preceq g + h$ for every $h \in X$ and $\lambda f \preceq \lambda g$ for every $\lambda \in [0, +\infty)$.

$$\begin{aligned} &\textit{for every sequence } (f_i)_{i\in\mathbb{N}} \subseteq X \textit{ such that } \|f_i\| \le a_i \textit{ for each } i \in \mathbb{N} \\ &\Longrightarrow \exists\, \gamma \in \big(0, (\log_2 C_0)^{-1}\big] \textit{ finite, with } \sum_{i=1}^{\infty} \|\mathcal{P}_\pm(f_i)\|^\gamma < +\infty. \end{aligned} \tag{5.108}$$

Let $\tau_{\|\cdot\|}$ *denote the topology induced by* $\|\cdot\|$ *on* $(X,+)$ *(in the sense of Definition 5.1). Then* $(X,\tau_{\|\cdot\|})$ *is a Hausdorff, complete, metrizable, topological vector space.*[12]

Before presenting the proof of this theorem, a few comments are in order.

Remark 5.10. (i) Since X^+ is the positive cone in $(X,\preceq)$, we clearly have $X^+ \cap (-X^+) = \{0\}$. Hence, (5.107) implies that $X^+ - X^+ = X$, so X^+ spans X. Parenthetically, we note that the latter property holds whenever X is a Riesz space, i.e., a partially ordered vector space $(X,\preceq)$ where the order structure is also a lattice. Indeed, in such a scenario, for any $f \in X$ we have that $f = f^+ - f^-$ where $f^\pm := \sup\{\pm f, 0\}$.

(ii) Hypothesis (5.108) may be regarded as a degenerate continuity condition for the operators $\mathcal{P}_\pm$. It amounts to the ability of specifying a rate of decay for the "norms" of a sequence of functions that ensures the membership of the sequences of "norms" of their positive and negative parts to the classical sequence space ℓ^γ (with γ playing the role of a fine-tune parameter of geometric character, as it relates to the quasisubadditivity constant for $\|\cdot\|$). Thus, $\mathcal{P}_\pm(f_i) \to 0$ whenever $f_i \to 0$ in $\tau_{\|\cdot\|}$ fast enough.

(iii) In the context of Theorem 5.9, if the condition

$$\begin{aligned} &\exists\, C \in [0,+\infty) \text{ such that } \forall\, f \in X \;\; \exists\, f^\pm \in X^+ \\ &\text{with } f = f^+ - f^- \text{ and } \|f^\pm\| \le C\|f\| \end{aligned} \tag{5.109}$$

holds, then property (6) in the statement of Theorem 5.9 is satisfied. Indeed, in such a case, we may take $\mathcal{P}_\pm(f) := f^\pm$ for every $f \in X$, and (5.108) is satisfied for every $\gamma \in (0,+\infty)$ by choosing, e.g., $a_i := 2^{-i}$ for each $i \in \mathbb{N}$.

(iv) In the converse direction, if all hypotheses of Theorem 5.9 with the exception of (5.108) are satisfied, the topological space $(X,\tau_{\|\cdot\|})$ is complete, and the positive cone X^+ is closed in $(X,\tau_{\|\cdot\|})$, then (5.109) holds. This is proved later, in Proposition 5.16, by relying on Baire's category theorem, and shows that (5.108) is a necessary condition for the conclusion of Theorem 5.9 in the class of partially ordered vector spaces with closed positive cones.

These comments also warrant recording the following utilitarian corollary of Theorem 5.9.

[12] That is, the vector addition and multiplication by scalars are continuous functions.

Corollary 5.11. *Let $(X, \preceq)$ be a Riesz space and assume that $\|\cdot\|$ is a quasinorm on X that satisfies the weak Fatou property (cf. (5) in Theorem 5.9) and for which there exists $C \in [0, +\infty)$ such that*[13]

$$\|f\| \leq C\|g\|, \quad \forall\, f, g \in X^+ \ \textit{with}\ \ f \preceq g, \tag{5.110}$$

$$\|\sup\{f, g\}\| \leq C \max\{\|f\|, \|g\|\}, \quad \forall\, f, g \in X. \tag{5.111}$$

Then $(X, \tau_{\|\cdot\|})$ is a Hausdorff, complete, metrizable, topological vector space.

Proof. This is a direct consequence of Theorem 5.9 as well as parts (i) and (iii) of Remark 5.10. □

We proceed now to presenting the proof of Theorem 5.9.

Proof of Theorem 5.9. That $(X, \tau_{\|\cdot\|})$ is a topological vector space is a straightforward consequence of (5.102), (5.103), and the last condition in (5.104). Furthermore, the fact that the topological space $(X, \tau_{\|\cdot\|})$ is Hausdorff is an immediate consequence of property (3) and (5.102). Also, that the topology $\tau_{\|\cdot\|}$ is metrizable is justified by observing that if $\|\cdot\|_{\#}$ is the regularization of $\|\cdot\|$ (as in Theorem 3.28, in the context of the Abelian group $(X, +)$), then, having fixed a finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$, we have that the mapping $X \times X \ni (f, g) \mapsto d(f, g) := \|f - g\|_{\#}^{\beta} \in [0, +\infty)$ is a genuine distance (thanks to (3.344)) that yields the same topology on X as $\tau_{\|\cdot\|}$ (thanks to (3.343)).

Capacitary estimates of the type (5.74) play an essential role in the proof of the completeness of the space $(X, \tau_{\|\cdot\|})$, and next we will set the stage for employing Theorem 5.8 in order to establish such estimates. To this end, let $f_\infty \notin X$, and consider $S := X^+ \bigcup \{f_\infty\}$. Then define

$$f * g := \begin{cases} f + g & \text{if} \quad f, g \in X^+, \\ f_\infty & \text{otherwise}, \end{cases} \qquad \forall\, f, g \in S. \tag{5.112}$$

It is straightforward to check that $*$ is an associative binary operation on S and, as such, $(S, *)$ is a semigroup. Going further, we extend $\preceq$ to a partial order on S by agreeing that $f_\infty \preceq f_\infty$ and

$$f \preceq f_\infty, \qquad \forall\, f \in X^+, \tag{5.113}$$

and extend $\|\cdot\|$ to a $\overline{\mathbb{R}}_+$-valued function on S by setting

$$\|f_\infty\| := +\infty. \tag{5.114}$$

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[13]Based on the formula $\sup\{f, g\} = f + \sup\{g - f, 0\}$ for all $f, g \in X$, it may be readily verified that (5.111) is equivalent to the condition that $\|\,|f|\,\| \leq C\|f\|$ for each $f \in X$, where $|f| := f^+ + f^-$.

Using the fact that $\preceq$ is a partial order on S, it is straightforward to see that property $(A2)$ stipulated in Theorem 5.8 is satisfied.

Next we claim that

$$\text{any monotone sequence in } (S, \preceq) \text{ has a least upper bound,} \tag{5.115}$$

which in turn guarantees that property $(A1)$ from the hypotheses of Theorem 5.8 is also satisfied. To show (5.115), let $(f_i)_{i\in\mathbb{N}} \subseteq S$ be such that $f_i \preceq f_{i+1}$ for all $i \in \mathbb{N}$. If $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$, then in fact $(f_i)_{i\in\mathbb{N}} \subseteq X^+$ and, by property (5) of the current hypotheses, there exists a vector $f \in X^+ \subseteq S$ that is the least upper bound for $(f_i)_{i\in\mathbb{N}}$ in $(S, \preceq)$, as desired. It remains to analyze the situation in which $\sup_{i\in\mathbb{N}} \|f_i\| = +\infty$, in which case we claim that $f_\infty \in S$ is the least upper bound for the family $(f_i)_{i\in\mathbb{N}} \subseteq S$ in $(S, \preceq)$. Indeed, in light of (5.113), f_∞ is an upper bound for the family $(f_i)_{i\in\mathbb{N}}$ and, reasoning by contradiction, assume that $g \in S \setminus \{f_\infty\}$ is another upper bound for $(f_i)_{i\in\mathbb{N}}$. Then $(f_i)_{i\in\mathbb{N}} \subseteq X^+$ and, by property (4) from the statement of the theorem, $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$, which is a contradiction. This shows that, in the current case, $f_\infty \in S$ is the least upper bound for the family $(f_i)_{i\in\mathbb{N}}$ in $(S, \preceq)$. This completes the proof of (5.115).

Going further and letting $\mathscr{C} := \|\cdot\| : S \to [0, +\infty]$, it is straightforward to see that this satisfies the quasisubadditivity, weak monotonicity, and weak Riesz–Fischer property, as formulated in $(i)-(iii)$ of the statement of Theorem 5.8. In addition, it is easy to check that in this setting, the definition of the constant c_N from the second part of (5.64) becomes

$$c_N = \sup_{f\in X^+\setminus\{0\}} \left(\frac{\|Nf\|}{\|f\|}\right), \quad \forall\, N \in \mathbb{N}, \tag{5.116}$$

as $f^N = Nf$, in this context, and $f \in S$ with $\mathscr{C}(f) \neq 0, +\infty$ is equivalent with the requirement that $f \in X^+ \setminus \{0\}$. Notice that (5.103) and (5.116) give that

$$c_N \leq \varphi(N), \quad \forall\, N \in \mathbb{N}. \tag{5.117}$$

Also, using again (5.103) and (5.116) we obtain

$$\begin{aligned} \|f\| &= \|N^{-1}(Nf)\| \leq \varphi(N^{-1})\|Nf\| \\ &\leq c_N\varphi(N^{-1})\|f\|, \qquad \forall f \in S \quad \text{and} \quad \forall\, N \in \mathbb{N}. \end{aligned} \tag{5.118}$$

This in turn implies (assuming that $X \neq \{0\}$, since otherwise there is nothing to prove) that for each $N \in \mathbb{N}$ there holds $c_N \geq \varphi(N^{-1})^{-1}$. Using the second part of (5.104), this further implies

$$\lim_{N\to\infty} c_N = +\infty. \tag{5.119}$$

In particular, this formula ensures that estimate (5.74) holds, i.e., for each finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ we have

$$\limsup_{N\to\infty}\left(\frac{\|N\big(\sum_{i=1}^{\infty} f_i\big)\|}{c_N}\right) \le 2^{\max\{1/\beta-1,0\}} C_0^2 \left\{\sum_{i=1}^{\infty} \|f_i\|^{\beta}\right\}^{\frac{1}{\beta}}, \quad \forall (f_i)_{i\in\mathbb{N}} \subseteq S. \tag{5.120}$$

Next, fix an arbitrary family $(f_i)_{i\in\mathbb{N}} \subseteq S$, and, using (5.103), write

$$\varphi(N^{-1})\Big\|N\big(\sum_{i=1}^{\infty} f_i\big)\Big\| \ge \Big\|\sum_{i=1}^{\infty} f_i\Big\|, \qquad \forall\, N \in \mathbb{N}. \tag{5.121}$$

This and (5.117) imply that

$$\frac{\|N\big(\sum_{i=1}^{\infty} f_i\big)\|}{c_N} \ge \frac{\|\sum_{i=1}^{\infty} f_i\|}{\varphi(N)\varphi(N^{-1})} \ge \frac{\|\sum_{i=1}^{\infty} f_i\|}{M}, \tag{5.122}$$

where we have set [recall (5.104)]

$$M := \sup_{\lambda>0}\big[\varphi(\lambda)\varphi(\lambda^{-1})\big] \in (0, +\infty). \tag{5.123}$$

Combining inequalities (5.120) and (5.122) proves that for each finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ there holds

$$\Big\|\sum_{i=1}^{\infty} f_i\Big\| \le M C_0^2\, 2^{\max\{1/\beta-1,0\}} \left\{\sum_{i=1}^{\infty} \|f_i\|^{\beta}\right\}^{1/\beta}, \qquad \forall\, (f_i)_{i\in\mathbb{N}} \subseteq S. \tag{5.124}$$

Estimate (5.124) is the key ingredient for establishing the completeness of the space $(X, \tau_{\|\cdot\|})$, an issue to which we now turn. Our goal is to show that any Cauchy sequence in $(X, \tau_{\|\cdot\|})$ is convergent and, to this end, consider an arbitrary Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $(X, \tau_{\|\cdot\|})$. In particular, there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that (with $(a_i)_{i\in\mathbb{N}}$ as in (5.108))

$$\|f_{n_{i+1}} - f_{n_i}\| < a_i, \qquad \forall\, i \in \mathbb{N}. \tag{5.125}$$

Recall the mappings $\mathcal{P}_{\pm}$ from property (6) in the statement of the theorem, and abbreviate

$$u_i^{\pm} := \mathcal{P}_{\pm}(f_{n_{i+1}} - f_{n_i}) \in X^{+}, \quad \forall\, i \in \mathbb{N}. \tag{5.126}$$

Notice that, in light of property (6) of the hypotheses and (5.125) and (5.126), there exists a finite number $\gamma \in \big(0, (\log_2 C_0)^{-1}\big]$ such that

$$\sum_{i=1}^{\infty} \|u_i^{\pm}\|^{\gamma} < +\infty. \tag{5.127}$$

In concert with (3.324) (used with $\psi := \|\cdot\|$ and $\beta := \gamma$), the finiteness condition (5.127) further entails

$$\sup_{N\in\mathbb{N}} \Big\| \sum_{i=1}^{N} u_i^{\pm} \Big\| \leq C_0^2 \Big\{ \sum_{i=1}^{\infty} \|u_i^{\pm}\|^{\gamma} \Big\}^{1/\gamma} < +\infty. \tag{5.128}$$

Consequently, from the weak-Fatou-type property (5) of the hypothesis it follows that

$$\begin{gathered}\text{the families } \Big\{ \sum_{i=1}^{N} u_i^{+} \Big\}_{N\in\mathbb{N}} \text{ and } \Big\{ \sum_{i=1}^{N} u_i^{-} \Big\}_{N\in\mathbb{N}} \text{ have least upper bounds} \\ \text{in } (X, \preceq)\text{, which we denote by } \sum_{i=1}^{\infty} u_i^{+} \text{ and } \sum_{i=1}^{\infty} u_i^{-}\text{, respectively.}\end{gathered} \tag{5.129}$$

In the notation introduced in Theorem 5.8 we have

$$\sum_{i=1}^{\infty} u_i^{\pm} = \sup_{N\in\mathbb{N}} \big(\sum_{i=1}^{N} u_i^{\pm} \big). \tag{5.130}$$

Next we claim that

$$\sum_{i=1}^{\infty} u_i^{\pm} - \sum_{i=1}^{N} u_i^{\pm} = \sum_{i=N+1}^{\infty} u_i^{\pm}, \quad \forall\, N \in \mathbb{N}, \tag{5.131}$$

where, much as in (5.129) and (5.130), we have set

$$\sum_{i=N+1}^{\infty} u_i^{\pm} :=\sup_{\substack{k\in\mathbb{N} \\ k\geq N+1}} \Bigg(\sum_{i=N+1}^{k} u_i^{\pm} \Bigg).$$

To prove (5.131), we start by making the observation that for each $N \in \mathbb{N}$ there holds

$$\sum_{i=1}^{k} u_i^{\pm} - \sum_{i=1}^{N} u_i^{\pm} = \sum_{i=N+1}^{k} u_i^{\pm} \preceq \sum_{i=N+1}^{\infty} u_i^{\pm}, \quad \forall\, k \in \mathbb{N}, \quad k \geq N+1, \tag{5.132}$$

and, consequently,

$$\sum_{i=1}^{k} u_i^{\pm} \preceq \sum_{i=1}^{N} u_i^{\pm} + \sum_{i=N+1}^{\infty} u_i^{\pm}, \qquad \forall\, k \in \mathbb{N}, \quad k \geq N+1. \tag{5.133}$$

Keeping $N \in \mathbb{N}$ fixed and taking the supremum over $k \geq N+1$ in (5.133) leads to

$$\sum_{i=1}^{\infty} u_i^{\pm} \preceq \sum_{i=1}^{N} u_i^{\pm} + \sum_{i=N+1}^{\infty} u_i^{\pm}, \tag{5.134}$$

establishing that the left-hand side in (5.131) is less than or equal to the right-hand side in (5.131). As far as the reverse inequality is concerned, we note that for each $N \in \mathbb{N}$ and each $k \in \mathbb{N}$ such that $k \geq N+1$ there holds

$$\sum_{i=1}^{N} u_i^{\pm} + \sum_{i=N+1}^{k} u_i^{\pm} = \sum_{i=1}^{k} u_i^{\pm} \preceq \sum_{i=1}^{\infty} u_i^{\pm}, \tag{5.135}$$

and then we take the supremum over k. This completes the proof of (5.131).

Next, use (5.131) and (5.124) (applied to the exponent γ as in (5.127)) to write, for each $N \in \mathbb{N}$, that

$$\left\| \sum_{i=1}^{\infty} u_i^{\pm} - \sum_{i=1}^{N} u_i^{\pm} \right\| = \left\| \sum_{i=N+1}^{\infty} u_i^{\pm} \right\| \leq C \left\{ \sum_{i=N+1}^{\infty} \|u_i^{\pm}\|^{\gamma} \right\}^{\frac{1}{\gamma}}, \tag{5.136}$$

where we have redenoted $MC_0^2 2^{\max\{1/\gamma-1,0\}}$ by C. Then (5.127) gives $\sum\limits_{i=N+1}^{\infty} \|u_i^{\pm}\|^{\gamma} \to 0$ as $N \to \infty$ and, hence, $\left\| \sum\limits_{i=1}^{\infty} u_i^{\pm} - \sum\limits_{i=1}^{N} u_i^{\pm} \right\| \longrightarrow 0$ as $N \to \infty$. Consequently,

$$\sum_{i=1}^{N} u_i^{\pm} \longrightarrow \sum_{i=1}^{\infty} u_i^{\pm} \in X^{+} \quad \text{as} \quad N \to \infty \quad \text{in} \quad \tau_{\|\cdot\|}. \tag{5.137}$$

Next, from (5.107) and (5.126) we see that for each $i \in \mathbb{N}$ we have

$$f_{n_{i+1}} - f_{n_i} = u_i^{+} - u_i^{-}, \tag{5.138}$$

and summing up over $i \in \{1, \dots, N\}$, where $N \in \mathbb{N}$ is arbitrary, we obtain

$$f_{n_{N+1}} = f_{n_1} + \left(\sum_{i=1}^{N} u_i^{+} \right) - \left(\sum_{i=1}^{N} u_i^{-} \right), \quad \forall\, N \in \mathbb{N}. \tag{5.139}$$

Using (5.137), we see that both $\sum\limits_{i=1}^{N} u_i^{+}$ and $\sum\limits_{i=1}^{N} u_i^{-}$ converge, as $N \to \infty$, in $\tau_{\|\cdot\|}$ to limits belonging to X^{+}; and hence identity (5.139) implies that

$$\text{the sequence } \{f_{n_i}\}_{i \in \mathbb{N}} \text{ is convergent in } (X, \tau_{\|\cdot\|}). \tag{5.140}$$

Finally, since $\{f_{n_i}\}_{i\in\mathbb{N}}$ is a subsequence of the original Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$, using (5.140) it is straightforward to see that $\{f_n\}_{n\in\mathbb{N}}$ is itself convergent in $(X, \tau_{\|\cdot\|})$, as desired. This completes the proof of the fact that $(X, \tau_{\|\cdot\|})$ is complete. $\square$

In Corollary 5.15 below, we single out a significant consequence of Theorem 5.9. This is formulated in terms of functional analytic jargon that, for the convenience of the reader, is clarified below.

Definition 5.12. Given a measure space $(\Sigma, \mathfrak{M}, \mu)$, let $L^0(\Sigma, \mathfrak{M}, \mu)$ stand for the vector space of (equivalence classes of) measurable functions that are finite μ-a.e. on Σ. In this context, call a linear subspace X of $L^0(\Sigma, \mathfrak{M}, \mu)$ an `order ideal in` $L^0(\Sigma, \mathfrak{M}, \mu)$ provided

$$f \in X \text{ and } g \in L^0(\Sigma, \mathfrak{M}, \mu) \text{ with } |g| \leq |f| \ \mu\text{-a.e. on } \Sigma \Longrightarrow g \in X. \tag{5.141}$$

Hence, an order ideal X in $L^0(\Sigma, \mathfrak{M}, \mu)$ is itself a vector lattice (when equipped with the μ-a.e. pointwise inequality on Σ), and its positive cone is

$$X^+ = \big\{f \in X : f \geq 0 \quad \mu\text{-a.e. on } \Sigma\big\}. \tag{5.142}$$

Definition 5.13. Let $(\Sigma, \mathfrak{M}, \mu)$ be a measure space. A quasinorm $\|\cdot\|$ on a linear subspace X of $L^0(\Sigma, \mathfrak{M}, \mu)$ is said to be a `lattice quasinorm` on X provided

$$\|f\| \leq \|g\| \quad \text{whenever} \quad f, g \in X \quad \text{are such that} \quad |f| \leq |g| \quad \mu\text{-a.e. on } \Sigma. \tag{5.143}$$

Definition 5.14. Let $(\Sigma, \mathfrak{M}, \mu)$ be a measure space. Call $(X, \|\cdot\|)$ a `quasinormed function space` based on $(\Sigma, \mathfrak{M}, \mu)$ if X is an order ideal in $L^0(\Sigma, \mathfrak{M}, \mu)$ and $\|\cdot\|$ is a lattice quasinorm on X.

The result below is a generalization of [91, Proposition 2.35, p. 54], where the case of a quasinormed function space based on finite measure spaces has been considered.

Corollary 5.15. *Any quasinormed function space satisfying the weak Fatou property (relative to the partial order induced by the pointwise a.e. inequality) is a quasi-Banach space.*

Proof. Note that if X is a quasinormed function space and $\preceq$ is the partial order on X induced by the pointwise a.e. inequality, then $(X, \preceq)$ is a partially ordered vector space. Then the fact that X is complete is a direct consequence of Theorem 5.9, taking $\mathcal{P}_\pm(f) := \frac{1}{2}(|f| \pm f) \in X^+$ for each $f \in X$. $\square$

Here is the result mentioned in part (iv) of Remark 5.10.

Proposition 5.16. *Assume that all hypotheses of Theorem 5.9, with the exception of (5.108), are satisfied, that the topological space $(X, \tau_{\|\cdot\|})$ is complete, and that the positive cone X^+ of $(X, \preceq)$ is closed in $(X, \tau_{\|\cdot\|})$. Then the claim made in (5.109) holds.*

Proof. For each $r \in (0, +\infty)$, set $\mathscr{B}_r := \{f \in X : \|f\| < r\}$. Based on (5.102), it is then not difficult to check that equation

$$\mathscr{B}_r + \mathscr{B}_r \subseteq \mathscr{B}_{C_0 r}, \quad \forall\, r \in (0, +\infty), \tag{5.144}$$

$$\theta \in (0, C_0^{-1}) \implies \overline{\mathscr{B}_{\theta r}} \subseteq \mathscr{B}_r \subseteq \big(\mathscr{B}_{\theta^{-1} r}\big)^{\circ}, \quad \forall\, r \in (0, +\infty), \tag{5.145}$$

where, generally speaking, $\overline{A}$ and A° denote, respectively, the closure and the interior of a set $A \subseteq X$ in the topology $\tau_{\|\cdot\|}$. Moreover, from definitions and (5.103) we see that

$$\lambda \mathscr{B}_r := \{\lambda f : f \in \mathscr{B}_r\} \subseteq \mathscr{B}_{\varphi(\lambda) r}, \quad \forall\, \lambda, r \in (0, +\infty), \tag{5.146}$$

which further entails

$$\mathscr{B}_r \subseteq \lambda \mathscr{B}_{\varphi(\lambda^{-1}) r} \text{ and } \mathscr{B}_{\varphi(\lambda^{-1})^{-1} r} \subseteq \lambda \mathscr{B}_r, \qquad \forall\, \lambda, r \in (0, +\infty). \tag{5.147}$$

Next, for each $n \in \mathbb{N}$, we consider

$$X_n := X^+ \cap (n\mathscr{B}_1) - X^+ \cap (n\mathscr{B}_1) := \big\{f - g : f, g \in X^+ \cap (n\mathscr{B}_1)\big\}. \tag{5.148}$$

Since $\lim_{\lambda \to 0+} \varphi(\lambda) = 0$, it is possible to choose some $k_o \in \mathbb{N}$ with the property that $\varphi(k_o^{-1}) < C_0^{-1}$, where $C_0 \in [1, +\infty)$ is as in (5.102). In relation to the family of sets X_n, $n \in \mathbb{N}$, and the number k_o, we make the following claims:

$$X_n - X_n \subseteq X_{nk_o}, \qquad \forall\, n \in \mathbb{N}, \tag{5.149}$$

$$X = \bigcup_{i=1}^{\infty} X_{n_i}, \quad \forall\, (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N} \text{ with } \lim_{i \to \infty} n_i = \infty. \tag{5.150}$$

To justify the inclusion (5.149), note that if $f, g \in X_n$, then there exist $f_1, f_2, g_1, g_2 \in X^+ \cap (n\mathscr{B}_1)$ such that $f = f_1 - f_2$ and $g = g_1 - g_2$. Hence, (5.149) is proved upon noting that, by (5.144), (5.147), and the choice of k_o, we have

$$\begin{aligned} f - g &= (f_1 + g_2) - (f_2 + g_1) \in X^+ \cap \big(n\mathscr{B}_{C_0}\big) - X^+ \cap \big(n\mathscr{B}_{C_0}\big) \\ &\subseteq X^+ \cap \big(nk_o \mathscr{B}_{C_0 \varphi(k_o^{-1})}\big) - X^+ \cap \big(nk_o \mathscr{B}_{C_0 \varphi(k_o^{-1})}\big) \\ &\subseteq X^+ \cap \big(nk_o \mathscr{B}_1\big) - X^+ \cap \big(nk_o \mathscr{B}_1\big) = X_{nk_o}. \end{aligned} \tag{5.151}$$

As for (5.150), assume that a sequence $(n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ satisfying $\lim_{i \to \infty} n_i = \infty$ is given and that an arbitrary $f \in X$ has been fixed. Then (5.107) shows that there exist $f^{\pm} \in X^+$ such that $f = f^+ - f^-$. Pick $r \in (0, +\infty)$ with the property that $r > \max\{\|f^+\|, \|f^-\|\}$ and, keeping in mind that φ vanishes at the origin in the limit, select $i \in \mathbb{N}$ large enough so that $\varphi(n_i^{-1}) < r^{-1}$. Then, thanks to (5.147), we may write

$$f^{\pm} \in X^+ \cap \mathscr{B}_r \subseteq X^+ \cap \big(n_i \mathscr{B}_{\varphi(n_i^{-1}) r}\big) \subseteq X^+ \cap \big(n_i \mathscr{B}_1\big), \tag{5.152}$$

and (5.150) readily follows from this. Recall that we are assuming that the topological space $(X, \tau_{\|\cdot\|})$ is complete and, as we showed in the course of the proof of Theorem 5.9 [without any reference to the condition (5.108), which is now omitted], the topology $\tau_{\|\cdot\|}$ is metrizable. Based on these, (5.150), and Baire's category theorem (cf., e.g., [74, Theorem 6, p. 27]), we then deduce that

$$\exists\, n_o \in \mathbb{N} \;\text{ and }\; \exists\, \mathcal{O} \subseteq X \;\text{ nonempty and open in }\; \tau_{\|\cdot\|} \;\text{ with }\; \mathcal{O} \subseteq \overline{X_{n_o k_o}}. \tag{5.153}$$

To proceed, recall from the proof of Theorem 5.9 that $(X, \tau_{\|\cdot\|})$ is a topological vector space (again, this has been established without any reference to the condition (5.108)). Consequently, for any set $A \subseteq X$ we have $\overline{A} - \overline{A} \subseteq \overline{A - A}$. Based on this, (5.153), and (5.149), we obtain

$$\mathcal{O} - \mathcal{O} \subseteq \overline{X_{n_o k_o}} - \overline{X_{n_o k_o}} \subseteq \overline{X_{n_o k_o} - X_{n_o k_o}} \subseteq \overline{X_{n_o k_o^2}}\,. \tag{5.154}$$

On the other hand, since $\mathcal{O}$ is nonempty and open in $\tau_{\|\cdot\|}$, it follows that there exists $r_1 > 0$ such that $\mathscr{B}_{r_1} \subseteq \mathcal{O} - \mathcal{O}$. Thanks to (5.154), this forces $\mathscr{B}_{r_1} \subseteq \overline{X_{n_o k_o^2}} = n_o k_o^2\, \overline{X_1}$, where the last equality also makes use of the fact that the operator of multiplication by a fixed positive scalar is a homeomorphism of $(X, \tau_{\|\cdot\|})$. In turn, based on this and (5.147), we may conclude that if $r_o := \varphi(n_o k_o^2)^{-1} r_1 \in (0, +\infty)$, then

$$\mathscr{B}_{r_o} \subseteq \overline{X_1}\,. \tag{5.155}$$

Hence, for every $\lambda \in (0, +\infty)$, from (5.147), (5.155), and (5.146) we see that

$$\begin{aligned} \mathscr{B}_{\varphi(\lambda^{-1})^{-1} r_o} &\subseteq \lambda \mathscr{B}_{r_o} \subseteq \overline{X^+ \cap (\lambda \mathscr{B}_1) - X^+ \cap (\lambda \mathscr{B}_1)} \\ &\subseteq \overline{X^+ \cap \mathscr{B}_{\varphi(\lambda)} - X^+ \cap \mathscr{B}_{\varphi(\lambda)}}\,. \end{aligned} \tag{5.156}$$

To continue, recall $M \in (0, +\infty)$ from (5.123) and consider the number $c := M^{-1} r_o \in (0, +\infty)$. From (5.123) we then deduce that $c\,\varphi(\lambda) \leq \varphi(\lambda^{-1})^{-1} r_o$ for each $\lambda \in (0, +\infty)$. In concert with (5.156), this ultimately allows us to conclude that

$$\mathscr{B}_{c\varphi(\lambda)} \subseteq \overline{X^+ \cap \mathscr{B}_{\varphi(\lambda)} - X^+ \cap \mathscr{B}_{\varphi(\lambda)}}\,, \qquad \forall\, \lambda \in (0, +\infty). \tag{5.157}$$

Recall next that the function φ vanishes at the origin in the limit, and choose a sequence $(\lambda_i)_{i \in \mathbb{N}} \subseteq (0, +\infty)$ with the property that $\varphi(\lambda_i) < 2^{-i}$ for each $i \in \mathbb{N}$. We claim that, for each fixed $f \in \mathscr{B}_{c\varphi(\lambda_1)}$, there exist two sequences $(f_i^{\pm})_{i \in \mathbb{N}} \subseteq X^+$ with the property that

$$\Big\| f - \sum_{i=1}^{n} (f_i^+ - f_i^-) \Big\| < c\, 2^{-n-1} \;\text{ and }\; \|f_n^{\pm}\| < 2^{-n}, \quad \forall\, n \in \mathbb{N}. \tag{5.158}$$

We will prove this claim by induction on $n \in \mathbb{N}$. When $n = 1$, from (5.157) (used with $\lambda = \lambda_1$) we see that there exist $f_1^{\pm} \in X^+ \cap \mathscr{B}_{\varphi(\lambda_1)}$ such that

$$\|f - (f_1^+ - f_1^-)\| < c\,\varphi(\lambda_2). \tag{5.159}$$

In particular, $f_1^{\pm} \in X^+$, $\|f_1^{\pm}\| < 2^{-1}$, and $\|f - (f_1^+ - f_1^-)\| < c\,2^{-2}$, as desired. Going further, the fact that $f - (f_1^+ - f_1^-) \in \mathscr{B}_{c\,\varphi(\lambda_2)}$ allows us to appeal to (5.157) (used with $\lambda = \lambda_2$) to conclude that there exist $f_2^{\pm} \in X^+ \cap \mathscr{B}_{\varphi(\lambda_2)}$ such that

$$\|f - (f_1^+ - f_1^-) - (f_2^+ - f_2^-)\| < c\,\varphi(\lambda_3). \tag{5.160}$$

Hence, $f_2^{\pm} \in X^+$, $\|f_1^{\pm}\| < 2^{-2}$, and $\|f - (f_1^+ - f_1^-) - (f_2^+ - f_2^-)\| < c\,2^{-3}$, as desired. Continuing this procedure in an inductive fashion yields two sequences $(f_i^{\pm})_{i\in\mathbb{N}} \subseteq X^+$ satisfying the properties listed in (5.158).

Moving on, fix a finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$, and invoke the capacitary estimate (3.324) (with $\psi := \|\cdot\|$) to conclude that

$$\begin{aligned} \Big\|\sum_{i=N}^{N+k} f_i^{\pm}\Big\| &\leq C_0^2 \Bigg\{\sum_{i=N}^{\infty} \|f_i^{\pm}\|^{\beta}\Bigg\}^{\frac{1}{\beta}} < C_0^2 \Bigg\{\sum_{i=N}^{\infty} 2^{-i\beta}\Bigg\}^{\frac{1}{\beta}} \\ &= \frac{C_0^2\, 2^{-N}}{(1-2^{-\beta})^{1/\beta}}, \qquad \forall\, N, k \in \mathbb{N}. \end{aligned} \tag{5.161}$$

This shows that the sequences $\Big\{\sum_{i=1}^{N} f_i^{\pm}\Big\}_{N\in\mathbb{N}} \subseteq X^+$ are Cauchy in $(X, \tau_{\|\cdot\|})$. Since the latter is complete, and since we are assuming that X^+ is closed in $(X, \tau_{\|\cdot\|})$, it follows that there exist $f^{\pm} \in X^+$ with the property that

$$\sum_{i=1}^{N} f_i^{\pm} \longrightarrow f^{\pm} \quad \text{in } \ \tau_{\|\cdot\|} \ \text{ as } \ N \to \infty. \tag{5.162}$$

On the other hand, the first inequality in (5.158) gives that

$$\Big(\sum_{i=1}^{N} f_i^+\Big) - \Big(\sum_{i=1}^{N} f_i^-\Big) \longrightarrow f \quad \text{in } \ \tau_{\|\cdot\|} \ \text{ as } \ N \to \infty. \tag{5.163}$$

Together, (5.162) and (5.163) prove that $f = f^+ - f^-$ since $(X, \tau_{\|\cdot\|})$ is a Hausdorff topological vector space. Let us also note that, if $R := \frac{C_0^2}{2(1-2^{-\beta})^{1/\beta}} > 0$, then (5.161) (used with $N = 1$ and $k = N - 1$) yields $\Big\|\sum_{i=1}^{N} f_i^{\pm}\Big\| < R$ for each $N \in \mathbb{N}$. Thus,

$$\Bigg\{\sum_{i=1}^{N} f_i^{\pm}\Bigg\}_{N\in\mathbb{N}} \subseteq \mathscr{B}_R, \tag{5.164}$$

and, hence, $f^{\pm} \in \overline{\mathscr{B}_R} \subseteq \mathscr{B}_{\theta R}$ for any $\theta > C_0$ (cf. (5.145)). This shows that $\|f^{\pm}\| < \theta R$ for any $\theta > C_0$; hence, ultimately, $\|f^{\pm}\| \leq C_0 R$. In summary, the analysis so far shows that any $f \in X$ with $\|f\| \leq C_1 := c\,\varphi(\lambda_1)$ can be decomposed as $f = f^+ - f^-$ for some $f^{\pm} \in X^+$ satisfying $\|f^{\pm}\| \leq C_0 R$.

At this stage, fix an arbitrary $f \in X \setminus \{0\}$ and consider

$$\lambda_o := \sup\big\{\lambda \in (0, +\infty) : \varphi(\lambda) < C_1\|f\|^{-1}\big\} \in (0, +\infty). \tag{5.165}$$

In particular, there exists $\widetilde{\lambda} \in (\lambda_o/2, \lambda_o]$ such that $\varphi(\widetilde{\lambda}) < C_1\|f\|^{-1}$. Then

$$\|\widetilde{\lambda} f\| \leq \varphi(\widetilde{\lambda})\|f\| \leq C_1. \tag{5.166}$$

Hence $\widetilde{\lambda} f \in \mathscr{B}_{c\varphi(\lambda_1)}$, and, based on what we have proved so far, we may decompose $\widetilde{\lambda} f = g^+ - g^-$ with $g^{\pm} \in X^+$ satisfying $\|g^{\pm}\| \leq C_0 R$. From this we then conclude that $f = f^+ - f^-$, where $f^{\pm} := \widetilde{\lambda}^{-1} g^{\pm} \in X^+$ satisfy (with $M \in (0, +\infty)$ as in (5.123))

$$\begin{aligned} \|f^{\pm}\| &= \big\|2(2\widetilde{\lambda})^{-1} g^{\pm}\big\| \leq \varphi(2)\big\|(2\widetilde{\lambda})^{-1} g^{\pm}\big\| \leq \varphi(2)\varphi\big((2\widetilde{\lambda})^{-1}\big)C_0 R \\ &\leq \varphi(2)\frac{MC_0R}{\varphi(2\widetilde{\lambda})} \leq \varphi(2)\Big(\frac{MC_0R}{C_1}\Big)\|f\| \end{aligned} \tag{5.167}$$

since the fact that $2\widetilde{\lambda} > \lambda_o$ forces, in light of (5.165), $\varphi(2\widetilde{\lambda}) \geq C_1\|f\|^{-1}$. This shows that the claim made in (5.109) holds with $C := \varphi(2)MRC_0C_1^{-1} \in (0, +\infty)$. □

With Theorem 5.9 at our disposal, we now turn to the task of providing the proof of Theorem 5.3.

Proof of Theorem 5.3. We start with the claim that

$$f \in \mathcal{L} \quad \Longrightarrow \quad |f| < +\infty \quad \mu\text{-a.e. on } \Sigma, \tag{5.168}$$

which ensures that the pointwise addition and multiplication by scalars induce a vector space structure on $\mathcal{L}$. Indeed, fix $f \in \mathcal{L}$ and let

$$A := \big\{x \in \Sigma : |f(x)| = +\infty\big\} \in \mathfrak{M}. \tag{5.169}$$

In particular, for each $n \in \mathbb{N}$ we have $|f| \geq n\mathbf{1}_A \geq 0$ on Σ. This and property (4) from the statement of Theorem 5.3 further imply that $\|f\|_{\mathcal{L}} = \|\,|f|\,\| \geq C_1^{-1}\|n\mathbf{1}_A\|$. Since $\mathbf{1}_A = n^{-1}n\mathbf{1}_A$, utilizing (5.9) we have $\|n\mathbf{1}_A\| \geq \varphi(n^{-1})^{-1}\|\mathbf{1}_A\|$, and thus

$$\frac{1}{C_1\varphi(n^{-1})}\|\mathbf{1}_A\| \leq \|f\|_{\mathcal{L}} < +\infty, \qquad \forall\, n \in \mathbb{N}. \tag{5.170}$$

Passing to the limit as $n \to \infty$ in the estimate $\|\mathbf{1}_A\| \leq C_1\varphi(n^{-1})\|f\|_{\mathcal{L}}$ and using the second part in (5.10), we obtain that necessarily

$$\|\mathbf{1}_A\| = 0. \tag{5.171}$$

We now make the claim that, in general,

$$E \in \mathfrak{M} \quad \text{and} \quad \|\mathbf{1}_E\| = 0 \Longrightarrow \mu(E) = 0. \tag{5.172}$$

Indeed, if $E \in \mathfrak{M}$ is such that $\|\mathbf{1}_E\| = 0$, then, on the grounds of (5.11), we deduce that $\mathbf{1}_E = 0$, which means that there exists $B \in \mathfrak{M}$ with $\mu(B) = 0$ and such that $\mathbf{1}_E(x) = 0$ for every $x \in \Sigma \setminus B$. This in turn forces $E \subseteq B$; hence, by the properties of a feeble measure, $\mu(E) = 0$. Having established (5.172), we then conclude from (5.171) that $\mu(A) = 0$, as desired.

Consider next the partially order vector space $(\mathcal{L}, \preceq)$, where $\preceq$ is the pointwise inequality $\leq$, μ-a.e. on Σ, and notice that properties (1)–(5) in the statement of Theorem 5.9 are direct consequences of properties (1)–(5) in the current hypotheses. Next define

$$\mathcal{P}_{\pm} : \mathcal{L} \to \mathcal{L}^{+}, \qquad \mathcal{P}_{\pm}(f) := \frac{1}{2}\big(|f| \pm f\big), \qquad \forall\, f \in \mathcal{L}, \tag{5.173}$$

and since $0 \leq \mathcal{P}_{\pm} f \leq |f|$ for all $f \in \mathcal{L}$, property (4) in the current statement guarantees that $\|\mathcal{P}_{\pm} f\| \leq C_1\|f\|$ for all $f \in \mathcal{L}$. Hence, using Remark 5.10 (iii), property (6) from the statement of Theorem 5.9 also holds for the partially ordered vector space $(\mathcal{L}, \preceq)$. Finally, the remaining conclusions in (5.13) follow directly from Theorem 5.9.

Here is the proposition dealing with the equivalence between the weak Fatou property and the condition expressed in (5.15).

Proposition 5.17. *Assume that $(\Sigma, \mathfrak{M})$ is a measurable space and that μ is a feeble measure on $\mathfrak{M}$. Suppose that $\|\cdot\| : \mathcal{M}_{+}(\Sigma, \mathfrak{M}, \mu) \to [0, +\infty]$ is a function for which there exist $C_0, C_1 \in [1, +\infty)$ with the property that*

$$\|f + g\| \leq C_0 \max\{\|f\|, \|g\|\} \qquad \textit{for all} \quad f, g \in \mathcal{M}_{+}(\Sigma, \mathfrak{M}, \mu), \tag{5.174}$$

$$\|f\| \leq C_1\|g\| \textit{ if } f, g \in \mathcal{M}_{+}(\Sigma, \mathfrak{M}, \mu) \textit{ are such that } f \leq g \ \mu\textit{-a.e. on } \Sigma. \tag{5.175}$$

Then the following two conditions are equivalent:

(i) If $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_{+}(\Sigma, \mathfrak{M}, \mu)$ is a sequence of functions satisfying $f_i \leq f_{i+1}$ μ-a.e. on Σ for each $i \in \mathbb{N}$ and $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$, then $\big\|\sup_{i\in\mathbb{N}} f_i\big\| < +\infty$.

(ii) If $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ *satisfies* $\liminf_{i\to\infty} \|f_i\| < +\infty$*, then* $\left\|\liminf_{i\to\infty} f_i\right\| < +\infty$.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial (and holds irrespective of the validity of the assumptions (5.174) and (5.175)), so we will concentrate on proving that also $(i) \Rightarrow (ii)$. With this goal in mind, assume that $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ is a sequence satisfying $\liminf_{i\to\infty} \|f_i\| < +\infty$, and for each $i \in \mathbb{N}$ introduce $g_i := \inf_{j\geq i} f_j$. Then

$$g_i \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \quad g_i \leq g_{i+1} \ \mu\text{-a.e. on } \Sigma, \quad g_i \leq f_i \ \mu\text{-a.e. on } \Sigma \quad \forall\, i \in \mathbb{N},$$

$$\text{and} \quad \sup_{i\in\mathbb{N}} g_i = \liminf_{i\to\infty} f_i. \tag{5.176}$$

To proceed, consider the semigroup $(S, *)$, where $S := \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$, and $f * g := f + g$ for every $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$. Then, taking $\preceq$ to be the partial order relation on S induced by the pointwise μ-a.e. inequality between functions, we define the compatibility condition (3.369) obviously holds. In addition, if we define the function $\psi : S \to [0, +\infty]$ by setting $\psi(f) := \|f\|$ for each $f \in S$, then properties (5.174) and (5.175) ensure that the quasisubadditivity and quasimonotonicity conditions for ψ stated in (3.370) and (3.371) are satisfied. Consequently, if $\psi_\star$ stands for the regularization of ψ considered in Theorem 3.38 relative to the current algebraic setting, then we have (cf. (3.375)–(3.377))

$$C_1^{-1} C_0^{-2} \|f\| \leq \psi_\star(f) \leq \|f\| \quad \text{for each} \quad f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \tag{5.177}$$

$$\psi_\star(f) \leq \psi_\star(g) \text{ for all } f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \text{ so that } f \leq g \ \mu\text{-a.e. on } \Sigma. \tag{5.178}$$

Moving on, based on (5.176)–(5.178) and assumptions, we estimate

$$\begin{aligned}\sup_{i\in\mathbb{N}} \|g_i\| &\leq C_1 C_0^2 \sup_{i\in\mathbb{N}} \psi_\star(g_i) = C_1 C_0^2 \liminf_{i\to\infty} \psi_\star(g_i) \leq C_1 C_0^2 \liminf_{i\to\infty} \psi_\star(f_i) \\ &\leq C_1 C_0^2 \liminf_{i\to\infty} \|f_i\| < +\infty. \end{aligned} \tag{5.179}$$

In turn, (5.179), (5.176), and condition (i) in the statement of the proposition imply that

$$\left\|\liminf_{i\to\infty} f_i\right\| = \left\|\sup_{i\in\mathbb{N}} g_i\right\| < +\infty, \tag{5.180}$$

as desired. □

5.4.2 Boolean Algebra Background

The capacitary estimates established in Sect. 5.3 are also useful for proving the completeness of certain classes of quasimetric spaces that are lacking a linear space structure. Indeed, as we will see in Theorem 5.18 below, this is the case for certain classes of Boolean algebras equipped with capacities compatible with a (sub-) lattice structure. Formulating this result in a precise manner requires a number of preliminaries, which we will deal with first.

Recall that a `Boolean algebra` $\mathscr{X}$ is a distributive lattice[14] $(\mathscr{X}, \preceq, \vee, \wedge)$ with (distinct) unit and zero[15] denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively, whose every element $A \in \mathscr{X}$ has a complement,[16] denoted by A^{c}. In such a setting, for each $n \in \mathbb{N}$ we set

$$\bigvee_{i=1}^{n} A_i := \sup\{A_1, \dots, A_n\}, \quad \bigwedge_{i=1}^{n} A_i := \inf\{A_1, \dots, A_n\}, \ \forall \{A_i\}_{1 \le i \le n} \subseteq \mathscr{X}. \tag{5.181}$$

Going further, call a Boolean algebra $(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\mathrm{c}})$ `sigma-complete` provided

$$\bigvee_{i=1}^{\infty} A_i := \sup\{A_i : i \in \mathbb{N}\} \quad \text{and} \quad \bigwedge_{i=1}^{\infty} A_i := \inf\{A_i : i \in \mathbb{N}\} \tag{5.182}$$

exist (in $(\mathscr{X}, \preceq)$) for every sequence $\{A_i\}_{i \in \mathbb{N}}$ of elements in $\mathscr{X}$. Next, given a Boolean algebra $(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\mathrm{c}})$, define the symmetric difference by the formula

$$A \triangle B := (A \wedge B^{\mathrm{c}}) \vee (A^{\mathrm{c}} \wedge B), \qquad \forall\, A, B \in \mathscr{X}, \tag{5.183}$$

and note that for every $A, B \in \mathscr{X}$ we have

$$A \triangle B = B \triangle A, \quad A \triangle B = \mathbf{0} \iff A = B, \quad \text{and} \quad A \preceq B \vee (A \triangle B), \tag{5.184}$$

$$B \preceq A \ \text{ and } \ A \wedge B^{\mathrm{c}} = \mathbf{0} \implies A = B \tag{5.185}$$

(cf., e.g., the discussion in [126, p. 11 and p. 23]). In fact,

$$\begin{gathered} (\mathscr{X}, \triangle) \text{ is an Abelian group, with neutral element } \mathbf{0}, \\ \text{and such that each element is its own inverse.} \end{gathered} \tag{5.186}$$

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[14]A partially ordered set $(\mathscr{X}, \preceq)$ is said to be a `lattice` if, for any elements $A, B \in X$, the set $\{A, B\}$ (we agree here that $\{A, B\} := \{A\}$ if $A = B$) has a least upper bound in $(\mathscr{X}, \preceq)$, denoted $A \vee B$, as well as a greatest lower bound in $(\mathscr{X}, \preceq)$, denoted $A \wedge B$. In turn, a lattice $(\mathscr{X}, \preceq, \vee, \wedge)$ is said to be `distributive` provided $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$ for any $A, B, C \in \mathscr{X}$.

[15]The unit and zero in a Boolean algebra are, respectively, the greatest and least elements in the partially ordered set $(\mathscr{X}, \preceq)$ (assumed to exist).

[16]The complement A^{c} of $A \in \mathscr{X}$ is (uniquely) characterized by the conditions $A \vee A^{\mathrm{c}} = \mathbf{1}$ and $A \wedge A^{\mathrm{c}} = \mathbf{0}$.

Also, assuming that the Boolean algebra in question is sigma-complete, define

$$\limsup_{n\to\infty} A_n := \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} A_m \quad \text{and} \quad \liminf_{n\to\infty} A_n := \bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} A_m \tag{5.187}$$

for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements in $\mathscr{X}$. Then it may be verified (cf., e.g., [126, Lemma 3, p. 191]) that

$$\Big(\limsup_{n\to\infty} A_n\Big) \wedge \Big(\liminf_{n\to\infty} A_n\Big)^{c} = \limsup_{n\to\infty}\big(A_{n+1} \triangle A_n\big) \tag{5.188}$$

for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements in the (sigma-complete) Boolean algebra $\mathscr{X}$. In a sigma-complete Boolean algebra $\mathscr{X}$, a sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to be `order-convergent` to $A \in \mathscr{X}$ provided[17]

$$\limsup_{n\to\infty} A_n = A = \liminf_{n\to\infty} A_n. \tag{5.189}$$

Finally, we note that in any sigma-complete Boolean algebra $\mathscr{X}$ one has (cf. [126, Theorem 3 and its corollary on p. 14])

$$A \wedge \Big(\bigvee_{n=1}^{\infty} A_n\Big) = \bigvee_{n=1}^{\infty} (A \wedge A_n) \quad \text{and} \quad A \vee \Big(\bigwedge_{n=1}^{\infty} A_n\Big) = \bigwedge_{n=1}^{\infty} (A \vee A_n) \tag{5.190}$$

for any $A \in \mathscr{X}$ and any sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{X}$. More details, as well as a multitude of examples of Boolean algebras, may be found in, e.g., [126, Sect. 2.4, pp. 15–22].

After this preamble, we are ready to formulate and prove the completeness result alluded to at the beginning of this subsection. Elements of the proof of Theorem 5.18 will also play an essential role in the proof of Theorem 5.4 given at the end of this subsection.

Theorem 5.18. *Suppose that* $(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{c})$ *is a sigma-complete Boolean algebra, and assume that* $\mathscr{C} : \mathscr{X} \to [0, +\infty]$ *is a mapping satisfying the following properties:*

(1) (Quasisubadditivity) There exists $C_0 \in [1, +\infty)$ *such that*

$$\mathscr{C}(A \vee B) \le C_0 \max\{\mathscr{C}(A), \mathscr{C}(B)\}, \qquad \forall\, A, B \in \mathscr{X}. \tag{5.191}$$

[17] This is not the typical definition of order convergence, though it is equivalent to it; see [126, Theorem 3, p. 187] for more details.

(2) (Quasimonotonicity) There exists $C_1 \in [1, +\infty)$ *such that*

$$\mathscr{C}(A) \leq C_1 \mathscr{C}(B), \qquad \forall\, A, B \in \mathscr{X} \quad \text{such that} \quad A \preceq B. \tag{5.192}$$

(3) (Quasi-Fatou property) There exists a constant $C_2 \in [0, +\infty)$ *such that for any sequence* $(A_n)_{n \in \mathbb{N}} \subseteq \mathscr{X}$ *such that* $A_n \preceq A_{n+1}$ *for every* $n \in \mathbb{N}$ *there holds*

$$\mathscr{C}\Big(\bigvee_{n=1}^{\infty} A_n\Big) \leq C_2 \limsup_{n \to \infty} \mathscr{C}(A_n). \tag{5.193}$$

Consider the relation

$$A \sim B \overset{\text{def}}{\iff} \mathscr{C}(A \triangle B) = 0, \qquad \forall\, A, B \in \mathscr{X}. \tag{5.194}$$

Then $\sim$ *is an equivalence relation on* $\mathscr{X}$, *and for every* $A, B \in \mathscr{X}$ *there holds*

$$(C_0 C_1)^{-2} \sup_{\substack{\mathscr{X} \ni A' \sim A \\ \mathscr{X} \ni B' \sim B}} \mathscr{C}(A' \triangle B') \leq \mathscr{C}(A \triangle B) \leq (C_0 C_1)^2 \inf_{\substack{\mathscr{X} \ni A' \sim A \\ \mathscr{X} \ni B' \sim B}} \mathscr{C}(A' \triangle B'). \tag{5.195}$$

Also, if

$$\mathscr{C}^{-1}(\{0\}) = \{\mathbf{0}\}, \tag{5.196}$$

and if one defines

$$\mathscr{X}_{\text{fin}} := \big\{A \in \mathscr{X} : \mathscr{C}(A) < +\infty\big\} \tag{5.197}$$

and considers the function $\rho : \mathscr{X} \times \mathscr{X} \to [0, +\infty]$ *given by the formula*

$$\rho\big(A, B\big) := \mathscr{C}(A \triangle B), \quad \forall\, A, B \in \mathscr{X}, \tag{5.198}$$

then $\big(\mathscr{X}_{\text{fin}}, \rho\big)$ *is a complete quasimetric space. In addition, any convergent sequence in* $\big(\mathscr{X}_{\text{fin}}, \rho\big)$ *contains a subsequence that is order-convergent to the limit of the original sequence in* $(\mathscr{X}_{\text{fin}}, \rho)$. *While, in general,* $\mathscr{C} : \big(\mathscr{X}_{\text{fin}}, \tau_\rho\big) \to [0, +\infty)$ *is not continuous, where* τ_ρ *denotes the topology canonically induced by the quasidistance* ρ *on* $\mathscr{X}_{\text{fin}}$, *one nonetheless has*

$$\begin{gathered} A \in \mathscr{X},\ (A_n)_{n \in \mathbb{N}} \subseteq \mathscr{X},\ A_n \to A \ \text{ in } \ \tau_\rho \\ \Longrightarrow (C_0 C_1)^{-2} \limsup_{n \to \infty} \mathscr{C}(A_n) \leq \mathscr{C}(A) \leq (C_0 C_1)^2 \liminf_{n \to \infty} \mathscr{C}(A_n). \end{gathered} \tag{5.199}$$

Finally, if in place of (1) *and* (2) *in the first part of the statement one assumes that*

$$\mathscr{C}(A \vee B) \leq \mathscr{C}(A) + \mathscr{C}(B), \qquad \forall\, A, B \in \mathscr{X}, \tag{5.200}$$

$$\mathscr{C}(A) \leq \mathscr{C}(B), \quad \forall\, A, B \in \mathscr{X} \quad \text{such that} \quad A \preceq B, \tag{5.201}$$

then, retaining (5.196) *and the quasi-Fatou property,* $(\mathscr{X}_{\text{fin}}, \rho)$ *becomes a complete metric space, and* (5.195) *improves to*

$$\mathscr{C}(A \triangle B) = \mathscr{C}(A' \triangle B'), \quad \forall\, A, B, A', B' \in \mathscr{X} \quad \textit{with} \ \ A' \sim A, \ \ B' \sim B. \tag{5.202}$$

Proof. We divide the proof into a number of steps.

`Step I:` *Formula* (5.194) *defines an equivalence relation on* $\mathscr{X}$. To justify this claim, note that if $A, B, C \in \mathscr{X}$, then, by (5.186) and (5.183),

$$A \triangle B = (A \triangle C) \triangle (C \triangle B) \preceq (A \triangle C) \vee (C \triangle B). \tag{5.203}$$

Hence, by (5.203), (5.192), and (5.191),

$$\mathscr{C}(A \triangle B) \leq C_0 C_1 \max\big\{\mathscr{C}(A \triangle C), \mathscr{C}(C \triangle B)\big\}, \quad \forall\, A, B, C \in \mathscr{X}. \tag{5.204}$$

In particular, this shows that if $A, B, C \in \mathscr{X}$ are such that $A \sim C$ and $C \sim B$, then $A \sim B$, so that the relation (5.194) is transitive. By (5.184), this relation is also reflexive and symmetric, concluding the discussion in Step I.

`Step II:` *For each* $A, B \in \mathscr{X}$ *there holds*

$$\sup_{\substack{\mathscr{X} \ni A' \sim A \\ \mathscr{X} \ni B' \sim B}} \mathscr{C}(A' \triangle B') \leq (C_0 C_1)^2 \mathscr{C}(A \triangle B). \tag{5.205}$$

Since for any $A, B \in \mathscr{X}$ we have $A \triangle B \preceq A \vee B$, from (5.191) and (5.192) we deduce that

$$\mathscr{C}(A \triangle B) \leq C_0 C_1 \max\{\mathscr{C}(A), \mathscr{C}(B)\}, \quad \forall\, A, B \in \mathscr{X}. \tag{5.206}$$

Hence, if $\beta \in \big(0, (\log_2 C_0 + \log_2 C_1)^{-1}\big]$ is a fixed finite number and $\mathscr{C}_\#$ is the regularization of $\mathscr{C}$ as described in Theorem 3.28, relative to the Abelian group $(\mathscr{X}, \triangle)$, then

$$(C_0 C_1)^{-2} \mathscr{C} \leq \mathscr{C}_\# \leq \mathscr{C} \quad \text{on} \ \ \mathscr{X}, \tag{5.207}$$

and $\big(\mathscr{C}_\#\big)^\beta$ is subadditive on $(\mathscr{X}, \triangle)$, i.e.,

$$\mathscr{C}_\#(A \triangle B)^\beta \leq \mathscr{C}_\#(A)^\beta + \mathscr{C}_\#(B)^\beta, \quad \forall\, A, B \in \mathscr{X}. \tag{5.208}$$

In turn, if $A, B, A' \in \mathscr{X}$ satisfy $A' \sim A$, then it follows from (5.208) and the equality in (5.203) (used with $C := A'$) that

$$\mathscr{C}_\#(A \triangle B)^\beta \leq \mathscr{C}_\#(A \triangle A')^\beta + \mathscr{C}_\#(B \triangle A')^\beta. \tag{5.209}$$

On the other hand, $0 \le \mathscr{C}_\#(A \triangle A') \le \mathscr{C}(A \triangle A') = 0$, thanks to (5.207), (5.194), and assumptions. In combination with (5.209), this shows that $\mathscr{C}_\#(A \triangle B) \le \mathscr{C}_\#(B \triangle A')$. Reversing the roles of A and A' we therefore obtain $\mathscr{C}_\#(A \triangle B) = \mathscr{C}_\#(B \triangle A')$, hence, ultimately,

$$\mathscr{C}_\#(A \triangle B) = \mathscr{C}_\#(A' \triangle B'), \quad \forall\, A, B, A', B' \in \mathscr{X} \ \text{ with } \ A' \sim A, \ \ B' \sim B. \tag{5.210}$$

To proceed, fix $A, B \in \mathscr{X}$ and assume that $A', B' \in \mathscr{X}$ are such that $A' \sim A$ and $B' \sim B$. Based on (5.210) and (5.207), we may then estimate

$$\begin{aligned} \mathscr{C}(A' \triangle B') &\le (C_0 C_1)^2\, \mathscr{C}_\#(A' \triangle B') = (C_0 C_1)^2\, \mathscr{C}_\#(A \triangle B) \\ &\le (C_0 C_1)^2\, \mathscr{C}(A \triangle B). \end{aligned} \tag{5.211}$$

Taking the supremum over all $A', B' \in \mathscr{X}$ such that $A' \sim A$ and $B' \sim B$ then yields (5.205).

`Step III:` *For each $A, B \in \mathscr{X}$ there holds*

$$(C_0 C_1)^{-2}\, \mathscr{C}(A \triangle B) \le \inf_{\substack{\mathscr{X} \ni A' \sim A \\ \mathscr{X} \ni B' \sim B}} \mathscr{C}(A' \triangle B'). \tag{5.212}$$

This is established using ideas similar to those in the proof of (5.205). In concert, (5.205) and (5.212) give (5.195).

`Step IV:` *The function ρ is a quasidistance on the set $\mathscr{X}_{\rm fin}$*. To begin with, we claim that the function $\rho : \mathscr{X}_{\rm fin} \times \mathscr{X}_{\rm fin} \to [0, +\infty)$ is well defined. To see this, note that if $A, B \in \mathscr{X}_{\rm fin}$, then we have $A \triangle B \in \mathscr{X}_{\rm fin}$, thanks to (5.191), (5.192), and the fact that $A \triangle B \preceq A \vee B$. Hence, $\rho(A, B) = \mathscr{C}(A \triangle B) < +\infty$. Next, for each $A, B \in \mathscr{X}$ we have $\rho(A, B) = 0$ if and only if $A = B$, by virtue of (5.198), (5.196), and (5.184). Also, $\rho(A, B) = \rho(B, A)$ for each $A, B \in \mathscr{X}$, thanks to (5.198) and (5.184). Finally, (5.204) and (5.198) yield

$$\rho(A, B) \le C_0 C_1 \max\big\{\rho(A, C), \rho(C, B)\big\}, \quad \forall\, A, B, C \in \mathscr{X}, \tag{5.213}$$

and the desired conclusion follows. In particular, $(\mathscr{X}_{\rm fin}, \rho)$ is a quasimetric space.

`Step V:` *If $(B_n)_{n \in \mathbb{N}} \subseteq \mathscr{X}$ is an arbitrary sequence, then*

$$\Big(\limsup_{m \to \infty} B_m\Big) \triangle B_n \preceq \bigvee_{m=n}^{\infty} (B_{m+1} \triangle B_m), \quad \forall\, n \in \mathbb{N}. \tag{5.214}$$

To prove this, consider $(B_n)_{n \in \mathbb{N}} \subseteq \mathscr{X}$, and abbreviate

$$B := \limsup_{n \to \infty} B_n = \bigwedge_{n=1}^{\infty} \Big(\bigvee_{m=n}^{\infty} B_m \Big). \tag{5.215}$$

We will actually show that

$$B \wedge B_n^{\mathrm{c}} \preceq \bigvee_{m=n}^{\infty} \big(B_{m+1} \wedge B_m^{\mathrm{c}}\big), \quad \forall\, n \in \mathbb{N}, \tag{5.216}$$

$$\mathscr{B}_k := \bigwedge_{m=k}^{\infty} B_m \preceq B, \qquad \forall\, k \in \mathbb{N}, \tag{5.217}$$

$$B_n \wedge B^{\mathrm{c}} \preceq \bigvee_{m=n}^{\infty} \big(B_m \wedge B_{m+1}^{\mathrm{c}}\big), \quad \forall\, n \in \mathbb{N}, \tag{5.218}$$

from which (5.214) readily follows. To get started, fix $n \in \mathbb{N}$ and note that

$$B \preceq \bigvee_{m=n}^{\infty} B_m \quad \text{and} \quad \bigvee_{m=n}^{\infty} B_m = \Big(\bigvee_{m=n}^{\infty} (B_{m+1} \wedge B_m^{\mathrm{c}})\Big) \vee B_n. \tag{5.219}$$

Indeed, the first inequality is a direct consequence of (5.215). Also, since we have $B_{m+1} \wedge B_m^{\mathrm{c}} \preceq B_{m+1}$ for every $m \in \mathbb{N}$, it follows that $\Big(\bigvee_{m=n}^{\infty} (B_{m+1} \wedge B_m^{\mathrm{c}})\Big) \vee B_n \preceq \bigvee_{m=n}^{\infty} B_m$. To prove the opposite inequality, it suffices to show that

$$\bigvee_{m=n}^{N} B_m \preceq \Big(\bigvee_{m=n}^{N} (B_{m+1} \wedge B_m^{\mathrm{c}})\Big) \vee B_n, \ \text{ for every } N \in \mathbb{N} \text{ satisfying } N \geq n. \tag{5.220}$$

In turn, inequality (5.220) follows by induction on N upon observing that for every $m \in \mathbb{N}$ we have $B_{m+1} \preceq B_m \vee B_{m+1} = B_m \vee (B_{m+1} \wedge B_m^{\mathrm{c}})$. Moving on, based on (5.219), we write

$$B \wedge B_n^{\mathrm{c}} \preceq \Bigg[\Big(\bigvee_{m=n}^{\infty} (B_{m+1} \wedge B_m^{\mathrm{c}})\Big) \vee B_n\Bigg] \wedge B_n^{\mathrm{c}} \preceq \bigvee_{m=n}^{\infty} (B_{m+1} \wedge B_m^{\mathrm{c}}), \quad \forall\, n \in \mathbb{N}, \tag{5.221}$$

which completes the proof of (5.216).

Turning our attention to proving (5.217), fix $k \in \mathbb{N}$, $k \geq 2$, and note that, since $\mathscr{B}_k \preceq B_k$, we have

$$\mathscr{B}_k \preceq \bigvee_{m=n}^{\infty} B_m, \qquad \forall\, n \in \{1, \dots, k-1\}. \tag{5.222}$$

Also, since for any $n \in \mathbb{N}$ we have $B_n \preceq \bigvee_{m=n}^{\infty} B_m$, we deduce that

$$\mathscr{B}_k = \bigwedge_{n=k}^{\infty} B_n \preceq \bigwedge_{n=k}^{\infty} \Big(\bigvee_{m=n}^{\infty} B_m\Big). \tag{5.223}$$

Hence, whenever $k \in \mathbb{N}$, $k \geq 2$, the inequality in (5.217) follows from (5.222), (5.223), and (5.215). If $k = 1$, then (5.217) is an immediate consequence of inequality (5.223), which holds when $k = 1$ as well. Thus, (5.217) is now proved. As for (5.218), fix again $n \in \mathbb{N}$, and start with the observation that by applying the second identity in (5.219) with B_m replaced by $B_n \wedge B_m^{\mathrm{c}}$ we obtain

$$\bigvee_{m=n+1}^{\infty} (B_n \wedge B_m^{\mathrm{c}}) = \bigvee_{m=n}^{\infty} \big(B_n \wedge (B_m \wedge B_{m+1}^{\mathrm{c}})\big) = B_n \wedge \Big(\bigvee_{m=n}^{\infty} (B_m \wedge B_{m+1}^{\mathrm{c}})\Big), \tag{5.224}$$

where the last equality is easily established (cf., e.g., [126, Theorem 3, p. 13]). Hence,

$$(5.218) \iff B_n \wedge B^{\mathrm{c}} \preceq \bigvee_{m=n+1}^{\infty} (B_n \wedge B_m^{\mathrm{c}}) \iff \Bigg[\bigvee_{m=n+1}^{\infty} (B_n \wedge B_m^{\mathrm{c}})\Bigg]^{\mathrm{c}} \preceq [B_n \wedge B^{\mathrm{c}}]^{\mathrm{c}}. \tag{5.225}$$

However, using De Morgan's laws – cf., e.g., [111, formula (3), p. 55] – and the definition of $\mathscr{B}_{n+1}$ from (5.217) we have

$$\Bigg[\bigvee_{m=n+1}^{\infty} (B_n \wedge B_m^{\mathrm{c}})\Bigg]^{\mathrm{c}} = \bigwedge_{m=n+1}^{\infty} (B_n^{\mathrm{c}} \vee B_m) = B_n^{\mathrm{c}} \vee \Big(\bigwedge_{m=n+1}^{\infty} B_m\Big) = B_n^{\mathrm{c}} \vee \mathscr{B}_{n+1}, \tag{5.226}$$

and $[B_n \wedge B^{\mathrm{c}}]^{\mathrm{c}} = B_n^{\mathrm{c}} \vee B$. When combined with (5.225) and (5.226), this gives that (5.218) is equivalent to showing that $B_n^{\mathrm{c}} \vee \mathscr{B}_{n+1} \preceq B_n^{\mathrm{c}} \vee B$, which follows immediately from inequalities (5.217). This completes the proof of (5.218).

`Step VI:` *Suppose that $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ is a fixed finite number. Then for every sequence $\{W_n\}_{n\in\mathbb{N}}$ in $\mathscr{X}$ there holds*

$$\sum_{n=1}^{\infty} \mathscr{C}(W_n)^{\beta} < +\infty \implies \mathscr{C}\Big(\limsup_{n\to\infty} W_n\Big) = 0. \tag{5.227}$$

To justify this claim, fix a sequence $\{W_n\}_{n\in\mathbb{N}} \subseteq \mathscr{X}$, and for each $n, k \in \mathbb{N}$ introduce

$$A_{n,k} := \bigvee_{i=n}^{n+k} W_i \in \mathscr{X}. \tag{5.228}$$

Then

$$A_{n,k} \preceq A_{n,k+1} \quad \forall\, n, k \in \mathbb{N}, \quad \text{and} \quad \bigvee_{k=1}^{\infty} A_{n,k} = \bigvee_{i=n}^{\infty} W_i \quad \forall\, n \in \mathbb{N}. \tag{5.229}$$

Therefore, for each fixed $n \in \mathbb{N}$

$$\begin{aligned}\mathscr{C}\Big(\bigvee_{i=n}^{\infty} W_i\Big) &= \mathscr{C}\Big(\bigvee_{k=1}^{\infty} A_{n,k}\Big) \le C_2 \limsup_{k\to\infty} \mathscr{C}(A_{n,k}) \\ &\le C_2 C_0^2 \Big\{\sum_{i=n}^{\infty} \mathscr{C}(W_i)^{\beta}\Big\}^{1/\beta},\end{aligned} \tag{5.230}$$

by the quasi-Fatou property (cf. (5.193)) and the capacitary estimate (3.324) (specialized to our current setting). Consequently, for each $n \in \mathbb{N}$

$$\begin{aligned}0 \le \mathscr{C}\Big(\limsup_{m\to\infty} W_m\Big) &= \mathscr{C}\Big(\bigwedge_{n=1}^{\infty}\bigvee_{i=n}^{\infty} W_i\Big) \\ &\le C_1 \mathscr{C}\Big(\bigvee_{i=n}^{\infty} W_i\Big) \le C_0^2 C_1 C_2 \Big\{\sum_{i=n}^{\infty} \mathscr{C}(W_i)^{\beta}\Big\}^{1/\beta},\end{aligned} \tag{5.231}$$

by (5.187), the quasimonotonicity of $\mathscr{C}$, and (5.230). From this, (5.227) readily follows.

Step VII: *Every Cauchy sequence in $(\mathscr{X}_{\text{fin}}, \rho)$ contains an order-convergent subsequence to an element in $\mathscr{X}_{\text{fin}}$.* Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{X}_{\text{fin}}, \rho)$. Then there exists a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that

$$\rho(A_{n_{k+1}}, A_{n_k}) < 2^{-k}, \qquad \forall\, k \in \mathbb{N}. \tag{5.232}$$

Fix a finite number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$, and note that this implies (cf. (5.198))

$$\sum_{k=1}^{\infty} \mathscr{C}\big(A_{n_{k+1}} \triangle A_{n_k}\big)^{\beta} < \sum_{k=1}^{\infty} 2^{-\beta k} < +\infty. \tag{5.233}$$

With this in hand, (5.227) yields $\mathscr{C}\Big(\limsup_{n\to\infty}\big(A_{n_{k+1}}\triangle A_{n_k}\big)\Big)=0$, hence

$$\limsup_{n\to\infty}\big(A_{n_{k+1}}\triangle A_{n_k}\big)=\mathbf{0}, \tag{5.234}$$

by (5.196). In turn, (5.234) and (5.188) entail

$$\Big(\limsup_{n\to\infty}A_{n_k}\Big)\wedge\Big(\liminf_{n\to\infty}A_{n_k}\Big)^{\mathrm{c}}=\mathbf{0}. \tag{5.235}$$

Now, (5.235) and (5.185) imply

$$\limsup_{n\to\infty}A_{n_k}=\liminf_{n\to\infty}A_{n_k}=:A\in\mathscr{X}, \tag{5.236}$$

and hence $\{A_{n_k}\}_{k\in\mathbb{N}}$ is order-convergent to A. From estimate (5.214) we also deduce, in a manner similar to the way in which (5.230) was derived, that

$$\begin{aligned}\mathscr{C}(A\triangle A_{n_1})&\leq C_1\mathscr{C}\Big(\bigvee_{k=1}^{\infty}\big(A_{n_{k+1}}\triangle A_{n_k}\big)\Big)\\&\leq C_1C_2\limsup_{N\to\infty}\mathscr{C}\Big(\bigvee_{k=1}^{N}\big(A_{n_{k+1}}\triangle A_{n_k}\big)\Big)\\&\leq C_1C_2C_0^2\Big\{\sum_{k=1}^{\infty}\mathscr{C}\big(A_{n_{k+1}}\triangle A_{n_k}\big)^{\beta}\Big\}^{1/\beta}\\&\leq C_1C_2C_0^2\Big\{\sum_{k=1}^{\infty}2^{-\beta k}\Big\}^{1/\beta}<+\infty.\end{aligned} \tag{5.237}$$

Hence, from the last formula in (5.184) and the quasimonotonicity of $\mathscr{C}$ we obtain

$$\begin{aligned}\mathscr{C}(A)&\leq C_1\mathscr{C}\big(A_{n_1}\vee(A\triangle A_{n_1})\big)\\&\leq C_0C_1\max\Big\{\mathscr{C}(A_{n_1}),\mathscr{C}\big(A\triangle A_{n_1}\big)\Big\}<+\infty,\end{aligned} \tag{5.238}$$

thanks to (5.237) and the fact that $A_{n_1}\in\mathscr{X}_{\rm fin}$. Thus, $A\in\mathscr{X}_{\rm fin}$, as desired.

`Step VIII:` *Every Cauchy sequence in* $\big(\mathscr{X}_{\rm fin},\rho\big)$ *contains a convergent subsequence to an element in* $\mathscr{X}_{\rm fin}$. Assume that $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathscr{X}_{\rm fin},\rho)$ and consider a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that (5.232) is satisfied. Also, set

$$A:=\limsup_{k\to\infty}A_{n_k}=\bigwedge_{k=1}^{\infty}\Big(\bigvee_{i=k}^{\infty}A_{n_i}\Big). \tag{5.239}$$

Then, in a similar fashion to (5.236)–(5.238), it follows that $A \in \mathscr{X}_{\text{fin}}$. Also, much as in (5.237), we may estimate

$$\begin{aligned}\rho(A, A_{n_i}) &= \mathscr{C}(A \triangle A_{n_i}) \leq C_1 \mathscr{C}\Big(\bigvee_{k=i}^{\infty}\big(A_{n_{k+1}} \triangle A_{n_k}\big)\Big)\\ &\leq C_1 C_2 \limsup_{N\to\infty} \mathscr{C}\Big(\bigvee_{k=i}^{N}\big(A_{n_{k+1}} \triangle A_{n_k}\big)\Big)\\ &\leq C_1 C_2 C_0^2 \Big\{\sum_{k=i}^{\infty} \mathscr{C}\big(A_{n_{k+1}} \triangle A_{n_k}\big)^{\beta}\Big\}^{1/\beta}\\ &\leq C_1 C_2 C_0^2 \Big\{\sum_{k=i}^{\infty} 2^{-\beta k}\Big\}^{1/\beta} \longrightarrow 0 \text{ as } i \to \infty. \end{aligned} \tag{5.240}$$

Having proved this, the fact that the sequence $(A_n)_{n\in\mathbb{N}}$ is Cauchy in $(\mathscr{X}_{\text{fin}}, \rho)$ allows us to obtain that $\lim_{n\to\infty} \rho(A_n, A) = 0$, i.e., the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A in $(\mathscr{X}_{\text{fin}}, \rho)$, as desired. This completes the proof of completeness of the quasimetric space $(\mathscr{X}_{\text{fin}}, \rho)$.

`Step IX:` *Given any $A \in \mathscr{X}$ and any $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{X}$ with the property that $A_n \to A$ in τ_ρ, we have*

$$(C_0C_1)^{-2} \limsup_{n\to\infty} \mathscr{C}(A_n) \leq \mathscr{C}(A) \leq (C_0C_1)^2 \liminf_{n\to\infty} \mathscr{C}(A_n). \tag{5.241}$$

Indeed, this is a direct consequence of (3.210) used for the Abelian group $(\mathscr{X}, \triangle)$ (cf. also (5.186) and (5.206)). Of course, this proves (5.199).

`Step X:` *Assume in place of* (1) *and* (2) *in the statement of the theorem that* (5.200) *and* (5.201) *hold. Then* $(\mathscr{X}_{\text{fin}}, \rho)$ *is a metric space and* (5.202) *holds.* In the setting just specified, from (5.183) we deduce that

$$\begin{aligned}\mathscr{C}(A \triangle B) &= \mathscr{C}\big((A \wedge B^{c}) \vee (A^{c} \wedge B)\big) \leq \mathscr{C}\big(A \wedge B^{c}\big) + \mathscr{C}\big(A^{c} \wedge B\big)\big)\\ &\leq \mathscr{C}(A) + \mathscr{C}(B), \quad \forall\, A, B \in \mathscr{X}. \end{aligned} \tag{5.242}$$

Hence, if $\mathscr{C}_{\#}$ is as in Step II, it follows from (3.319) that $\mathscr{C} = \mathscr{C}_{\#}$ on $\mathscr{X}$ (given that, in the application of Theorem 3.28 to the present setting, (3.314) holds with $C_1 := 2$, hence $\alpha := (\log_2 C_1)^{-1} = 1$). Consequently, the validity of (5.202) is guaranteed by (5.210). From (5.203) and the current assumptions we also deduce that

$$\mathscr{C}(A \triangle B) \leq \mathscr{C}(A \triangle C) + \mathscr{C}(C \triangle B), \quad \forall\, A, B, C \in \mathscr{X}, \tag{5.243}$$

which shows that $(\mathscr{X}_{\text{fin}}, \rho)$ is a metric space. This completes the treatment of Step X and completes the proof of the theorem. □

The following useful consequence of Theorem 5.18, itself a generalization of the classic result corresponding to the case when μ is a measure, deserves a brief discussion.

Corollary 5.19. *Let Σ be an arbitrary given set, let $\mathfrak{M}$ be a sigma-algebra of subsets of Σ, and assume that $\mu : \mathfrak{M} \to [0, +\infty]$ is a function that satisfies the following properties:*

$$\mu(\emptyset) = 0, \quad \mu(A) \leq \mu(B) \quad \textit{whenever} \quad \forall\, A, B \in \mathfrak{M} \;\; \textit{with} \;\; A \subseteq B, \tag{5.244}$$

$$\mu(A \cup B) \leq \mu(A) + \mu(B) \quad \textit{whenever} \quad \forall\, A, B \in \mathfrak{M}, \qquad \textit{and} \tag{5.245}$$

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \lim_{n\to\infty} \mu(A_n), \quad \forall\, (A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M} \; \textit{with} \; A_n \subseteq A_{n+1} \quad \forall\, n \in \mathbb{N}. \tag{5.246}$$

Next, consider the equivalence relation on $\mathfrak{M}$ given by

$$A \sim B \overset{\text{def}}{\Longleftrightarrow} \mu(A \triangle B) = 0, \tag{5.247}$$

where "$\triangle$" denotes the set-theoretic symmetric difference, and denote by $[A]$ the equivalence class of a generic set $A \in \mathfrak{M}$. Then

$$\big\{[A] : \; A \in \mathfrak{M}, \; \mu(A) < +\infty\big\} \quad \textit{equipped with the distance} \tag{5.248}$$

$$\big([A], [B]\big) \mapsto \mu(A \triangle B) \quad \textit{is a complete metric space.} \tag{5.249}$$

Proof. Define a partial order relation on $\mathscr{X} := \{[A] : \; A \in \mathfrak{M}\}$ by setting

$$[A] \preceq [B] \overset{\text{def}}{\Longleftrightarrow} \exists\, E \in \mathfrak{M} \quad \text{with} \quad A \subseteq B \cup E \quad \text{and} \quad \mu(E) = 0. \tag{5.250}$$

Then, if $[A] \vee [B] := [A \cup B]$, $[A] \wedge [B] := [A \cap B]$, $[A]^{\text{c}} := [\Sigma \setminus A]$, for every $A, B \in \mathfrak{M}$, and $\mathbf{0} := [\emptyset]$, $\mathbf{1} := [\Sigma]$, then it is straightforward to check that

$$\big(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\text{c}}\big) \quad \text{is a sigma-complete Boolean algebra.} \tag{5.251}$$

In this setting, if $\mathscr{C} : \mathscr{X} \to [0, +\infty]$ is given by $\mathscr{C}\big([A]\big) := \mu(A)$ for every $A \in \mathfrak{M}$, then Theorem 5.18 applies and shows that (5.248) holds. □

Parenthetically, we wish to note that any measure on $\mathfrak{M}$, as well as any outer measure on Σ, satisfies (5.244) (in the latter case taking $\mathfrak{M} := 2^{\Sigma}$).

We will now set out the following definition, which will be relevant for us later, in Sect. 5.7.

Definition 5.20. Given a measure space $(\Sigma, \mathfrak{M}, \mu)$, call the measure μ `separable` provided the (complete) metric space (5.248) is separable.

We proceed now to presenting the proof of Theorem 5.4.

Proof of Theorem 5.4. We will work in the context of Theorem 5.3. In the first part of the proof, the goal is to show that the weak Fatou property stated in Theorem 5.3 implies a quantitative version of itself. Specifically, we claim that

$$\left.\begin{array}{l}\exists\, C \in [0,\infty) \ \text{ so that } \ \forall\, (f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \\ \quad \text{with } \ f_i \leq f_{i+1} \ \ \mu\text{-a.e. on } \ \Sigma \ \text{ for each } \ i \in \mathbb{N}\end{array}\right\} \Rightarrow \Big\|\sup_{i\in\mathbb{N}} f_i\Big\| \leq C \sup_{i\in\mathbb{N}} \|f_i\|. \tag{5.252}$$

Seeking a contradiction, assume that (5.252) fails. Then, for every number $n \in \mathbb{N}$, there exists a sequence of functions $(f_{i,n})_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ satisfying

$$f_{i,n} \leq f_{i+1,n} \quad \mu\text{-a.e. on } \ \Sigma \ \text{ for each } \ i \in \mathbb{N} \tag{5.253}$$

and such that if $f_n \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ is given by the formula

$$f_n := \sup_{i\in\mathbb{N}} f_{i,n} \qquad \text{for each } \ n \in \mathbb{N}, \tag{5.254}$$

then

$$\|f_n\| > 2^n \sup_{i\in\mathbb{N}} \|f_{i,n}\| \qquad \text{for each } \ n \in \mathbb{N}. \tag{5.255}$$

Trivially, for each $n \in \mathbb{N}$, the strict inequality (5.255) implies that $\|f_n\| > 0$ and $\sup_{i\in\mathbb{N}} \|f_i\| < +\infty$. In turn, when combined with the weak Fatou property (cf. item (5) in the statement of Theorem 5.3), the latter condition implies that for each $n \in \mathbb{N}$ we have $\|f_n\| = \big\|\sup_{i\in\mathbb{N}} f_{i,n}\big\| < +\infty$. Hence, all together,

$$\|f_n\| \in (0, +\infty) \qquad \text{for each } \ n \in \mathbb{N}. \tag{5.256}$$

To proceed, note that since there is nothing to prove in the case when $\mathcal{L} = \{0\}$, we may assume that there exists $f_o \in \mathcal{L}$ with $\|f_o\|_{\mathcal{L}} > 0$. Then $f := |f_o| \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ satisfies $\|f\| \in (0, +\infty)$ while, for each number $\lambda \in (0, +\infty)$, condition (5.9) yields $\|f\| = \|\lambda(\lambda^{-1} f)\| \leq \varphi(\lambda)\varphi(\lambda^{-1})\|f\|$. Consequently, with $M \in (0, +\infty)$ as in (5.123), we have

$$1 \leq \varphi(\lambda)\varphi(\lambda^{-1}) \leq M, \qquad \forall\, \lambda \in (0, +\infty). \tag{5.257}$$

In particular, $\varphi(\lambda) \geq \varphi(\lambda^{-1})^{-1}$ for each $\lambda \in (0, +\infty)$, from which we deduce (upon recalling that φ vanishes in the limit at the origin) that $\lim_{\lambda \to +\infty} \varphi(\lambda) = +\infty$. Having noticed this and keeping in mind that φ vanishes in the limit at the origin, for each $n \in \mathbb{N}$ it follows from (5.256) that

$$t_n := \inf\big\{\lambda \in (0, +\infty) : \varphi(\lambda^{-1}) \leq n^{-1}\|f_n\|\big\} \tag{5.258}$$

is a well-defined number belonging to $(0, +\infty)$. Next, if for each $n \in \mathbb{N}$ we define $\lambda_n := t_n/2 \in (0, +\infty)$, then the fact that $\lambda_n < t_n$ forces

$$\varphi(\lambda_n^{-1}) > n^{-1}\|f_n\|, \qquad \forall\, n \in \mathbb{N}. \tag{5.259}$$

Moreover, the definition of t_n from (5.258) entails that for each $n \in \mathbb{N}$ there exists $\widetilde{\lambda}_n \in [t_n, 2t_n)$ such that $\varphi(\widetilde{\lambda}_n^{-1}) \leq n^{-1}\|f_n\|$. Together with (5.9), for each $n \in \mathbb{N}$ this allows us to estimate

$$\|f_n\| = \|\widetilde{\lambda}_n^{-1}(\widetilde{\lambda}_n f_n)\| \leq \varphi(\widetilde{\lambda}_n^{-1})\|\widetilde{\lambda}_n f_n\| \leq n^{-1}\|f_n\|\|\widetilde{\lambda}_n f_n\|, \tag{5.260}$$

hence $n \leq \|\widetilde{\lambda}_n f_n\|$, thanks to (5.256). Making use of this, the quasimonotonicity condition for $\|\cdot\|$, (5.9), and the definitions of $\widetilde{\lambda}_n, \lambda_n$, we may then write (with $C_1 \in [1, +\infty)$ as in hypothesis (4) of Theorem 5.4)

$$n \leq \|\widetilde{\lambda}_n f_n\| \leq C_1\|2t_n f_n\| = C_1\|4\lambda_n f_n\| \leq C_1\varphi(4)\|\lambda_n f_n\|, \quad \forall\, n \in \mathbb{N}. \tag{5.261}$$

Introducing $\theta := \big[C_1\varphi(4)\big]^{-1} \in (0, +\infty)$ we therefore arrive at the conclusion that

$$\theta\, n \leq \|\lambda_n f_n\|, \qquad \forall\, n \in \mathbb{N}. \tag{5.262}$$

Moving on, define

$$\widetilde{f}_{i,n} := \lambda_n f_{i,n} \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \quad \forall\, i \in \mathbb{N}, \ \forall\, n \in \mathbb{N}, \tag{5.263}$$

and note that, by virtue of (5.263), (5.9), (5.255), and (5.259), for each $i, n \in \mathbb{N}$ we have

$$\|\widetilde{f}_{i,n}\| \leq \varphi(\lambda_n)\|f_{i,n}\| < \varphi(\lambda_n)2^{-n}\|f_n\| < 2^{-n} n\,\varphi(\lambda_n)\varphi(\lambda_n^{-1}) \leq M\, n 2^{-n}. \tag{5.264}$$

At this stage, for each $i \in \mathbb{N}$ introduce

$$g_i := \lambda_1 f_{i,1} + \lambda_2 f_{i,2} + \cdots + \lambda_i f_{i,i} = \sum_{n=1}^{i} \widetilde{f}_{i,n}. \tag{5.265}$$

Hence, from (5.265) and (5.253) we see that

$$g_i \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \quad \text{and} \quad g_i \leq g_{i+1} \ \ \mu\text{-a.e. on } \Sigma, \quad \forall\, i \in \mathbb{N}. \tag{5.266}$$

In addition, from (5.265), (5.8), the capacitary estimate (3.324) (used in the current context for $\psi := \|\cdot\|$), and (5.264), we obtain that for each fixed number $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$,

$$\sup_{i\in\mathbb{N}} \|g_i\| = \sup_{i\in\mathbb{N}} \Big\|\sum_{n=1}^{i} \widetilde{f}_{i,n}\Big\| \leq C_0^2 \left\{\sum_{n=1}^{\infty} \|\widetilde{f}_{i,n}\|^{\beta}\right\}^{\frac{1}{\beta}}$$

$$\leq C_0^2\, M \left\{\sum_{n=1}^{\infty} (n2^{-n})^{\beta}\right\}^{\frac{1}{\beta}} < +\infty. \tag{5.267}$$

Having established (5.266) and (5.267), the weak Fatou property then ensures that, on the one hand,

$$\Big\|\sup_{i\in\mathbb{N}} g_i\Big\| < +\infty, \tag{5.268}$$

while, on the other hand, the fact that [cf. (5.265)]

$$\sup_{j\in\mathbb{N}} g_j \geq g_i \geq \lambda_n f_{i,n} \quad \mu\text{-a.e. on } \Sigma, \quad \forall\, i \in \mathbb{N} \text{ and } \forall\, n \in \{1, \dots, i\}, \tag{5.269}$$

entails (recalling (5.253) and (5.254))

$$\sup_{j\in\mathbb{N}} g_j \geq \lambda_n \sup_{i\in\mathbb{N}, i\geq n} f_{i,n} = \lambda_n \sup_{i\in\mathbb{N}} f_{i,n} = \lambda_n f_n \quad \mu\text{-a.e. on } \Sigma, \quad \forall\, n \in \mathbb{N}. \tag{5.270}$$

In concert with the quasimonotonicity of $\|\cdot\|$ and (5.262), the pointwise estimate (5.270) forces

$$\theta\, n \leq \|\lambda_n f_n\| \leq C_1 \Big\|\sup_{j\in\mathbb{N}} g_j\Big\|, \quad \forall\, n \in \mathbb{N}, \tag{5.271}$$

and, hence,

$$\Big\|\sup_{i\in\mathbb{N}} g_i\Big\| = +\infty. \tag{5.272}$$

This contradicts (5.268), hence (5.252) is proved.

To set the stage for the subsequent discussion, define $\mathscr{C} : \mathfrak{M} \to [0, +\infty]$ by setting

$$\mathscr{C}(A) := \|\mathbf{1}_A\|, \quad \forall\, A \in \mathfrak{M}. \tag{5.273}$$

We claim that the function $\mathscr{C}$ is a quasimonotone capacity, i.e., it satisfies the following two properties:

$$\mathscr{C}(A) \leq C_1 \mathscr{C}(B) \quad \text{for any } A, B \in \mathfrak{M} \text{ such that } A \subseteq B, \tag{5.274}$$

$$\mathscr{C}(A \cup B) \leq C_0 C_1 \max\{\mathscr{C}(A), \mathscr{C}(B)\}, \quad \forall\, A, B \in \mathfrak{M}. \tag{5.275}$$

Indeed, as regards the quasimonotonicity property (5.274), if $A, B \in \mathfrak{M}$ are such that $A \subseteq B$, then $\mathbf{1}_A \leq \mathbf{1}_B$ on Σ. In concert with the definition of $\mathscr{C}$ and the quasimonotonicity property of $\|\cdot\|$, this implies that $\mathscr{C}(A) = \|\mathbf{1}_A\| \leq C_1\|\mathbf{1}_B\| = C_1\mathscr{C}(B)$, as desired. As far as the quasisubadditivity property (5.275) is concerned, pick two arbitrary sets $A, B \in \mathfrak{M}$. Then, since $\mathbf{1}_{A\cup B} \leq \mathbf{1}_A + \mathbf{1}_B$ on Σ, it follows from this and the quasimonotonicity of $\|\cdot\|$ that $\|\mathbf{1}_{A\cup B}\| \leq C_1\|\mathbf{1}_A + \mathbf{1}_B\|$. Based on this and (5.8), we may then write

$$\begin{aligned}\mathscr{C}(A\cup B) = \|\mathbf{1}_{A\cup B}\| &\leq C_0C_1 \max\Big\{\|\mathbf{1}_A\|, \|\mathbf{1}_B\|\Big\}\\ &= C_0C_1 \max\big\{\mathscr{C}(A), \mathscr{C}(B)\big\},\end{aligned} \tag{5.276}$$

completing the proof of the quasisubadditivity property for $\mathscr{C}$.

In relation to the capacity $\mathscr{C}$ introduced in (5.273), we also make the claim that if $\beta \in \big(0, (\log_2 C_0 + \log_2 C_1)^{-1}\big]$ is a fixed finite number, then for every sequence of sets $\{W_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{M}$ there holds

$$\begin{aligned}&\sum_{n=1}^{\infty}\mathscr{C}(W_n)^\beta < +\infty \Longrightarrow \mu\Big(\limsup_{n\to\infty} W_n\Big) = 0,\\ &\qquad\text{where}\quad \limsup_{n\to\infty} W_n := \bigcap_{n=1}^{\infty}\bigcup_{i\geq n} W_i.\end{aligned} \tag{5.277}$$

To justify this claim, consider the semigroup $(S, *)$, where $S := \mathfrak{M}$ and where we have set $A * B := A \cup B$ for all $A, B \in \mathfrak{M}$. Then the inclusion of sets induces a partial order relation on S that satisfies condition (3.369). Denote by $\mathscr{C}_\star$ the regularization of $\mathscr{C}$ defined as in Theorem 3.38 relative to the algebraic setting just described. In particular, properties (3.375) and (3.377) translate (in light of (5.274) and (5.275)) to

$$C_1^{-3}C_0^{-2}\,\mathscr{C}(A) \leq \mathscr{C}_\star(A) \leq \mathscr{C}(A) \quad \text{for each} \quad A \in \mathfrak{M}, \tag{5.278}$$

$$\mathscr{C}_\star(A) \leq \mathscr{C}_\star(B) \quad \text{for all} \quad A, B \in \mathfrak{M} \quad \text{such that} \quad A \subseteq B. \tag{5.279}$$

Next, we note that if a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M}$ is such that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$, then, with $C \in [0, +\infty)$ as in (5.252), we have

$$\begin{aligned}\mathscr{C}\Big(\bigcup_{n=1}^{\infty} A_n\Big) &= \Big\|\mathbf{1}_{\cup_{n=1}^{\infty}A_n}\Big\| = \Big\|\sup_{n\in\mathbb{N}}\mathbf{1}_{A_n}\Big\| \leq C\sup_{n\in\mathbb{N}}\big\|\mathbf{1}_{A_n}\big\| = C\sup_{n\in\mathbb{N}}\mathscr{C}(A_n)\\ &\leq CC_1^3C_0^2\sup_{n\in\mathbb{N}}\mathscr{C}_\star(A_n) = CC_1^3C_2\lim_{n\to\infty}\mathscr{C}_\star(A_n)\\ &\leq CC_1^3C_0^2\limsup_{n\to\infty}\mathscr{C}(A_n).\end{aligned} \tag{5.280}$$

Indeed, the first inequality is a consequence of (5.252), the second inequality is implied by (the first part in) (5.278), the last equality holds by virtue of (5.279) and the fact that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$, and the last inequality follows from (the second part in) (5.278). To proceed, observe that $\big(\mathfrak{M}, \subseteq, \cup, \cap, \emptyset, \Sigma, (\cdot)^{\mathrm{c}}\big)$ is a sigma-complete Boolean algebra and that $\mathscr{C}$ from (5.273) satisfies conditions (1)–(3) stipulated in the statement of Theorem 5.18, thanks to (5.274), (5.275), and (5.280). Then the claim established in Step VI of the proof of Theorem 5.18 ensures that, in the context of (5.277), the finiteness of the series on the left-hand side implies $\mathscr{C}\big(\limsup_{n\to\infty} W_n\big) = 0$. With this in hand, the implication in (5.277) follows with the help of (5.172).

Turning now in earnest to the proof of the conclusion in Theorem 5.4, assume that the sequence $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}$ converges to some $f \in \mathcal{L}$ in the topology $\tau_{\|\cdot\|_{\mathcal{L}}}$. Then it is possible to find integer numbers $1 \leq n_1 < n_2 < \cdots < n_k < \cdots$ with the property that

$$\|f_{n_k} - f\|_{\mathcal{L}} < 2^{-k}\varphi(2^k)^{-1}, \quad \forall\, k \in \mathbb{N}. \tag{5.281}$$

Next, for each $k \in \mathbb{N}$ introduce

$$A_k := \big\{x \in \Sigma : |f_{n_k}(x) - f(x)| > 2^{-k}\big\} \in \mathfrak{M}, \tag{5.282}$$

and consider

$$A := \limsup_{k\to\infty} A_k := \bigcap_{k=1}^{\infty} \bigcup_{i\geq n} A_i \in \mathfrak{M}. \tag{5.283}$$

Then for each point $x \in \Sigma \setminus A$ there exists $k_o \in \mathbb{N}$ with the property that $x \notin \bigcup\limits_{k\geq k_o} A_k$ and, hence, $|f_{n_k}(x) - f(x)| \leq 2^{-k}$ for each $k \geq k_o$. Consequently,

$$\lim_{k\to\infty} |f_{n_k}(x) - f(x)| = 0, \quad \forall\, x \in \Sigma \setminus A. \tag{5.284}$$

Thus, we may conclude that the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise μ-a.e. on Σ as soon as we show that

$$\mu(A) = 0. \tag{5.285}$$

In turn, by (5.277), matters may be further reduced to proving that

$$\sum_{k=1}^{\infty} \mathscr{C}(A_k)^{\beta} < +\infty. \tag{5.286}$$

To this end, for each fixed $k \in \mathbb{N}$, using the fact that $\|\cdot\|$ is quasimonotone, (5.9), and (5.281), we may estimate

$$\mathscr{C}(A_k) = \|\mathbf{1}_{A_k}\| \leq C_1 \big\| 2^k \cdot |f_{n_k} - f| \,\big\| \leq C_1 \varphi(2^k) \| f_{n_k} - f \|_{\mathcal{L}} < C_1 2^{-k}, \qquad \forall\, k \in \mathbb{N}. \tag{5.287}$$

Hence, $\sum\limits_{k=1}^{\infty} \mathscr{C}(A_k)^{\beta} < C_1^{\beta} \sum\limits_{k=1}^{\infty} 2^{-k\beta} < +\infty$, proving (5.286) and completing the proof of the theorem. □

We conclude this subsection by recording a notable consequence of the proofs of Theorem 5.4 and Proposition 5.17.

Corollary 5.21. *In the context of Theorem 5.3, the quantitative Fatou property formulated in (5.16) holds.*

Proof. This is proved much as the implication $(i) \Rightarrow (ii)$ in Proposition 5.17, making use of (5.252). □

5.5 Absolute Continuity of a Measure with Respect to a Capacity

We start by making the following definition.

Definition 5.22. (i) Given a lattice $(\mathscr{X}, \preceq, \vee, \wedge)$, call $A, B \in \mathscr{X}$ `disjoint` provided $A \wedge B = \mathbf{0}$.

(ii) Given a sigma-complete Boolean algebra $(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\mathrm{c}})$, call a mapping $\mu : \mathscr{X} \to [0, +\infty]$ a `measure` provided

$$\mu\Big(\bigvee_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) \tag{5.288}$$

for every sequence $(A_n)_{n\in\mathbb{N}}$ consisting of pairwise disjoint elements in the lattice $(\mathscr{X}, \preceq, \vee, \wedge)$. Such a measure is called `finite` if $\mu(\mathbf{1}) < +\infty$.

The next lemma summarizes some of the main properties of finite measures on sigma-complete Boolean algebras.

Lemma 5.23. *Let $(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\mathrm{c}})$ be a sigma-complete Boolean algebra, and assume that μ is a measure on it. Then μ satisfies the following properties:*

(1) $\mu(\mathbf{0}) = 0$.
(2) $\mu(A) \leq \mu(B)$ *for every* $A, B \in \mathscr{X}$ *with* $A \preceq B$.

(3) *If* $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{X}$ *satisfy* $A_n \preceq A_{n+1}$ *for every* $n \in \mathbb{N}$ *and* $A := \bigvee_{n=1}^{\infty} A_n$*, then* $\mu(A_n) \nearrow \mu(A)$ *as* $n \to \infty$.

(4) *If* $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{X}$ *satisfy* $A_{n+1} \preceq A_n$ *for every* $n \in \mathbb{N}$ *and* $A := \bigwedge_{n=1}^{\infty} A_n$*, then* $\mu(A_n) \searrow \mu(A)$ *as* $n \to \infty$*, provided* $\mu(A_1) < +\infty$.

The proof of this result is analogous to the proof of the version of this lemma corresponding to finite measures on ordinary measure spaces (see, e.g., [107, Theorem 1.19, p. 16]; in the current context, the identity in (5.219) is also useful). We omit the routine details.

The following theorem, which constitutes the main result in this subsection, extends the classical result regarding the ε-δ characterization of the absolute continuity of measures (cf., e.g., [107, Theorem 6.11, p. 124] for the familiar formulation) by considering the setting of sigma-complete Boolean algebras and allowing the role of the dominating measure to be played by a quasimonotone capacity.

Theorem 5.24. *Suppose* $\big(\mathscr{X}, \preceq, \vee, \wedge, \mathbf{0}, \mathbf{1}, (\cdot)^{\rm c}\big)$ *is a sigma-complete Boolean algebra. Let* μ *be a finite measure on* $\mathscr{X}$*. Assume that the function*

$$\mathscr{C} : \mathscr{X} \longrightarrow [0, +\infty] \tag{5.289}$$

satisfies the following properties:

(i) *(Quasisubadditivity) There exists* $C_0 \in [1, +\infty)$ *such that*

$$\mathscr{C}(A \vee B) \leq C_0 \max\{\mathscr{C}(A), \mathscr{C}(B)\}, \quad \forall\, A, B \in \mathscr{X}. \tag{5.290}$$

(ii) *(Quasimonotonicity) There exists* $C_1 \in [1, +\infty)$ *such that*

$$\mathscr{C}(A) \leq C_1 \mathscr{C}(B), \quad \forall\, A, B \in \mathscr{X} \quad \text{such that} \quad A \preceq B. \tag{5.291}$$

Then the following statements are equivalent:

(1) *There holds* $\mu \ll \mathscr{C}$ *in the sense that whenever* $A \in \mathscr{X}$ *satisfies* $\mathscr{C}(A) = 0$*, then necessarily* $\mu(A) = 0$.
(2) *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\mu(A) < \varepsilon$ *for each* $A \in \mathscr{X}$ *with* $\mathscr{C}(A) < \delta$.
(3) *For any family* $\{A_j\}_{j\in\mathbb{N}} \subseteq \mathscr{X}$ *such that* $\lim_{j\to\infty} \mathscr{C}(A_j) = 0$ *there holds* $\lim_{j\to\infty} \mu(A_j) = 0$.

Compared to the classical setting of measures, there are two novel aspects of the proof of this theorem. First, the results from Sect. 1 play a basic role here and, second, the overall strategy of the proof has been designed to cope with the lack of countable additivity of the capacity.

Proof of Theorem 5.24. The implications (2) ⇒ (1) and (3) ⇒ (1) are obvious, and the implication (2) ⇒ (3) is easy. Thus, we only have to prove that (1) ⇒ (2), and we will do so by reasoning by contradiction. To this end, assume that property (1) holds and that property (2) fails, i.e., there exist $\varepsilon_o > 0$ and a family $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{X}$ such that

$$\mathscr{C}(A_n) < 2^{-n} \quad \text{and} \quad \mu(A_n) \geq \varepsilon_o, \quad \forall\, n \in \mathbb{N}. \tag{5.292}$$

Introduce $A \in \mathscr{X}$ defined by

$$A := \bigwedge_{n=1}^{\infty}\Big(\bigvee_{k=n}^{\infty} A_k\Big). \tag{5.293}$$

Then, using (4) and (2) in Lemma 5.23, as well as (5.292), we may write

$$\mu(A) = \lim_{n\to\infty} \mu\Big(\bigvee_{k=n}^{\infty} A_k\Big) \geq \varepsilon_o. \tag{5.294}$$

For each $k \in \mathbb{N}$ define

$$\widetilde{A}_k := A_k \wedge A, \tag{5.295}$$

and note that for each $j \in \mathbb{N}$ we have

$$\bigvee_{k=j}^{\infty} \widetilde{A}_k = A \wedge \Big(\bigvee_{k=j}^{\infty} A_k\Big) = A \tag{5.296}$$

since the definition of the set A in (5.293) entails $A \preceq \bigvee\limits_{k=j}^{\infty} A_k$ for each $j \in \mathbb{N}$.

Going further, for each $n, j \in \mathbb{N}$ with $n \geq j$ define the set

$$V_{j,n} := \bigvee_{k=j}^{n} \widetilde{A}_k, \tag{5.297}$$

and note that (5.296) forces

$$V_{j,n} \preceq A, \quad \forall\, n, j \in \mathbb{N} \;\text{ with }\; n \geq j. \tag{5.298}$$

In addition, from the definition of the sets $\widetilde{A}_k$ we deduce that

$$V_{j,n} = A \wedge \Big(\bigvee_{k=j}^{n} A_k\Big) \preceq \bigvee_{k=j}^{n} A_k \quad \forall\, n, j \in \mathbb{N}, \quad \text{with} \quad n \geq j. \tag{5.299}$$

To proceed, fix $\beta \in \big(0, (\log_2 C_0)^{-1}\big]$ and, for each $n, j \in \mathbb{N}$ with $n \geq j$, write

$$\begin{aligned}\mathscr{C}(V_{j,n}) &\leq C_1 \mathscr{C}\Big(\bigvee_{k=j}^{n} A_k\Big) \leq C_0^2 C_1 \Big(\sum_{k=j}^{n} \mathscr{C}(A_k)^{\beta}\Big)^{1/\beta} \\ &\leq C_0^2 C_1 \Big(\sum_{k=j}^{\infty} 2^{-k\beta}\Big)^{1/\beta} = C_0^2 C_1 \frac{2^{-j}}{(1-2^{-\beta})^{1/\beta}}, \end{aligned} \tag{5.300}$$

where the first inequality follows from (5.299) and the quasimonotonicity property (5.291) of $\mathscr{C}$, the second one is a consequence of estimate (3.322) (written for $\psi := \mathscr{C}$), and the third inequality is a consequence of the first part of (5.292).

Since for each $j \in \mathbb{N}$ fixed we have

$$V_{j,n} \preceq V_{j,n+1} \quad \text{for every} \quad n \in \mathbb{N} \quad \text{with} \quad n \geq j \tag{5.301}$$

and

$$\bigvee_{n=j}^{\infty} V_{j,n} = A, \tag{5.302}$$

by property (3) in Lemma 5.23 we may conclude that (recall that the measure μ is finite)

$$\forall\, j \in \mathbb{N} \quad \exists\, n_j \in \mathbb{N} \quad \text{such that} \quad n_j \geq j \quad \text{and} \quad \mu(A) - \mu(V_{j,n_j}) < 2^{-j}. \tag{5.303}$$

Moreover, for each $j \in \mathbb{N}$ fixed we have $A = V_{j,n} \vee (A \wedge V_{j,n}^{c})$ for every $n \in \mathbb{N}$ satisfying $n \geq j$, thanks to (5.298) and the distributivity laws in $\mathscr{X}$. Thus, based on this and (5.288), we deduce that $\mu(A) = \mu(V_{j,n}) + \mu(A \wedge V_{j,n}^{c})$ for every $n \in \mathbb{N}$ with $n \geq j$. Utilizing this back in (5.303) then yields

$$\forall\, j \in \mathbb{N} \quad \exists\, n_j \in \mathbb{N} \quad \text{such that} \quad n_j \geq j \quad \text{and} \quad \mu\big(A \wedge V_{j,n_j}^{c}\big) < 2^{-j}. \tag{5.304}$$

Next, for each $k \in \mathbb{N}$ we define the set $W_k \in \mathscr{X}$ by setting $W_k := \bigwedge_{j=k}^{\infty} V_{j,n_j}$. Using again the quasimonotonicity property (5.291) of $\mathscr{C}$, along with (5.300), for each $k \in \mathbb{N}$ and $j \in \mathbb{N}$ with $j \geq k$ we have

$$\mathscr{C}(W_k) \leq C_1 \mathscr{C}(V_{j,n_j}) \leq C_0^2 C_1^2 \frac{2^{-j}}{(1-2^{-\beta})^{1/\beta}} \longrightarrow 0 \quad \text{as} \quad j \to \infty. \tag{5.305}$$

Thus, for each $k \in \mathbb{N}$ we have $\mathscr{C}(W_k) = 0$, which, granted property (1) from the statement of the theorem, implies that $\mu(W_k) = 0$ for each $k \in \mathbb{N}$. In concert with property (3) in Lemma 5.23, this implies

$$\mu\Big(\bigvee_{k=1}^{\infty} W_k\Big) = 0. \tag{5.306}$$

Consider now the set $E \in \mathscr{X}$ defined by $E := \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (A \wedge V_{j,n_j}^{\mathrm{c}})$, and observe that

$$\mu(E) \le \mu\Big(\bigvee_{j=k}^{\infty} (A \wedge V_{j,n_j}^{\mathrm{c}})\Big) \le \sum_{j=k}^{\infty} \mu(A \wedge V_{j,n_j}^{\mathrm{c}}) \quad \text{for each } k \in \mathbb{N}. \tag{5.307}$$

Combining (5.307) with the second part of (5.304) we obtain $\mu(E) \le \sum_{j=k}^{\infty} 2^{-j}$ for each $k \in \mathbb{N}$. Hence,

$$\mu(E) = 0. \tag{5.308}$$

It remains to observe that, based on the definition of the sets E and W_k, $k \in \mathbb{N}$, (5.190), (5.298), the distributivity, and De Morgan's laws in $\mathscr{X}$, we have

$$E \vee \Big(\bigvee_{k=1}^{\infty} W_k\Big) = \left(A \wedge \Big(\bigvee_{k=1}^{\infty} \bigwedge_{j=k}^{\infty} V_{j,n_j}\Big)^{\mathrm{c}}\right) \vee \Big(\bigvee_{k=1}^{\infty} \bigwedge_{j=k}^{\infty} V_{j,n_j}\Big) = A. \tag{5.309}$$

Thus, ultimately,

$$\mu(A) \le \mu(E) + \mu\Big(\bigvee_{k=1}^{\infty} W_k\Big) = 0, \tag{5.310}$$

which contradicts the fact that $\mu(A) \ge \varepsilon_o > 0$ from (5.294). This completes the proof by contradiction of the fact that (1) $\Rightarrow$ (2) and completes the proof of the theorem. □

Specializing the previous theorem to the case when the sigma-complete Boolean algebra in question is a sigma-algebra of subsets of a given background set, equipped with the standard set-theoretic operations, readily yields the following corollary.

Corollary 5.25. *Suppose that $(\Sigma, \mathfrak{M}, \mu)$ is a complex measure space, and assume that $\mathscr{C} : \mathfrak{M} \to [0, +\infty]$ has the property that for some fixed constant $C_0 \in [1, +\infty)$ one has $\mathscr{C}(A \cup B) \le C_0 \max\{\mathscr{C}(A), \mathscr{C}(B)\}$ for all $A, B \in \mathfrak{M}$, and $\mathscr{C}(A) \le C_0 \mathscr{C}(B)$ for all $A, B \in \mathfrak{M}$ with $A \subseteq B$.*

Then the following statements are equivalent:

(1) There holds $\mu \ll \mathscr{C}$ in the sense that if $A \in \mathfrak{M}$ satisfies $\mathscr{C}(A) = 0$, then $\mu(A) = 0$.
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mu(A)| < \varepsilon$ for each $A \in \mathfrak{M}$ with $\mathscr{C}(A) < \delta$.
(3) For any family $(A_j)_{j \in \mathbb{N}} \subseteq \mathfrak{M}$ such that $\lim_{j\to\infty} \mathscr{C}(A_j) = 0$ there holds $\lim_{j\to\infty} \mu(A_j) = 0$.

The conclusion in Corollary 5.25 is false for μ in the class of sigma-finite measures, even when $\mathscr{C}$ is a finite measure. For example, if for each $k \in \mathbb{N}$ we let δ_k denote the Dirac measure on the real line with mass at k, then $\mu := \sum_{k=1}^{\infty} \delta_k$ and $\mathscr{C} := \sum_{k=1}^{\infty} 2^{-k} \delta_k$.

Then, obviously, $\mu \ll \mathscr{C}$, and yet the sequence $(A_j)_{j\in\mathbb{N}}$ with $A_j := \{j\}$, $j \in \mathbb{N}$, satisfies $\mathscr{C}(A_j) = 2^{-j} \to 0$ and $\mu(A_j) = 1 \to 1$ as $j \to \infty$.

In turn, Corollary 5.25 is the key ingredient in the proof of the embedding result from Theorem 5.26, stated in the next section.

5.6 Embeddings and Pointwise Convergence

Let $(\Sigma, \mathfrak{M}, \mu)$ be a sigma-finite measure space. Hence, there exists a sequence $(K_j)_{j\in\mathbb{N}}$ satisfying

$$(K_j)_{j\in\mathbb{N}} \subseteq \mathfrak{M}, \quad K_j \subseteq K_{j+1}, \quad \mu(K_j) < +\infty, \quad \forall\, j \in \mathbb{N}, \quad \bigcup_{j=1}^{\infty} K_j = \Sigma. \tag{5.311}$$

In this scenario, we equip the space $L^0(\Sigma, \mathfrak{M}, \mu)$ with the topology τ_μ in which a fundamental system of neighborhoods of an arbitrary $f \in L^0(\Sigma, \mathfrak{M}, \mu)$ is given by $\big(\mathcal{V}_{\varepsilon,j}(f)\big)_{\varepsilon>0\,,\,j\in\mathbb{N}}$ where, for each $\varepsilon > 0$ and $j \in \mathbb{N}$, we have the set

$$\mathcal{V}_{\varepsilon,j}(f) := \big\{g \in L^0(\Sigma, \mathfrak{M}, \mu) : \mu\big(\{x \in K_j : |f(x)-g(x)| > \varepsilon\}\big) < \varepsilon\big\}. \tag{5.312}$$

As is well known, the topology τ_μ is metrizable, and a sequence $(f_j)_{j\in\mathbb{N}} \subseteq L^0(\Sigma, \mathfrak{M}, \mu)$ converges to some $f \in L^0(\Sigma, \mathfrak{M}, \mu)$ in τ_μ if and only if $(f_j)_{j\in\mathbb{N}}$ converges to f in measure on sets of finite measure, i.e., for each $\varepsilon > 0$ there holds

$$\forall\, A \in \mathfrak{M} \text{ with } \mu(A) < +\infty \Longrightarrow \lim_{j\to\infty} \mu\big(\{x \in A : |f_j(x) - f(x)| > \varepsilon\}\big) = 0. \tag{5.313}$$

Furthermore, $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$ is a complete topological vector space if the measure μ is complete.

The stage has been set for formulating and proving our main embedding theorem in this chapter. Specifically, the following result holds.

Theorem 5.26. *Let $(\Sigma, \mathfrak{M}, \mu)$ be a background sigma-finite measure space. Assume that X is a linear subspace of $L^0(\Sigma, \mathfrak{M}, \mu)$ and $\|\cdot\| : X \to [0, +\infty)$ is a function satisfying the following properties:*

(1) (Quasitriangle inequality) There exists a constant $C_0 \in [1, +\infty)$ with the property that

$$\|f + g\| \le C_0 \max\{\|f\|, \|g\|\}, \quad \forall\, f, g \in X. \tag{5.314}$$

(2) (Weak pseudohomogeneity) There exists a function $\varphi : (0, +\infty) \to (0, +\infty)$ satisfying

$$\|\lambda f\| \le \varphi(\lambda)\|f\|, \quad \forall\, f \in X, \quad \forall\, \lambda \in (0, +\infty). \tag{5.315}$$

(3) (Nondegeneracy) There holds

$$\|f\| = 0 \iff f = 0, \quad \forall\, f \in X. \tag{5.316}$$

(4) (Quasimonotonicity) There exists a constant $C_1 \in [1, +\infty)$ such that whenever $f \in X$ and $g \in L^0(\Sigma, \mathfrak{M}, \mu)$ satisfy $|g| \leq |f|$ pointwise μ-a.e. on Σ, then $g \in X$ and $\|g\| \leq C_1 \|f\|$.

(5) (Locality) There exists a sequence $(K_j)_{j\in\mathbb{N}}$ as in (5.311) such that

$$\mathbf{1}_{K_j} \in X, \quad \forall\, j \in \mathbb{N}. \tag{5.317}$$

Then, with the topology $\tau_{\|\cdot\|}$ on X considered in the sense of Definition 5.1, there holds

$$\big(X, \tau_{\|\cdot\|}\big) \hookrightarrow \big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big) \quad \textit{continuously.} \tag{5.318}$$

Proof. Note that, thanks to (5.317) and the fact that $\|\cdot\|$ is quasimonotone, we have $\mathbf{1}_E \in X$ whenever the set $E \in \mathfrak{M}$ has the property that there exists a number $j \in \mathbb{N}$ for which $E \subseteq K_j$. This observation allows us to (meaningfully) consider the family of functions $\mathscr{C}_j : \mathfrak{M} \to [0, +\infty]$, defined by setting

$$\mathscr{C}_j(A) := \|\mathbf{1}_{A\cap K_j}\|, \quad \forall\, A \in \mathfrak{M}, \quad \forall\, j \in \mathbb{N}. \tag{5.319}$$

Fix $j \in \mathbb{N}$ arbitrary. Then, much as in (5.274) and (5.275), the function $\mathscr{C}_j$ is a quasimonotone capacity; more specifically, $\mathscr{C}_j$ satisfies the following properties:

- (Quasimonotonicity) For any sets $A, B \in \mathfrak{M}$ with the property that $A \subseteq B$ there holds $\mathscr{C}_j(A) \leq C_1 \mathscr{C}_j(B)$.
- (Quasisubadditivity) For any sets $A, B \in \mathfrak{M}$ there holds

$$\mathscr{C}_j(A \cup B) \leq C_0 C_1 \max\{\mathscr{C}_j(A), \mathscr{C}_j(B)\}. \tag{5.320}$$

Next, consider the finite measure $\mu_j : \mathfrak{M} \to [0, +\infty)$ defined by $\mu_j(E) := \mu(E \cap K_j)$ for each $E \in \mathfrak{M}$. Whenever the set $A \in \mathfrak{M}$ is such that $\mathscr{C}_j(A) = 0$, then $\|\mathbf{1}_{A\cap K_j}\| = 0$, hence $\mu_j(A) = \mu(A \cap K_j) = 0$ by (5.172). Thus, property (1) from the statement of Corollary 5.25 is satisfied, i.e., $\mu_j \ll \mathscr{C}_j$. With this in hand, the implication (1) $\Rightarrow$ (2) from Corollary 5.25 guarantees that

$$\begin{gathered} \forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \ \text{ such that } \ \mu(A \cap K_j) < \varepsilon \\ \text{whenever } \ A \in \mathfrak{M} \ \text{ satisfies } \ \mathscr{C}_j(A) < \delta. \end{gathered} \tag{5.321}$$

Moving on, fix $\varepsilon > 0$, and let $\delta = \delta(\varepsilon, j) > 0$ be as in (5.321). Given an arbitrary, fixed function $f \in X$, introduce

$$A := \{x \in K_j : |f(x)| > \varepsilon\} \in \mathfrak{M}. \tag{5.322}$$

Note that this definition ensures that $\varepsilon\,\mathbf{1}_A \leq |f|$ pointwise on Σ. Using this, the definition of $\mathscr{C}_j$, the weak pseudohomogeneity condition, and the quasimonotonicity property of $\|\cdot\|$, we therefore obtain

$$\mathscr{C}_j(A) = \|\mathbf{1}_A\| \leq \varphi(\varepsilon^{-1})\|\varepsilon\,\mathbf{1}_A\| \leq C_1\varphi(\varepsilon^{-1})\|f\|. \tag{5.323}$$

Consequently, if $\|f\| \leq \delta_o$ where $\delta_o \in \big(0, \delta\varphi(\varepsilon^{-1})^{-1}C_1^{-1}\big)$, then $\mathscr{C}_j(A) < \delta$, which, in light of (5.321), forces $\mu(A) < \varepsilon$.

In summary, given any numbers $\varepsilon > 0$ and $j \in \mathbb{N}$, if $\delta = \delta(\varepsilon, j) > 0$ is as in (5.321), then taking $\delta_o \in (0, \delta\varphi(\varepsilon^{-1})^{-1}C_1^{-1})$ guarantees that

$$f \in X \quad \text{such that} \quad \|f\| \leq \delta_o \implies \mu\big(\{x \in K_j : |f(x)| > \varepsilon\}\big) < \varepsilon. \tag{5.324}$$

Granted the nature of the fundamental system of neighborhoods of the origin in the space $L^0(\Sigma, \mathfrak{M}, \mu)$ equipped with the topology τ_μ described in (5.312), the continuity of the embedding operator in (5.318) is readily seen from (5.324). □

The nature of the range of the inclusion operator in (5.318) naturally invites revisiting the issue of pointwise convergence of sequences of functions from the vector space X. In this regard, we have the following result.

Theorem 5.27. *In the context of Theorem 5.26, any sequence $(f_j)_{j\in\mathbb{N}} \subseteq X$ that is convergent to some $f \in X$ in the topology $\tau_{\|\cdot\|}$ has a subsequence that converges to f pointwise μ-a.e. on Σ.*

If, in addition, the measure μ is assumed to be complete, then any Cauchy sequence in $(X, \tau_{\|\cdot\|})$ has a subsequence taht converges pointwise μ-a.e. on Σ.

Proof. The first claim in the statement of the theorem follows from (5.318) given that any sequence $(f_j)_{j\in\mathbb{N}} \subseteq L^0(\Sigma, \mathfrak{M}, \mu)$ that is convergent to some $f \in L^0(\Sigma, \mathfrak{M}, \mu)$ in the topology τ_μ has a subsequence that converges to f pointwise μ-a.e. on Σ.

The second claim in the statement of the theorem follows also from (5.318) by invoking the well-known fact (cf., e.g., [26, pp. 3–4]) that, if μ is a complete measure, then, as was mentioned, $\big(L^0(\Sigma, \mathfrak{M}, \mu), \tau_\mu\big)$ is a complete, metrizable, topological vector space. □

5.7 Separability

Here we consider the issue of the separability of certain classes of topological vector spaces. In particular, Theorem 5.6 is an immediate consequence of the following more general result.

Theorem 5.28. *Let $(\Sigma, \mathfrak{M}, \mu)$ be a sigma-finite measure space with the property that the measure μ is separable (in the sense of Definition 5.20). Assume that X is* *a linear subspace of* $L^0(\Sigma, \mathfrak{M}, \mu)$ *and* $\|\cdot\| : X \to [0, +\infty)$ *is a function satisfying the following properties:*

(1) (Quasitriangle inequality) There exists a constant $C_0 \in [1, +\infty)$ *with the property that*

$$\|f + g\| \leq C_0 \max\{\|f\|, \|g\|\}, \quad \forall\, f, g \in X. \tag{5.325}$$

(2) (Pseudohomogeneity) There exists a function $\varphi : (0, +\infty) \to (0, +\infty)$ *satisfying*

$$\|\lambda f\| \leq \varphi(\lambda)\|f\|, \quad \forall\, f \in X, \quad \forall\, \lambda \in (0, +\infty), \tag{5.326}$$

and such that

$$\sup_{\lambda>0}\big[\varphi(\lambda)\varphi(\lambda^{-1})\big] < +\infty \quad \textit{and} \quad \lim_{\lambda\to 0^+} \varphi(\lambda) = 0. \tag{5.327}$$

(3) (Nondegeneracy) There holds

$$\|f\| = 0 \iff f = 0, \quad \forall\, f \in X. \tag{5.328}$$

(4) (Quasi-order ideal) There exists a constant $C_1 \in [1, +\infty)$ *such that whenever* $f \in X$ *and* $g \in L^0(\Sigma, \mathfrak{M}, \mu)$ *satisfy* $|g| \leq |f|$ *pointwise* μ*-a.e. on* Σ*, then* $g \in X$ *and* $\|g\| \leq C_1\|f\|$*.*

(5) (Locality) There exists a sequence $(K_j)_{j\in\mathbb{N}}$ *as in* (5.311) *such that*

$$\mathbf{1}_{K_j} \in X, \qquad \forall\, j \in \mathbb{N}. \tag{5.329}$$

(6) (Absolute continuity) There holds

$$\begin{aligned} &\forall\, f \in X,\ \forall\, (A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M} \ \textit{ such that } \ \mathbf{1}_{A_n} \to 0 \ \ \mu\textit{-a.e. on } \Sigma \textit{ as } n \to \infty \\ &\qquad \Longrightarrow \|f \cdot \mathbf{1}_{A_n}\| \to 0 \ \textit{ as } \ n \to \infty. \end{aligned} \tag{5.330}$$

Then, if the topology $\tau_{\|\cdot\|}$ *on* X *is as in Definition 5.1, it follows that*

$$\big(X, \tau_{\|\cdot\|}\big) \ \textit{ is a separable topological space.} \tag{5.331}$$

Proof. Introduce $X^+ := \big\{f \in X : f \geq 0\ \mu\text{-a.e. on } \Sigma\big\}$, and define

$$\begin{aligned} X^+_{bc} := \Big\{&f \in X : \exists\, j \in \mathbb{N} \text{ and } \exists\, M \in [0, +\infty) \text{ such that} \\ &0 \leq f \leq M\ \mu\text{-a.e. on } \Sigma \text{ and } f = 0\ \mu\text{-a.e. on } \Sigma \setminus K_j\Big\}. \end{aligned} \tag{5.332}$$

We claim that

$$X_{bc}^{+} - X_{bc}^{+} \text{ is dense in } \big(X, \tau_{\|\cdot\|}\big). \tag{5.333}$$

Given an arbitrary function $f \in X$, decompose it into $f = f^{+} - f^{-}$ with

$$f^{\pm} := \frac{1}{2}(|f| \pm f) \in L^{0}(\Sigma, \mathfrak{M}, \mu), \tag{5.334}$$

and note that, since $\|\cdot\|$ is quasimonotone, we have $f^{\pm} \in X^{+}$. Hence, as far as (5.333) is concerned, matters have been reduced to proving that

$$X_{bc}^{+} \text{ is dense in } \big(X^{+}, \tau_{\|\cdot\|}\big). \tag{5.335}$$

To this end, given $f \in X^{+} \subseteq L_{+}^{0}(\Sigma, \mathfrak{M}, \mu)$, define $f_n := f \cdot \mathbf{1}_{\{f<n\}} \in X^{+}$ for each $n \in \mathbb{N}$, and note that $\|f - f_n\| = \|f \cdot \mathbf{1}_{\{f\geq n\}}\| \to 0$ as $n \to \infty$ by (5.330) and the fact that $\mathbf{1}_{\{f\geq n\}} \to 0$ pointwise μ-a.e. on Σ as $n \to \infty$. The latter condition is justified by observing that, much as in the proof of (5.168), the fact that $f \in X^{+}$ entails $f < +\infty$ μ-a.e. on Σ. Consequently,

$$f_n \to f \quad \text{in } \tau_{\|\cdot\|} \text{ as } n \to \infty. \tag{5.336}$$

Next, for each fixed $f \in X^{+}$, a similar type of argument also gives that

$$f \cdot \mathbf{1}_{K_j} \to f \quad \text{in } \tau_{\|\cdot\|} \text{ as } n \to \infty. \tag{5.337}$$

In concert, (5.336) and (5.337) readily imply (5.335), and this completes the proof of (5.333).

Moving on, fix an arbitrary $f \in X_{bc}^{+}$, and construct (cf., e.g., [107, Theorem 1.17, p. 15]) a sequence $(s_n)_{n\in\mathbb{N}} \subseteq L_{+}^{0}(\Sigma, \mathfrak{M}, \mu)$ of nonnegative simple functions on the measurable space $(\Sigma, \mathfrak{M})$ with the property that

$$0 \leq f - s_n \leq 2^{-n} \cdot \mathbf{1}_{\{f>0\}} \text{ on } \Sigma, \quad \text{for each } n \in \mathbb{N} \text{ sufficiently large.} \tag{5.338}$$

Then, $\|f - s_n\| \leq C_1\|2^{-n} \cdot \mathbf{1}_{\{f>0\}}\| \leq C_1\varphi(2^{-n})\|\mathbf{1}_{\{f>0\}}\|$ for each $n \in \mathbb{N}$ sufficiently large, and since $\varphi(2^{-n}) \to 0$ as $n \to \infty$, while $\|\mathbf{1}_{\{f>0\}}\| < +\infty$ given that $f \in X_{bc}^{+}$ (keeping in mind (5.332) and (5.329)), we may ultimately conclude that $s_n \to f$ in $\tau_{\|\cdot\|}$ as $n \to \infty$. In addition, since $0 \leq s_n \leq f$, there exists $j \in \mathbb{N}$ with the property that $s_n = 0$ on $\Sigma \setminus K_j$ whenever n is sufficiently large.

Next, assume that a simple function $s = \sum_{i=1}^{N} \lambda_i \mathbf{1}_{E_i}$ has been fixed, with the property that $\lambda_i \in (0, +\infty)$ and $E_i \in \mathfrak{M}$ for each $i \in \{1, \ldots, N\}$, and such that there exists $j \in \mathbb{N}$ for which $\cup_{i=1}^{N} E_i \subseteq K_j$. Then, choosing for each $i \in \{1, \ldots, N\}$ a sequence $(\lambda_{i,n})_{n\in\mathbb{N}} \in \mathbb{Q}$ with the property that $\lambda_{i,n} \nearrow \lambda_i$ as $n \to \infty$, it follows that

$$\|\lambda_i \mathbf{1}_{E_i} - \lambda_{i,n}\mathbf{1}_{E_i}\| \leq \varphi(\lambda_i - \lambda_{i,n})\|\mathbf{1}_{E_i}\| \to 0 \quad \text{as } n \to \infty. \tag{5.339}$$

At this stage in the proof, the goal is to identify a countable set $\mathscr{D} \subseteq X$ whose closure in $\big(X, \tau_{\|\cdot\|}\big)$ contains

$$\big\{\lambda \cdot \mathbf{1}_E : \lambda \in \mathbb{Q}_+,\ E \in \mathfrak{M} \text{ such that } \exists\, j \in \mathbb{N} \text{ with } E \subseteq K_j\big\}. \tag{5.340}$$

By assumption, the measure μ is separable, and as such there exists a sequence of sets $(E_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M}$ with the property that

$$\begin{gathered}\forall\, A \in \mathfrak{M} \text{ with } \mu(A) < +\infty, \quad \exists\, (E_{n_k})_{k\in\mathbb{N}} \text{ subsequence of } (E_n)_{n\in\mathbb{N}} \\ \text{satisfying } \mu(A \triangle E_{n_k}) \to 0 \text{ as } k \to \infty.\end{gathered} \tag{5.341}$$

We then define

$$\mathscr{D} := \big\{\lambda \cdot \mathbf{1}_{E_n \cap K_j} : \lambda \in \mathbb{Q}_+,\ n, j \in \mathbb{N}\big\} \subseteq X^+ \tag{5.342}$$

and claim that this satisfies the desired property (formulated earlier, in relation to (5.340)). To justify this claim, consider a function of the form $\lambda \cdot \mathbf{1}_E$, where $\lambda \in \mathbb{Q}_+$ and $E \in \mathfrak{M}$ is such that $E \subseteq K_j$ for some $j \in \mathbb{N}$. By (5.341), there exists a subsequence $(E_{n_k})_{k\in\mathbb{N}}$ of $(E_n)_{n\in\mathbb{N}}$ with the property that

$$0 = \lim_{k\to\infty} \mu(A \triangle E_{n_k}) = \lim_{k\to\infty} \int_\Sigma \big|\mathbf{1}_{E_{n_k}} - \mathbf{1}_E\big|\, d\mu. \tag{5.343}$$

Hence, there exists a subsequence $(E_{n_{k_i}})_{i\in\mathbb{N}}$ of $(E_{n_k})_{k\in\mathbb{N}}$ such that

$$\mathbf{1}_{E_{n_{k_i}}} \longrightarrow \mathbf{1}_E \text{ pointwise } \mu\text{-a.e. on } \Sigma, \quad \text{as } i \to \infty. \tag{5.344}$$

In fact, since $E \subseteq K_j$, it follows that

$$\begin{gathered}\mathbf{1}_{E_{n_{k_i}} \cap K_j} \in \mathscr{D} \ \text{ for each } i \in \mathbb{N}, \text{ and} \\ \mathbf{1}_{E_{n_{k_i}} \cap K_j} = \mathbf{1}_{E_{n_{k_i}}} \cdot \mathbf{1}_{K_j} \to \mathbf{1}_E \text{ pointwise } \mu\text{-a.e. on } \Sigma, \quad \text{as } i \to \infty.\end{gathered} \tag{5.345}$$

It remains to observe that

$$\big\|\mathbf{1}_{E_{n_{k_i}} \cap K_j} - \mathbf{1}_E\big\| = \big\|\mathbf{1}_{(E_{n_{k_i}} \cap K_j)\triangle E} \cdot \mathbf{1}_{K_j}\big\| \longrightarrow 0, \quad \text{as } i \to \infty, \tag{5.346}$$

by virtue of (5.329), (5.330), and (5.345). From this the desired conclusion readily follows, and this completes the proof of the theorem. □

Chapter 6
Functional Analysis on Quasi-Pseudonormed Groups

The aim in this chapter is to explore the extent to which a significant portion of classical functional analysis can be carried out on groups equipped with topologies that are only partially compatible with the underlying algebraic structure. More specifically, the settings we consider here will frequently (though not always) be more general than those of topological groups (recall that a topological group is a group endowed with a topology with the property that both the group's binary operation and the group's inverse function are continuous with respect to the given topology).

One case of particular interest for us is that in which the topology on a given group G is naturally induced by a nonnegative function ψ defined on G exhibiting properties similar to, yet weaker than, those satisfied by a genuine norm on a vector space. In such a case, we will refer to G as a quasi-pseudonormed group. As such, our earlier analysis pertaining to the nature of the topology induced by ψ on G, addressing issues such as quantitative metrizability, will play a basic role in this setting.

A specific goal is to revisit the fundamental trilogy in functional analysis, i.e., the open mapping theorem (OMT), the closed graph theorem (CGT), and the uniform boundedness principle (UBP), typically formulated in the context of Banach spaces, or at least complete metrizable topological groups, and identify the specific format of each of these fundamental theorems that permits the consideration of the more general setting alluded to earlier. In this vein, it is worth recalling that relaxing the demands on the environment from Banach to quasi-Banach already has major functional analytic consequences. Specifically, as proved in [66], the Hahn–Banach extension theorem (the cornerstone of the Banach space theory) fails in *all* quasi-Banach spaces that are not Banach spaces.

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6.1 Topological and Algebraical Preliminaries

We start out by introducing some useful notation and basic results from topology. As in the past, given an arbitrary topological space (X, τ), we will denote by $\mathscr{N}(x;\tau)$ the family of neighborhoods of the point $x \in X$, relative to the topology τ. Also, for any $A \subseteq X$ we will let $\text{Int}(A;\tau)$ and $\text{Clo}(A;\tau)$ denote, respectively, the interior and closure of the set A, relative to the topology τ. Finally, for any set $Y \subseteq X$ we let $\tau\big|_Y$ denote the topology induced on Y by τ, i.e.,

$$\tau\big|_Y := \{\mathcal{O} \cap Y : \mathcal{O} \in \tau\}. \tag{6.1}$$

We will make frequent use of the following result.

Lemma 6.1. *Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces, and suppose that the function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is given. Then*

$$f \ \textit{ continuous and closed} \iff f\big(\text{Clo}(A;\tau_1)\big) = \text{Clo}(f(A);\tau_2), \ \ \forall\, A \subseteq X_1, \tag{6.2}$$

$$f \ \textit{ injective, continuous, and open} \Rightarrow f\big(\text{Int}(A;\tau_1)\big) = \text{Int}(f(A);\tau_2), \ \ \forall\, A \subseteq X_1, \tag{6.3}$$

$$f \ \textit{ open} \iff f(U) \in \mathscr{N}\big(f(x);\tau_2\big), \quad \forall\, x \in X_1 \ \textit{ and } \ \forall\, U \in \mathscr{N}(x;\tau_1). \tag{6.4}$$

Proof. On the one hand, if the function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is continuous, then for any $A \subseteq X_1$ the set $f^{-1}\big(\text{Clo}(f(A);\tau_2)\big)$ is closed in the topology τ_1 and contains A. Hence, $\text{Clo}(A;\tau_1) \subseteq f^{-1}\big(\text{Clo}(f(A);\tau_2)\big)$, which ultimately implies that

$$f\big(\text{Clo}(A;\tau_1)\big) \subseteq \text{Clo}(f(A);\tau_2). \tag{6.5}$$

If, on the other hand, the function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is closed, then for any $A \subseteq X_1$ the set $f\big(\text{Clo}(A;\tau_1)\big)$ is closed in τ_2 and contains $f(A)$. As such, we necessarily have $\text{Clo}(f(A);\tau_2) \subseteq f\big(\text{Clo}(A;\tau_1)\big)$. This, together with (6.5), proves the left-to-right implication in (6.1).

In the converse direction, suppose that $f : X_1 \to X_2$ is function with the property that

$$f\big(\text{Clo}(A;\tau_1)\big) = \text{Clo}(f(A);\tau_2), \quad \forall\, A \subseteq X_1. \tag{6.6}$$

This clearly entails that $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is closed, so there remains to prove that this function is also continuous. Reasoning by contradiction, assume that there exists $x_0 \in X_1$ such that f is not continuous at x_0. Then there exists $U \in \mathscr{N}(f(x_0);\tau_2)$ with the property that

$$\forall\, V \in \mathscr{N}(x_0;\tau_1) \ \exists\, x_V \in V \ \text{ such that } \ f(x_V) \notin U. \tag{6.7}$$

Next, consider the set $A := \{x_V : V \in \mathcal{N}(x_0; \tau_1)\}$, and note that we have $x_0 \in \mathrm{Clo}(A; \tau_1)$ since, by design, for any $V \in \mathcal{N}(x_0; \tau_1)$ one has $x_V \in V \cap A$, hence $V \cap A \neq \emptyset$ for each $V \in \mathcal{N}(x_0; \tau_1)$. Thus, $f(x_0) \in f(\mathrm{Clo}(A; \tau_1))$, and using (6.6) we may therefore conclude that $f(x_0) \in \mathrm{Clo}(f(A); \tau_2)$. In particular, this shows that $U \cap f(A) \neq \emptyset$, and, based on the definition of A, this implies that there exists $V \in \mathcal{N}(x_0; \tau_1)$ such that $f(x_V) \in U$, contradicting (6.7). This completes the proof of the right-to left implication in (6.2).

Turning our attention to proving the implication formulated in (6.3), assume that the function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is injective, continuous, and open. Since for each $A \subseteq X_1$ one has $f(\mathrm{Int}(A; \tau_1)) \subseteq f(A)$, the fact that the function f is open implies that

$$f(\mathrm{Int}(A; \tau_1)) \subseteq \mathrm{Int}(f(A); \tau_2), \qquad \forall\, A \subseteq X_1. \tag{6.8}$$

Going further, the fact that the function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is also continuous gives

$$\forall\, A \subseteq X_1 \ \text{ the set } \ f^{-1}\big(\mathrm{Int}(f(A); \tau_2)\big) \ \text{ is open in } \tau_1. \tag{6.9}$$

However, the function $f : X_1 \to X_2$ is injective, which allows us to write

$$f^{-1}\big(\mathrm{Int}(f(A); \tau_2)\big) \subseteq f^{-1}\big(f(A)\big) \subseteq A. \tag{6.10}$$

In combination with (6.9), this implies that $f^{-1}\big(\mathrm{Int}(f(A); \tau_2)\big) \subseteq \mathrm{Int}(A; \tau_1)$. In turn, this gives

$$\mathrm{Int}(f(A); \tau_2) \subseteq f(\mathrm{Int}(A; \tau_1)), \qquad \forall\, A \subseteq X_1. \tag{6.11}$$

Then (6.8) and (6.11) prove the implication in (6.3).

Finally, (6.4) is essentially a direct consequence of the definition of an open function. □

The equivalence in (6.4) suggests adopting the following piece of terminology (which will become relevant later on).

Definition 6.2. Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces, and let $f : X_1 \to X_2$ be a given function. Call f `almost open at` $x \in X_1$ (relative to τ_1, τ_2) provided

$$\mathrm{Clo}\big(f(U); \tau_2\big) \in \mathcal{N}\big(f(x); \tau_2\big), \qquad \forall\, U \in \mathcal{N}(x; \tau_1). \tag{6.12}$$

For further use, let us also note here that if (X_1, τ_1) and (X_2, τ_2) are topological spaces, then for every function $f : X_1 \to X_2$ we have

$$\left.\begin{array}{c} (X_1, \tau_1) \text{ compact} \\ \text{and} \\ (X_2, \tau_2) \text{ Hausdorff} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} f : (X_1, \tau_1) \to (X_2, \tau_2) \text{ is a homeomorphism} \\ \qquad\qquad\qquad \text{if and only if} \\ f : (X_1, \tau_1) \to (X_2, \tau_2) \text{ is continuous and bijective.} \end{array}\right. \tag{6.13}$$

Indeed, this follows by observing that if (X_1, τ_1) is compact and (X_2, τ_2) is Hausdorff, then any continuous function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is closed (since it maps compacts to compacts).

Finally, later on we will need the following elementary lemma.

Lemma 6.3. *Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces, with τ_2 Hausdorff, and assume that $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is a continuous function. Then $\mathscr{G}_f$, the graph of the function f, is a closed subset of $(X_1 \times X_2, \tau_1 \times \tau_2)$.*

Proof. Let $(a, b) \in (X_1 \times X_2) \setminus \mathscr{G}_f$. Then $f(a) \neq b$ and, since (X_2, τ_2) is Hausdorff, there exist $U \in \mathscr{N}(f(a); \tau_2)$ and $V \in \mathscr{N}(b; \tau_2)$ such that $U \cap V = \emptyset$. Given that $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is continuous, there exists $O \in \mathscr{N}(a; \tau_1)$ such that $f(O) \subseteq U$. As such, $O \times V \in \mathscr{N}\big((a, b); \tau_1 \times \tau_2\big)$ and $(O \times V) \cap \mathscr{G}_f = \emptyset$. This shows that $\mathscr{G}_f$ is a closed subset of $(X_1 \times X_2, \tau_1 \times \tau_2)$. □

We continue by recording a useful lemma describing the manner in which a topology can be assigned starting from a system of axioms mimicking the properties of the neighborhood filter associated with a given topology.

Lemma 6.4. *Let X be a nonempty set, and suppose that $X \ni x \mapsto \mathcal{N}_x \subseteq 2^X$ assigns to each point in X a nonempty collection of subsets of X satisfying the following properties:*

$$(i) \qquad V \cap W \in \mathcal{N}_x \ \ \textit{whenever} \ \ x \in X \ \ \textit{and} \ \ V, W \in \mathcal{N}_x; \tag{6.14}$$

$$(ii) \qquad \begin{aligned} &W \in \mathcal{N}_x \ \ \textit{whenever} \ \ x \in X \ \ \textit{and} \ \ W \subseteq X \\ &\textit{is such that} \ \ \exists\, V \in \mathcal{N}_x \ \ \textit{with} \ \ V \subseteq W. notag \end{aligned} \tag{6.15}$$

Then

$$\tau := \big\{O \subseteq X : O \in \mathcal{N}_x \ \ \textit{for each} \ \ x \in O\big\} \tag{6.16}$$

is a topology on X satisfying

$$\mathscr{N}(x; \tau) \subseteq \mathcal{N}_x \quad \textit{for each} \ \ x \in X \tag{6.17}$$

and with the property that it is the largest topology on X for which (6.17) holds.

Furthermore, for τ as in (6.16) one actually has

$$\mathscr{N}(x; \tau) = \mathcal{N}_x \quad \textit{for each} \ \ x \in X \tag{6.18}$$

if and only if, in addition to (i) and (ii), the assignment $X \ni x \mapsto \mathcal{N}_x \subseteq 2^X$ also satisfies the following two properties:

$$(iii) \qquad x \in V \ \ \textit{for each} \ \ x \in X \ \ \textit{and each} \ \ V \in \mathcal{N}_x; \tag{6.19}$$

$$(iv) \qquad \begin{aligned} &\forall\, x \in X \ \ \textit{and} \ \ \forall\, V \in \mathcal{N}_x \ \ \textit{there exists} \ \ W \in \mathcal{N}_x \\ &\textit{so that} \ \ W \subseteq V \ \ \textit{and} \ \ W \in \mathcal{N}_y \ \ \textit{for each} \ \ y \in W. \end{aligned} \tag{6.20}$$

Proof. First assume that the assignment $X \ni x \mapsto \mathcal{N}_x \subseteq 2^X$ is such that

$$\mathcal{N}_x \neq \emptyset, \qquad \forall\, x \in X, \tag{6.21}$$

and it satisfies properties *(i)* and *(ii)*. Our goal is to show that τ introduced in (6.16) is a topology on X. Note first that $\emptyset \in \tau$, tautologically. Next, (6.21) and property *(ii)* imply that $X \in \mathcal{N}_x$ for any $x \in X$. Thus, by (6.16), we obtain that $X \in \tau$. Going further, consider $\{\mathcal{O}_i\}_{i\in I} \subseteq \tau$ and let $\mathcal{O} := \bigcup_{i\in I} \mathcal{O}_i$. With an eye toward showing that

$$\mathcal{O} \in \tau, \tag{6.22}$$

pick an arbitrary $x \in \mathcal{O}$ and observe that that there exists $i_o \in I$ such that $x \in \mathcal{O}_{i_o} \in \tau$. Hence, using property *(ii)* along with the fact that $\mathcal{O}_{i_o} \subseteq \mathcal{O}$, this further implies that (6.22) holds. Moving on, let $\mathcal{O}_j \in \tau$, $j = 1, 2$, and pick an arbitrary point $x \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then, by (6.16), we obtain that $\mathcal{O}_j \in \mathcal{N}_x$ for $j \in \{1, 2\}$, and consequently, using property *(i)*, we obtain that $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{N}_x$. Since $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, this further yields $\mathcal{O}_1 \cap \mathcal{O}_2 \in \tau$ and completes the proof of the fact that τ is a topology.

Next we turn our attention to proving (6.17) and start by picking a point $x \in X$ and $V \in \mathscr{N}(x;\tau)$. Then, there exists $\mathcal{O} \in \tau$ such that $x \in \mathcal{O} \subseteq V$. Consequently, from (6.16) and property *(ii)* it follows that $V \in \mathcal{N}_x$, completing the proof of (6.17). Moving on assume that $\widetilde{\tau}$ is a topology on X such that

$$\mathscr{N}(x;\widetilde{\tau}) \subseteq \mathcal{N}_x, \qquad \forall\, x \in X. \tag{6.23}$$

Pick next $\mathcal{O} \in \widetilde{\tau}$ and $x \in \mathcal{O}$. Since in this scenario $\mathcal{O} \in \mathscr{N}(x;\widetilde{\tau})$, this and (6.23) allow us to conclude that $\mathcal{O} \in \mathcal{N}_x$. Thanks to the fact that $x \in \mathcal{O}$ was arbitrary this further implies that $\mathcal{O} \in \tau$. Consequently, $\widetilde{\tau} \subseteq \tau$, completing the proof that τ is the largest topology on X for which (6.17) holds.

We are left with proving the last claim in the lemma having to do with the equivalence between (6.18) and (6.19), (6.20) whenever the assignment $X \ni x \mapsto \mathcal{N}_x \subseteq 2^X$ satisfies *(i)* and *(ii)*. First assume that (6.18) holds. Then, for any $x \in X$ and any $V \in \mathcal{N}_x$, thanks to (6.18) we have that $V \in \mathscr{N}(x;\tau)$, and consequently $x \in V$. Thus *(iii)* holds. Furthermore, if $x \in X$ and $V \in \mathcal{N}_x$, then, as above, $V \in \mathscr{N}(x;\tau)$, and consequently (by the definition of $\mathscr{N}(x;\tau)$) there exists $W \in \tau$ such that $W \subseteq V$. In turn, the membership $W \in \tau$ guarantees that $W \in \mathcal{N}_y$ for any $y \in W$. Thus *(iv)* holds as well.

Finally, assume that the assignment $X \ni x \mapsto \mathcal{N}_x \subseteq 2^X$ satisfies *(i)–(iv)*. Our goal is to prove the identity from (6.18). Thanks to (6.17), it suffices to show that the right-to-left inclusion in (6.18) holds. To see this, let $x \in X$ and $\mathcal{O} \in \mathcal{N}_x$. Using property *(iv)*, there exists $W \in \mathcal{N}_x$ satisfying $W \subseteq \mathcal{O}$ and $W \in \mathcal{N}_y$ for every $y \in W$. However, on the one hand, *(i)* and $W \in \mathcal{N}_x$ imply that $x \in W$, and on the other hand, $W \in \mathcal{N}_y$ for every $y \in W$ guarantees that $W \in \tau$. Combining these with the fact that $W \subseteq \mathcal{O}$ we obtain that $\mathcal{O} \in \mathcal{N}(x;\tau)$, establishing the right-to-left inclusion in (6.18) and completing the proof of the lemma. □

Remark 6.5. It is instructive to note that, under the assumption that (6.15) holds, condition (6.20) is further equivalent to the demand that

$$(iv)' \quad \forall\, x \in X \quad \text{and} \quad \forall\, V \in \mathcal{N}_x \quad \exists\, W \in \mathcal{N}_x \quad \text{such that} \quad V \in \mathcal{N}_y \quad \forall\, y \in W. \tag{6.24}$$

For the reader's convenience, we also record the following definition.

Definition 6.6. A topological space (X, τ) is called `Lindelöf` provided every open cover of X has a countable subcover.

Moving on, let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group, i.e., G is a set, $*$ is an associative binary operation on G, a^{-1} is the inverse of $a \in G$, and e_G is the neutral (or identity) element. In the subsequent discussion, we will often abbreviate $\big(G, *, (\cdot)^{-1}, e_G\big)$ by $(G, *)$ and, occasionally, simply by G. Given two groups $(G, *)$, $(S, \circ)$, denote by $\mathrm{Hom}\,(G, S)$ the collection of all (group) homomorphisms from G to S. That is, $T \in \mathrm{Hom}\,(G, S)$ if and only if $T : G \to S$ satisfies

$$T(a * b) = (Ta) \circ (Tb), \qquad \forall\, a, b \in G. \tag{6.25}$$

In turn, (6.25) used with $a = b = e_G$ forces

$$Te_G = e_S, \quad \forall\, T \in \mathrm{Hom}\,(G, S). \tag{6.26}$$

For each $T \in \mathrm{Hom}\,(G, S)$ we will denote its graph by

$$\mathscr{G}_T := \{(a, Ta) : a \in G\} \subseteq G \times S \tag{6.27}$$

and define the kernel and image of T, respectively, according to

$$\mathrm{Ker}\, T := \{a \in G : Ta = e_G\}, \quad \mathrm{Im}\, T := \{Ta : a \in G\}. \tag{6.28}$$

Fix now a group $(G, *)$. For each $a \in G$ define

$$a^n := \underbrace{a * a * \cdots * a}_{n \text{ factors}}, \quad \forall\, n \in \mathbb{N}. \tag{6.29}$$

Also, for any two arbitrary subsets A, B of G introduce the notation

$$A * B := \{a * b : (a, b) \in A \times B\}, \qquad A^{-1} := \big\{a^{-1} : a \in A\big\}, \tag{6.30}$$

and call $A \subseteq G$ symmetric if $A = A^{-1}$. In particular, for $A \subseteq G$ and $a \in G$ we will abbreviate

$$A * a := A * \{a\}, \quad a * A := \{a\} * A, \tag{6.31}$$

$$A^n := \underbrace{A * A * \cdots * A}_{n \text{ factors}}, \quad \forall\, n \in \mathbb{N}. \tag{6.32}$$

For each $A \subseteq G$ and each $n \in \mathbb{N}$, we will also use the abbreviation

$$A^{(n)} := \{a^n : a \in A\}. \tag{6.33}$$

We stress that (6.33) is not to be confused with (6.32). Indeed, we have $A^{(n)} \subseteq A^n$ for every $A \subseteq G$, but in general the inclusion is strict.

We continue by recording a couple of definitions that will be relevant in Sect. 4. First, we recall the following standard piece of terminology.

Definition 6.7. Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group. Define the order of an element $a \in G$ as

$$\operatorname{ord}(a) := \inf\{n \in \mathbb{N} : a^n = e_G\}, \tag{6.34}$$

with the convention that $\inf \emptyset := +\infty$. An element $a \in G$ is said to be a torsion element if it has finite order. If the only torsion element in G is the identity element, then the group G is said to be torsion-free.

Our next definition elaborates on the possibility of taking roots (with respect to the group multiplication) in a given group.

Definition 6.8. (i) Call a group G divisible[1] if for every $n \in \mathbb{N}$ and every $a \in G$ there exists $x \in G$ such that $x^n = a$.

(ii) Call a group G uniquely divisible if for every $n \in \mathbb{N}$ and every $a \in G$ there exists a unique $x \in G$ such that $x^n = a$. Henceforth, given $n \in \mathbb{N}$ and $a \in G$, the notation $\sqrt[n]{a}$ stands for the unique element $x \in G$ such that $x^n = a$.

It is then clear from Definitions 6.7 and 6.8 that

$$G \text{ torsion-free, divisible, Abelian group} \implies G \text{ uniquely divisible}, \tag{6.35}$$

and

$$G \text{ uniquely divisible} \implies G \text{ torsion-free and divisible}. \tag{6.36}$$

Next, given a group $(G, *)$, for each $a \in G$ denote by s_a^{R} the right-shift (or right-translation) by a, i.e.,

$$s_a^{\mathrm{R}} : G \longrightarrow G, \qquad s_a^{\mathrm{R}}(x) := x * a, \quad \forall\, x \in G, \tag{6.37}$$

[1] In contrast to the more common practice, here we do *not* make the background assumption that the group in question is Abelian.

and by s_a^{L} the left-shift (or left-translation) by a, i.e.,

$$s_a^{\mathrm{L}} : G \longrightarrow G, \qquad s_a^{\mathrm{L}}(x) := a * x, \quad \forall\, x \in G. \tag{6.38}$$

Then, obviously,

$$s_{e_G}^{\mathrm{R}} = s_{e_G}^{\mathrm{L}} = \mathrm{id}_G, \text{ the identity mapping of } G; \tag{6.39}$$

$$s_a^{\mathrm{R}} \circ s_b^{\mathrm{R}} = s_{b*a}^{\mathrm{R}} \text{ and } s_a^{\mathrm{L}} \circ s_b^{\mathrm{L}} = s_{a*b}^{\mathrm{L}}, \quad \forall\, a, b \in G; \tag{6.40}$$

$$s_a^{\mathrm{R}}, s_a^{\mathrm{L}} \text{ are bijective, } \big(s_a^{\mathrm{R}}\big)^{-1} = s_{a^{-1}}^{\mathrm{R}} \text{ and } \big(s_a^{\mathrm{L}}\big)^{-1} = s_{a^{-1}}^{\mathrm{L}}, \quad \forall\, a \in G. \tag{6.41}$$

Also, the following intertwining identities hold:

$$s_a^{\mathrm{R}} \circ (\cdot)^{-1} = (\cdot)^{-1} \circ s_{a^{-1}}^{\mathrm{L}}, \qquad s_a^{\mathrm{L}} \circ (\cdot)^{-1} = (\cdot)^{-1} \circ s_{a^{-1}}^{\mathrm{R}}, \qquad \forall\, a \in G. \tag{6.42}$$

Of particular interest is the case when a group is equipped with a topology that is compatible with the underlying algebraic structures, as described in the definition below.

Definition 6.9. A `topological group` is a group $\big(G, *, (\cdot)^{-1}, e_G\big)$ endowed with a topology τ on the set G with the property that the group operations $(\cdot)^{-1} : G \to G$ and $* : G \times G \to G$ are continuous functions (in the latter case, considering the product topology $\tau \times \tau$ on $G \times G$).

Alternatively, τ is called a group topology on G provided $(G, *, \tau)$ is a topological group.

Of course, given a group $(G, *)$ and a topology τ on G, in order for $(G, *, \tau)$ to be a topological group, it suffices to have

$$\big(G \times G, \tau \times \tau\big) \ni (x, y) \mapsto x * y^{-1} \in (G, \tau) \quad \text{continuous.} \tag{6.43}$$

Nonetheless, in the sequel we will work with groups equipped with topologies that satisfy weaker conditions than those required to render them topological groups. This is made precise in our next definition.

Definition 6.10. Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group, and assume that τ is a topology on the set G.

(i) Call τ a `symmetric topology` provided

$$\forall\, V \in \mathscr{N}(e_G; \tau)\ \exists\, W \in \mathscr{N}(e_G; \tau) \text{ such that } W \text{ is symmetric and } W \subseteq V. \tag{6.44}$$

(ii) Call τ a `right-invariant topology` provided

$$s_a^{\mathrm{R}} : (G, \tau) \longrightarrow (G, \tau) \quad \text{is continuous for every } a \in G. \tag{6.45}$$

Also, call τ a `left-invariant topology` if $s_a^{\mathrm{L}} : (G, \tau) \to (G, \tau)$ is continuous for every $a \in G$.

(iii) Call (G, τ) a `semitopological group` provided the inverse operation and all right-shifts are continuous, i.e.,

$$(\cdot)^{-1} : (G, \tau) \longrightarrow (G, \tau) \;\text{ and }\; s_a^{\mathrm{R}} : (G, \tau) \longrightarrow (G, \tau), \quad \forall\, a \in G, \tag{6.46}$$

are continuous functions.

Obviously, any topological group is a semitopological group. Other properties of interest are collected in the lemma below.

Lemma 6.11. *Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group, and assume that τ is a topology on the set G.*

(i) The following equivalences hold:

τ is a symmetric topology on G

$$\begin{aligned} &\Longleftrightarrow (\cdot)^{-1} : (G, \tau) \longrightarrow (G, \tau) \;\text{ is continuous at } e_G \\ &\Longleftrightarrow V^{-1} \in \mathscr{N}(e_G; \tau) \;\text{ for every }\; V \in \mathscr{N}(e_G; \tau), \end{aligned} \tag{6.47}$$

and

τ is a right-invariant topology on G

$$\Longleftrightarrow s_a^{\mathrm{R}} : (G, \tau) \longrightarrow (G, \tau) \;\text{ is a homeomorphism, }\; \forall\, a \in G, \tag{6.48}$$

and

τ is a left-invariant topology on G

$$\Longleftrightarrow s_a^{\mathrm{L}} : (G, \tau) \longrightarrow (G, \tau) \;\text{ is a homeomorphism, }\; \forall\, a \in G. \tag{6.49}$$

(ii) If (G, τ) is a semitopological group, then τ is symmetric and both left-invariant and right-invariant.

(iii) In any semitopological group, the inverse operation and all right-shifts and left-shifts are in fact homeomorphisms.

Proof. To justify the sequence of equivalences in *(i)*, assume first that τ is a symmetric topology on G and pick an arbitrary $V \in \mathscr{N}(e_G; \tau)$. Then we may find some set $W \in \mathscr{N}(e_G; \tau)$ such that W is symmetric and $W \subseteq V$. This implies that $W^{-1} = W \subseteq V$, which shows that the function $(\cdot)^{-1} : (G, \tau) \to (G, \tau)$ is continuous at e_G. Next, if $(\cdot)^{-1} : (G, \tau) \to (G, \tau)$ is continuous at e_G, then for any $V \in \mathscr{N}(e_G; \tau)$ there exists $U \in \mathscr{N}(e_G; \tau)$ such that $U^{-1} \subseteq V$. This forces $U \subseteq V^{-1}$ and, hence, $V^{-1} \in \mathscr{N}(e_G; \tau)$. Finally, assume that $V^{-1} \in \mathscr{N}(e_G; \tau)$

for every $V \in \mathscr{N}(e_G;\tau)$ and pick an arbitrary $U \in \mathscr{N}(e_G;\tau)$. Then $W := U \cap U^{-1} \in \mathscr{N}(e_G;\tau)$ is symmetric and $W \subseteq U$, which proves that τ is a symmetric topology on G. This concludes the proof of the equivalences in (6.47). Of course, the equivalences in (6.49) are dealt with similarly.

As regards part (ii), thanks to the intertwining identities from (6.42), the fact that the functions in (6.46) are continuous implies that the left-shift functions $s_a^{\mathrm{L}} : (G,\tau) \to (G,\tau)$, $a \in G$, are continuous as well. Finally, the claim in part (iii) is readily seen from (6.41). □

Lemma 6.12. *Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group, and assume that τ is a topology on the set G. Then the following assertions are true:*

(1) If τ is symmetric, then

$$U^{-1} \in \mathscr{N}(e_G;\tau) \text{ and } U * V \in \mathscr{N}(e_G;\tau), \quad \forall\, U, V \in \mathscr{N}(e_G;\tau). \tag{6.50}$$

(2) If τ is symmetric and right-invariant, then

$$\mathrm{Clo}(A;\tau) = \bigcap_{V \in \mathscr{N}(e_G;\tau)} V * A, \qquad \forall\, A \subseteq G, \tag{6.51}$$

whereas if τ is symmetric and left-invariant, then

$$\mathrm{Clo}(A;\tau) = \bigcap_{V \in \mathscr{N}(e_G;\tau)} A * V, \qquad \forall\, A \subseteq G. \tag{6.52}$$

(3) If the topology τ is symmetric and two-sided invariant (in particular if (G,τ) is a semitopological group), then

$$\mathrm{Clo}(A;\tau) = \bigcap_{V,W \in \mathscr{N}(e_G;\tau)} V * A * W, \qquad \forall\, A \subseteq G, \tag{6.53}$$

$$\mathrm{Clo}(A;\tau) * \mathrm{Clo}(B;\tau) \subseteq \mathrm{Clo}(A * B;\tau), \qquad \forall\, A, B \subseteq G. \tag{6.54}$$

(4) If τ is symmetric, then the mapping

$$\big(G \times G, \tau \times \tau\big) \ni (x,y) \mapsto x * y \in (G,\tau) \quad \textit{is continuous}, \tag{6.55}$$

and if τ is either right-invariant or left-invariant, then

$$\forall\, V \in \mathscr{N}(e_G;\tau)\ \exists\, W \in \mathscr{N}(e_G;\tau) \text{ such that } \mathrm{Clo}(W;\tau) \subseteq V. \tag{6.56}$$

Proof. Given any $U \in \mathscr{N}(e_G;\tau)$, part (iii) in Remark 6.11, together with (6.4), proves that $U^{-1} \in \mathscr{N}(e_G;\tau)$. For any sets $U, V \in \mathscr{N}(e_G;\tau)$, observe that $U * V \in \mathscr{N}(e_G;\tau)$ since $\mathscr{N}(e_G;\tau) \ni U \subseteq U * V$, given that $e_G \in V$. This proves (6.50).

Next, if $A \subseteq G$ is arbitrary, then for each $a \in G$ we may write (by once again relying on part (iii) in Remark 6.11 and (6.4)) that

$$\begin{aligned} a \notin \mathrm{Clo}(A;\tau) &\Longleftrightarrow \exists\, V \in \mathscr{N}(e_G;\tau) \text{ such that } (V * a) \cap A = \emptyset \\ &\Longleftrightarrow a \notin V^{-1} * A. \end{aligned} \tag{6.57}$$

In concert with the first formula in (6.50), this proves the first equality in (6.51). The second equality in (6.51) is justified in a similar manner. Moreover, having established these two equalities, we may now make use of them and the second formula in (6.50) to write

$$\begin{aligned} \bigcap_{V,W\in\mathscr{N}(e_G;\tau)} V * A * W &= \bigcap_{V\in\mathscr{N}(e_G;\tau)} \Big(\bigcap_{W\in\mathscr{N}(e_G;\tau)} (V * A) * W \Big) \\ &= \bigcap_{V\in\mathscr{N}(e_G;\tau)} \mathrm{Clo}\,(V * A;\tau) \\ &= \bigcap_{V\in\mathscr{N}(e_G;\tau)} \Big(\bigcap_{W\in\mathscr{N}(e_G;\tau)} W * (V * A) \Big) \\ &= \bigcap_{V,W\in\mathscr{N}(e_G;\tau)} W * V * A \subseteq \bigcap_{U\in\mathscr{N}(e_G;\tau)} U * A \\ &= \mathrm{Clo}\,(A;\tau). \end{aligned} \tag{6.58}$$

On the other hand, since $e_G \in W$ for every $W \in \mathscr{N}(e_G;\tau)$, it follows that

$$\mathrm{Clo}\,(A;\tau) = \bigcap_{V\in\mathscr{N}(e_G;\tau)} V * A \subseteq \bigcap_{V,W\in\mathscr{N}(e_G;\tau)} V * A * W. \tag{6.59}$$

Now, the last equality in (6.51) is a consequence of (6.58) and (6.59). Next, given two arbitrary sets $A, B \subseteq G$, we have, thanks to (6.51),

$$\begin{aligned} \mathrm{Clo}(A;\tau) * \mathrm{Clo}(B;\tau) &= \Big(\bigcap_{V\in\mathscr{N}(e_G;\tau)} V * A \Big) * \Big(\bigcap_{W\in\mathscr{N}(e_G;\tau)} B * W \Big) \\ &\subseteq \bigcap_{V,W\in\mathscr{N}(e_G;\tau)} V * A * B * W = \mathrm{Clo}(A * B;\tau), \end{aligned} \tag{6.60}$$

and (6.54) follows. Finally, assume that the topology τ is as in part (4) in the statement of the lemma. Then for each $V \in \mathscr{N}(e_G;\tau)$ the continuity of the mapping (6.55) entails the existence of some $W \in \mathscr{N}(e_G;\tau)$ such that $W * W \subseteq V$.

Since $\mathrm{Clo}(W;\tau) \subseteq W * W$ by (2) and the properties of τ, we deduce that $\mathrm{Clo}(W;\tau) \subseteq V$, and this completes the proof of the lemma. ☐

We next describe the topology induced on a given group G by an arbitrary nonnegative (and possibly infinite) function ψ defined on G. Recall that, in the more general context of groupoids, this was done in Definition 2.62. For the reader's convenience, below we specialize the latter definition to the particular case of a group G (in which scenario $G^{(2)} = G \times G$).

Definition 6.13. Given a group $(G, *, (\cdot)^{-1}, e_G)$ and a function $\psi : G \to [0, +\infty]$, define the right and left ψ-balls centered at $a \in G$ and with radius $r \in (0, +\infty)$ respectively as

$$B^{\mathrm{R}}_{\psi}(a, r) := \big\{x \in G : \psi(a * x^{-1}) < r\big\}, \tag{6.61}$$

$$B^{\mathrm{L}}_{\psi}(a, r) := \big\{x \in G : \psi(x^{-1} * a) < r\big\}. \tag{6.62}$$

Then the `right-topology` τ^{R}_{ψ} induced by ψ on G is defined by

$$\tau^{\mathrm{R}}_{\psi} := \big\{O \subseteq G : \forall\, a \in O \ \exists\, r \in (0, +\infty) \text{ such that } B^{\mathrm{R}}_{\psi}(a, r) \subseteq O\big\}, \tag{6.63}$$

whereas the `left-topology` τ^{L}_{ψ} induced by ψ on G is defined by

$$\tau^{\mathrm{L}}_{\psi} := \big\{O \subseteq G : \forall\, a \in O \ \exists\, r \in (0, +\infty) \text{ such that } B^{\mathrm{L}}_{\psi}(a, r) \subseteq O\big\}. \tag{6.64}$$

In our next lemma, we explore the extent to which the topology induced by a nonnegative function defined on a given group is compatible with the algebraic structure. Before stating this, we remind the reader that, given a topological space (X, τ) and a point $x \in X$, a subset $\mathscr{U}$ of $\mathcal{N}(x;\tau)$ is called a fundamental system of neighborhoods of x (in the topology τ) if for every $V \in \mathcal{N}(x;\tau)$ there exists $U \in \mathscr{U}$ such that $U \subseteq V$.

Lemma 6.14. *Assume that $(G, *)$ is a group and that $\psi : G \to [0, +\infty]$ is an arbitrary function.*

(i) Both τ^{R}_{ψ} and τ^{L}_{ψ} are topologies on G, having the following properties:

$$\mathrm{Int}\big(A; \tau^{\mathrm{R}}_{\psi}\big) \subseteq \big\{a \in A : \exists\, r \in (0, +\infty) \ \textit{such that} \ B^{\mathrm{R}}_{\psi}(a, r) \subseteq A\big\}, \tag{6.65}$$

$$\mathrm{Int}\big(A; \tau^{\mathrm{L}}_{\psi}\big) \subseteq \big\{a \in A : \exists\, r \in (0, +\infty) \ \textit{such that} \ B^{\mathrm{L}}_{\psi}(a, r) \subseteq A\big\} \tag{6.66}$$

for every $A \subseteq G$ and

$$\forall\, V \in \mathscr{N}\big(a;\tau_\psi^{\mathrm{R}}\big)\ \ \exists\, r \in (0,+\infty)\ \ \textit{such that}\ \ B_\psi^{\mathrm{R}}(a,r) \subseteq V, \tag{6.67}$$

$$\forall\, V \in \mathscr{N}\big(a;\tau_\psi^{\mathrm{L}}\big)\ \ \exists\, r \in (0,+\infty)\ \ \textit{such that}\ \ B_\psi^{\mathrm{L}}(a,r) \subseteq V \tag{6.68}$$

for each $a \in X$.

(ii) For each $a \in G$ the shifts

$$s_a^{\mathrm{R}} : \big(G,\tau_\psi^{\mathrm{R}}\big) \longrightarrow \big(G,\tau_\psi^{\mathrm{R}}\big), \qquad s_a^{\mathrm{L}} : \big(G,\tau_\psi^{\mathrm{L}}\big) \longrightarrow \big(G,\tau_\psi^{\mathrm{L}}\big) \tag{6.69}$$

are homeomorphisms. Furthermore, for each $x \in G$ and $r \in (0,+\infty)$

$$s_a^{\mathrm{R}}\Big(B_\psi^{\mathrm{R}}(x,r)\Big) = B_\psi^{\mathrm{R}}\big(s_a^{\mathrm{R}}(x),r\big) \quad \textit{and} \quad s_a^{\mathrm{L}}\Big(B_\psi^{\mathrm{L}}(x,r)\Big) = B_\psi^{\mathrm{L}}\big(s_a^{\mathrm{L}}(x),r\big). \tag{6.70}$$

(iii) One has

$$a \in B_\psi^{\mathrm{R}}(a,r)\ \ \forall\, a \in G,\ \ \forall\, r > 0 \ \Leftrightarrow\ \psi(e_G) = 0$$
$$\Leftrightarrow\ a \in B_\psi^{\mathrm{L}}(a,r)\ \ \forall\, a \in G,\ \ \forall\, r > 0. \tag{6.71}$$

(iv) The mappings

$$(\cdot)^{-1} : \big(G,\tau_\psi^{\mathrm{R}}\big) \longrightarrow \big(G,\tau_\psi^{\mathrm{L}}\big), \qquad (\cdot)^{-1} : \big(G,\tau_\psi^{\mathrm{L}}\big) \longrightarrow \big(G,\tau_\psi^{\mathrm{R}}\big) \tag{6.72}$$

are homeomorphisms (that are inverse to one another) provided the function ψ has the property that

$$\forall\, \varepsilon > 0\ \ \exists\, \delta > 0,\ \ \textit{so that if}\ \ x \in G\ \ \textit{satisfies}\ \ \psi(x) < \delta,\ \ \textit{then}\ \ \psi(x^{-1}) < \varepsilon. \tag{6.73}$$

(v) Assume that the function ψ satisfies

$$\begin{aligned} &\forall\, \varepsilon > 0\ \ \exists\, \delta > 0\ \textit{with the property that}\ \psi(x * y) < \varepsilon \\ &\quad \textit{whenever}\ \ x,y \in G\ \ \textit{satisfy}\ \ \psi(x) < \delta,\ \ \psi(y) < \delta. \end{aligned} \tag{6.74}$$

Then for every $A \subseteq G$ one has

$$\mathrm{Int}\big(A;\tau_\psi^{\mathrm{R}}\big) = \big\{a \in A : \exists\, r \in (0,+\infty)\ \ \textit{such that}\ \ B_\psi^{\mathrm{R}}(a,r) \subseteq A\big\}, \tag{6.75}$$

$$\mathrm{Int}\big(A;\tau_\psi^{\mathrm{L}}\big) = \big\{a \in A : \exists\, r \in (0,+\infty)\ \ \textit{such that}\ \ B_\psi^{\mathrm{L}}(a,r) \subseteq A\big\}. \tag{6.76}$$

Furthermore, if ψ satisfies (6.74) and $\psi(e_G) = 0$, then

$$B_\psi^{\mathrm{R}}(a,r) \in \mathscr{N}\big(a;\tau_\psi^{\mathrm{R}}\big)\ \ \textit{and}\ \ B_\psi^{\mathrm{L}}(a,r) \in \mathscr{N}\big(a;\tau_\psi^{\mathrm{L}}\big), \quad \forall\, a \in G,\ \ \forall\, r > 0. \tag{6.77}$$

In fact, in this scenario, for each $a \in G$

$$\left\{B^{\mathrm{R}}_{\psi}(a,r)\right\}_{r>0} \quad \textit{is a fundamental system of neighborhoods of } a \textit{ in } \tau^{\mathrm{R}}_{\psi} \tag{6.78}$$

and

$$\left\{B^{\mathrm{L}}_{\psi}(a,r)\right\}_{r>0} \quad \textit{is a fundamental system of neighborhoods of } a \textit{ in } \tau^{\mathrm{L}}_{\psi}. \tag{6.79}$$

(vi) If the function ψ satisfies (6.73), (6.74), and

$$\psi^{-1}(\{0\}) = \{e_G\}, \tag{6.80}$$

then the topological spaces $(G, \tau^{\mathrm{R}}_{\psi})$ and $(G, \tau^{\mathrm{L}}_{\psi})$ are Hausdorff.

(vii) If the function ψ has the property that

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \textit{so that} \;\; \psi(y * x) < \varepsilon \;\; \textit{if} \;\; x, y \in G \;\; \textit{satisfy} \;\; \psi(x * y) < \delta; \tag{6.81}$$

then

$$\tau^{\mathrm{R}}_{\psi} = \tau^{\mathrm{L}}_{\psi}. \tag{6.82}$$

(viii) Suppose that (6.81) holds and, in such a scenario, set $\tau_{\psi} := \tau^{\mathrm{R}}_{\psi} (= \tau^{\mathrm{L}}_{\psi})$. In addition, assume that the function ψ satisfies (6.73) and (6.74). Then

$$\big(G, *, \tau_{\psi}\big) \quad \textit{is a topological group.} \tag{6.83}$$

Proof. It is clear from definitions that τ^{R}_{ψ} and τ^{L}_{ψ} are topologies on G and that (6.65) and (6.66) hold. In turn, the latter formulas readily imply (6.67) and (6.68). The fact that the mappings in (6.69) are homeomorphisms follows from (6.41) and (6.63), (6.64) after observing that (recall (2.151) and the convention made in (6.31))

$$B^{\mathrm{R}}_{\psi}(a,r) * b = B^{\mathrm{R}}_{\psi}(a * b, r), \qquad b * B^{\mathrm{L}}_{\psi}(a,r) = B^{\mathrm{L}}_{\psi}(b * a, r) \tag{6.84}$$

for all $a, b \in G$ and $r \in (0, +\infty)$. As a byproduct, the identities in (6.70) also follow. In addition, the equivalences in (6.71) are clear from definitions. Going further, one can check without difficulty that [recall the piece of notation introduced in (6.30)]

$$\Big(B^{\mathrm{R}}_{\psi}(a,r)\Big)^{-1} = B^{\mathrm{L}}_{\psi\circ(\cdot)^{-1}}(a^{-1}, r) \quad \text{and} \quad \Big(B^{\mathrm{L}}_{\psi}(a,r)\Big)^{-1} = B^{\mathrm{R}}_{\psi\circ(\cdot)^{-1}}(a^{-1}, r) \tag{6.85}$$

for every $a \in G$ and $r \in (0, +\infty)$. In light of (6.85), condition (6.73) then becomes equivalent to the demand that

$$\begin{gathered}\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \text{such that for each} \;\; a \in G \;\; \text{one has} \\ B^{\rm L}_\psi(a^{-1}, \delta) \subseteq \Big(B^{\rm R}_\psi(a, \varepsilon)\Big)^{-1} \;\; \text{and} \;\; B^{\rm R}_\psi(a^{-1}, \delta) \subseteq \Big(B^{\rm L}_\psi(a, \varepsilon)\Big)^{-1}.\end{gathered} \tag{6.86}$$

With this in hand, it readily follows that the maps in (6.72) are homeomorphisms if (6.73) holds.

Next, consider (6.75) for some fixed $A \subseteq G$. To facilitate the presentation, we will temporarily use the notation

$$\widetilde{A} := \big\{a \in A : \exists\, r \in (0, +\infty) \;\; \text{such that} \;\; B^{\rm R}_\psi(a, r) \subseteq A\big\}. \tag{6.87}$$

In this regard, we make the claim that $\widetilde{A} \in \tau^{\rm R}_\psi$. To justify this claim, pick some $a \in \widetilde{A}$. Then there exists $r > 0$ such that $B^{\rm R}_\psi(a, r) \subseteq A$, and (6.74) guarantees the existence of some $\delta > 0$ with the property that

$$\psi(x * y) < r \;\; \text{whenever} \;\; x, y \in G \;\; \text{are such that } \psi(x) < \delta, \;\; \psi(y) < \delta. \tag{6.88}$$

Consider now two arbitrary elements, $b \in B^{\rm R}_\psi(a, \delta)$ and $c \in B^{\rm R}_\psi(b, \delta)$. Then we may write $\psi(a * c^{-1}) = \psi\big((a * b^{-1}) * (b * c^{-1})\big) < r$ by (6.88) and the fact that $\psi(a * b^{-1}) < \delta$ and $\psi(b * c^{-1}) < \delta$. This proves that $B^{\rm R}_\psi(b, \delta) \subseteq B^{\rm R}_\psi(a, r)$, hence $B^{\rm R}_\psi(b, \delta) \subseteq A$ for every element $b \in B^{\rm R}_\psi(a, \delta)$. Thus, ultimately, $B^{\rm R}_\psi(a, \delta) \subseteq \widetilde{A}$, proving that $\widetilde{A} \in \tau^{\rm R}_\psi$. Since by design $\widetilde{A} \subseteq A$, we may therefore conclude that $\widetilde{A} \subseteq {\rm Int}\big(A; \tau^{\rm R}_\psi\big)$. Since the converse inclusion is contained in (6.65), this completes the proof of (6.75). Of course, (6.76) is established similarly.

If ψ satisfies (6.74) and $\psi(e_G) = 0$, then, thanks to (6.75)–(6.76), we have

$$a \in {\rm Int}\big(B^{\rm R}_\psi(a, r); \tau^{\rm R}_\psi\big) \cap {\rm Int}\big(B^{\rm L}_\psi(a, r); \tau^{\rm L}_\psi\big), \qquad \forall\, a \in G, \;\; \forall\, r > 0, \tag{6.89}$$

from which (6.77) follows. In turn, (6.78) and (6.79) are immediate consequences of (6.77).

Suppose now that the function ψ satisfies (6.73), (6.74), and (6.80), and pick $a, b \in G$ with $a \neq b$. Then $a * b^{-1} \neq e_G$ and, as such, $\psi(a * b^{-1}) > 0$. Pick $\varepsilon \in \big(0, \psi(a * b^{-1})\big)$, and let $\delta > 0$ be associated with this ε as in (6.74). Also, making use of (6.73), select $\delta_1 \in (0, \delta)$ such that

$$\psi(x^{-1}) < \delta \;\; \text{if } x \in G \text{ satisfies } \psi(x) < \delta_1. \tag{6.90}$$

We claim that

$$B^{\rm R}_\psi(a, \delta_1) \cap B^{\rm R}_\psi(b, \delta_1) = \emptyset. \tag{6.91}$$

To see this, reason by contradiction and assume that there exists $c \in B^{\mathrm{R}}_{\psi}(a, \delta_1) \cap B^{\mathrm{R}}_{\psi}(b, \delta_1)$. Then, since $\psi(b * c^{-1}) < \delta_1$, it follows from (6.90) that $\psi(c * b^{-1}) < \delta$. With this in hand, we have, thanks to our choice of δ and the fact that $\delta_1 < \delta$,

$$\psi(a * b^{-1}) = \psi\big((a * c^{-1}) * (c * b^{-1})\big) < \varepsilon. \tag{6.92}$$

This contradiction proves (6.91). Keeping in mind (6.77), we may then conclude from (6.91) that the topological space $(G, \tau^{\mathrm{R}}_{\psi})$ is Hausdorff. Moreover, a similar reasoning applies in the case of $(G, \tau^{\mathrm{L}}_{\psi})$.

Next, condition (6.81) is equivalent to the demand that

$$\begin{gathered} \forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that for each } \ a \in G \ \text{ one has} \\ B^{\mathrm{L}}_{\psi}(a, \delta) \subseteq B^{\mathrm{R}}_{\psi}(a, \varepsilon) \ \text{ and } \ B^{\mathrm{R}}_{\psi}(a, \delta) \subseteq B^{\mathrm{L}}_{\psi}(a, \varepsilon), \end{gathered} \tag{6.93}$$

which, in turn, readily yields (6.82). Finally, if (6.81) and (6.73) hold, then from (6.82) and (6.72) we know that the mapping $(\cdot)^{-1} : \big(G, \tau_{\psi}\big) \to \big(G, \tau_{\psi}\big)$ is a homeomorphism. Thus, as far as (6.83) is concerned, there remains to show that

$$p : (G \times G, \tau_{\psi} \times \tau_{\psi}) \longrightarrow (G, \tau_{\psi}), \ \ p(x, y) := x * y, \ \ \forall\, x, y \in G, \ \text{ is continuous,} \tag{6.94}$$

i.e., that $p^{-1}(O) \in \tau_{\psi} \times \tau_{\psi}$ for each $O \in \tau_{\psi}$. Unraveling definitions, it is apparent that it suffices to check that for every $x_o, y_o \in G$ and any $r > 0$ there exists $\varepsilon > 0$ with the property that

$$B^{\mathrm{R}}_{\psi}(x_o, \varepsilon) * B^{\mathrm{R}}_{\psi}(y_o, \varepsilon) \subseteq B^{\mathrm{R}}_{\psi}(x_o * y_o, r). \tag{6.95}$$

To this end, fix $x_o, y_o \in G$ along with $r > 0$ and, for some $\varepsilon > 0$ to be specified later, select two arbitrary elements $x \in B^{\mathrm{R}}_{\psi}(x_o, \varepsilon)$ and $y \in B^{\mathrm{R}}_{\psi}(y_o, \varepsilon)$. The goal is to specify ε such that we necessarily have $\psi\big(x_o * y_o * (x * y)^{-1}\big) < r$. The latter condition is equivalent to $\psi\big(x_o * y_o * y^{-1} * x^{-1}\big) < r$, and, granted (6.81), there exists $\delta_1 > 0$ such that this is true provided the inequality $\psi\big(x^{-1} * x_o * y_o * y^{-1}\big) < \delta_1$ holds. In turn, granted (6.74), there exists $\delta_2 > 0$ such this inequality holds if

$$\psi\big(x^{-1} * x_o\big) < \delta_2 \ \text{ and } \ \psi\big(y_o * y^{-1}\big) < \delta_2. \tag{6.96}$$

Note that $y \in B^{\mathrm{R}}_{\psi}(y_o, \varepsilon)$ forces $\psi\big(y_o * y^{-1}\big) < \varepsilon$, so the second inequality in (6.96) is automatically satisfied if $\varepsilon < \delta_2$. Moreover, thanks to (6.81), there exists $\delta_3 > 0$ such that the first inequality in (6.96) holds provided $\psi\big(x_o * x^{-1}\big) < \delta_3$. However, given that $x \in B^{\mathrm{R}}_{\psi}(x_o, \varepsilon)$, this last inequality will hold provided $\varepsilon < \delta_3$. All in all, (6.95) is verified if we choose $\varepsilon \in \big(0, \min\{\delta_2, \delta_3\}\big)$. This completes the proof of (6.83). □

We conclude this section by making a series of definitions that will be relevant in the next section.

Definition 6.15. Let $(G, *)$ be a group, and suppose that $\psi : G \to [0, +\infty]$ is an arbitrary function. Then for each $\varepsilon > 0$ define the `right ε-enhancement` (with respect to ψ) of a given set $A \subseteq G$ as

$$[A]^{\mathrm{R}}_{\psi,\varepsilon} := \bigcup_{a \in A} B^{\mathrm{R}}_{\psi}(a, \varepsilon), \tag{6.97}$$

and define the `left ε-enhancement` (with respect to ψ) of a given set $A \subseteq G$ as

$$[A]^{\mathrm{L}}_{\psi,\varepsilon} := \bigcup_{a \in A} B^{\mathrm{L}}_{\psi}(a, \varepsilon). \tag{6.98}$$

Definition 6.16. Let $(G, *)$ be a group, and suppose that τ is a topology on the set G. Also, suppose that a sequence $(a_i)_{i \in \mathbb{N}} \subseteq G$ and an element $a \in G$ have been given. Call the sequence $(a_i)_{i \in \mathbb{N}}$ `right-convergent to` a with respect to τ provided

$$\forall\, O \in \mathscr{N}(e_G; \tau)\ \exists\, n_O \in \mathbb{N} \ \text{ such that } \ a_i * a^{-1} \in O, \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq n_O. \tag{6.99}$$

Also, call the sequence $(a_i)_{i \in \mathbb{N}}$ `left-convergent to` a with respect to τ provided

$$\forall\, O \in \mathscr{N}(e_G; \tau)\ \exists\, n_O \in \mathbb{N} \ \text{ such that } \ a^{-1} * a_i \in O, \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq n_O. \tag{6.100}$$

In connection with Definition 6.16, we wish to note that

$$(a_i)_{i \in \mathbb{N}} \text{ is left-convergent to } e_G \iff (a_i)_{i \in \mathbb{N}} \text{ is right-convergent to } e_G. \tag{6.101}$$

In such a case, we will drop the adjectives left/right and simply refer to $(a_i)_{i \in \mathbb{N}}$ as being convergent to e_G (with respect to τ).

Comment 6.17. Attention should be paid to the fact that there is yet another type of convergence, namely, the ordinary notion of a sequence to a limit in a topological space. To distinguish this from the notions introduced in Definition 6.16, we will refer to it as `ordinary convergence`. Hence, given a group $(G, *)$ and a topology τ on the set G, a sequence $(a_i)_{i \in \mathbb{N}} \subseteq G$ is ordinarily convergent to an element $a \in G$ provided

$$\forall\, V \in \mathscr{N}(a; \tau)\ \exists\, n_V \in \mathbb{N} \ \text{ such that } \ a_i \in V, \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq n_V. \tag{6.102}$$

In particular,

$$\begin{aligned}&\text{right-convergence reduces to ordinary convergence}\\&\text{if the topology in question is right-invariant.}\end{aligned} \tag{6.103}$$

Of course, similar considerations apply to left-convergence. ■

Definition 6.18. Consider a group $(G, *, (\cdot)^{-1}, e_G)$ together with a topology τ on the set G. Then $(G, *, (\cdot)^{-1}, e_G, \tau)$ is called `topologically divisible` provided the group G is uniquely divisible and

$$\text{for each } a \in G \text{ the sequence } (\sqrt[n]{a})_{n\in\mathbb{N}} \text{ is convergent to } e_G \text{ in the topology } \tau \tag{6.104}$$

(recall that for each $n \in \mathbb{N}$ and each $a \in G$, $\sqrt[n]{a}$ denotes the unique element $x \in G$ such that $x^n = a$).

The following lemma will be useful in the treatment of the OMT later on.

Lemma 6.19. *Let $(G, *)$ and $(S, \circ)$ be two groups, and assume that τ is a topology on the set S with the property that $(S, \circ, (\cdot)^{-1}, e_S, \tau)$ is topologically divisible. If $T \in \mathrm{Hom}\,(G, S)$ is such that $\mathrm{Im}\,T$ is open in (S, τ), then T is surjective.*

Proof. Start by fixing an element $a \in G$. Since the set $\mathrm{Im}\,T$ is open in (S, τ), it follows that $T(G) \in \mathcal{N}(e_S; \tau)$. This, combined with the fact that $(S, \circ, (\cdot)^{-1}, e_S, \tau)$ is topologically divisible, guarantees that there exists $n \in \mathbb{N}$ such that $\sqrt[n]{a} \in T(G)$. Consequently, there exists $x \in G$ such that $Tx = \sqrt[n]{a}$. From this and the fact that $T \in \mathrm{Hom}\,(G, S)$ it immediately follows that $T(x^n) = a$, which proves that T is surjective. □

Definition 6.20. Let $(G, *)$ be a group, and suppose that τ is a topology on the set G. Call a sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ `right-Cauchy with respect to` τ provided

$$\begin{aligned}&\forall\, O \in \mathcal{N}(e_G; \tau)\ \exists\, n_O \in \mathbb{N} \ \text{ with the property that}\\&a_i * a_j^{-1} \in O, \ \ \forall\, i, j \in \mathbb{N} \ \text{ with } \ \min\{i, j\} \geq n_O.\end{aligned} \tag{6.105}$$

Also, call a sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ `left-Cauchy with respect to` τ provided

$$\begin{aligned}&\forall\, O \in \mathcal{N}(e_G; \tau)\ \exists\, n_O \in \mathbb{N} \ \text{ with the property that}\\&a_i^{-1} * a_j \in O, \ \ \forall\, i, j \in \mathbb{N} \ \text{ with } \ \min\{i, j\} \geq n_O.\end{aligned} \tag{6.106}$$

Definition 6.21. Let $(G, *)$ be a group, and suppose that τ is a topology on the set G. Call G `right-complete with respect to` τ provided every sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ that is right-Cauchy with respect to τ is also right-convergent in the topology τ to some element $a \in G$.

Likewise, call the group G `left-complete with respect to` τ provided every sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ that is left-Cauchy with respect to τ is also left-convergent in the topology τ to some element $a \in G$.

6.2 Quasi-Pseudonormed Groups and an Extension of the Birkhoff–Kakutani Theorem

Recall that, given a group $\big(G, *, (\cdot)^{-1}, e_G\big)$, a function $\psi : G \to [0, +\infty)$ is called a `norm on` G provided

$$(i)\ \psi^{-1}\big(\{0\}\big) = \{e_G\}; \tag{6.107}$$

$$(ii)\ \psi(x^{-1}) = \psi(x), \quad \forall\, x \in G; \tag{6.108}$$

$$(iii)\ \psi(x * y) \le \psi(x) + \psi(y), \quad \forall\, x, y \in G. \tag{6.109}$$

Moreover, a norm ψ on G is said to be `invariant` provided[2]

$$\psi(x^{-1} * y * x) = \psi(y), \quad \forall\, x, y \in G. \tag{6.110}$$

When axiom (i) is relaxed to $\psi(e_G) = 0$, while (ii) and (iii) as stated previously are retained, the corresponding function ψ is typically called a `pseudonorm` (in the sense of Markov). In many situations of practical interest, it is desirable to also weaken axioms (ii) and (iii) and to allow the function ψ to eventually be infinite. We thus arrive at the notion of quasi-pseudonorm on a group, formally introduced below.

Definition 6.22. Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a given group.

(i) Call a function $\psi : G \to [0, +\infty]$ a `quasi-pseudonorm on` G provided there exist constants $C_0, C_1 \in [1, +\infty)$ with the property that

$$\texttt{[vanishing condition]}\quad \psi(e_G) = 0, \tag{6.111}$$

$$\texttt{[quasi-symmetry]}\quad \psi(x^{-1}) \le C_0\, \psi(x), \quad \forall\, x \in G, \tag{6.112}$$

$$\texttt{[quasi-subadditivity]}\quad \psi(x * y) \le C_1\, \big(\psi(x)+\psi(y)\big), \forall\, x, y \in G. \tag{6.113}$$

[2]Formula (6.110) is equivalent to $\psi(y * x) = \psi(x * y)$ for all $x, y \in G$, a condition also referred to as Abelian norm property.

The pair (C_0, C_1) appearing in (6.112) and (6.113) will be referred to as the constants of the quasi-pseudonorm ψ. Also, a quasi-pseudonorm ψ on G is called a `quasinorm` provided (6.111) is strengthened to $\psi^{-1}(\{0\}) = \{e_G\}$.

(ii) A function $\psi : G \to [0, +\infty]$ is said to be `quasi-invariant` if there exists $C_2 \in [1, +\infty)$ with the property that

$$\psi(x^{-1} * y * x) \leq C_2\, \psi(y), \quad \forall\, x, y \in G. \tag{6.114}$$

(iii) A quasi-pseudonorm ψ on G is said to be `finite` provided ψ actually takes values in $[0, +\infty)$. Finally, a quasinorm is said to be `finite` provided it is so when viewed as a quasi-pseudonorm.

Comment 6.23. Compared with Definition 6.22, a slightly more economical (though ultimately equivalent) way of introducing the notion of quasi-pseudonorm on a group $\big(G, *, (\cdot)^{-1}, e_G\big)$ is by stipulating that $\psi : G \to (-\infty, +\infty]$ is a function satisfying (6.111)–(6.113) for some constants $C_0, C_1 \in [1, +\infty)$. The fact that such a function is necessarily nonnegative is seen by writing, for each $x \in G$,

$$0 = \psi(e_G) = \psi(x * x^{-1}) \leq C_1 \big(\psi(x) + \psi(x^{-1})\big) \leq C_1(1 + C_0)\psi(x), \tag{6.115}$$

which forces $\psi(x) \geq 0$, as desired. ■

Remark 6.24. A simple but useful observation is that the quasi-invariance condition (6.114) for a quasi-pseudonorm ψ on a group G may be equivalently rephrased as (with $C_2 \in [1, +\infty)$ the same constant as in (6.114))

$$\psi(y * x) \leq C_2\, \psi(x * y), \quad \forall\, x, y \in G. \tag{6.116}$$

Moreover, conditions (6.112) and (6.116) may be consolidated into just one condition, demanding the existence of a finite constant $C > 0$ with the property that

$$\psi(x * y) \leq C\, \psi(x^{-1} * y^{-1}), \qquad \forall\, x, y \in G. \tag{6.117}$$

Indeed, making $x := e_G$ in (6.117) we obtain that ψ is quasisymmetric with quasisymmetry constant $C_0 = C$. In addition, using this and (6.117) we further obtain

$$\psi(y * x) \leq C(y^{-1} * x^{-1}) = C\,\psi((x * y)^{-1}) \leq C^2 \psi(x * y), \quad \forall\, x, y \in G. \tag{6.118}$$

Thus ψ is also quasi-invariant (with constant $C_2 = C^2$).

Conversely, if ψ is quasisymmetric (with constant C_0) and quasi-invariant (with constant C_2), then

$$\begin{aligned} \psi(x * y) &\leq C_0 \psi((x * y)^{-1}) = C_0 \psi(y^{-1} * x^{-1}) \\ &\leq C_0 C_2 \psi(x^{-1} * y^{-1}), \qquad \forall\, x, y \in G, \end{aligned} \tag{6.119}$$

i.e., (6.117) holds with $C = C_0 C_2$.

Later on, we will need the notion of the restriction of a quasi-pseudonorm to a subgroup. In anticipation of this, below we collect the main properties preserved by taking such restrictions (the reader may find it useful to recall (6.1)).

Lemma 6.25. *Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group equipped with a quasi-pseudonorm ψ with constants (C_0, C_1). Also, assume that H is a subgroup of G. Then $\psi\big|_H$, the restriction of the function $\psi : G \to [0, +\infty]$ to the set $H \subseteq G$, is a quasi-pseudonorm on the group H with the same constants (C_0, C_1). Furthermore,*

$$\psi \text{ finite} \Longrightarrow \psi\big|_H \text{ finite}, \text{ and } \psi \text{ quasinorm} \Longrightarrow \psi\big|_H \text{ quasinorm}; \tag{6.120}$$

$$\psi \text{ quasi-invariant} \Longrightarrow \psi\big|_H \text{ quasi-invariant (with the same constant)}; \tag{6.121}$$

$$\tau^{\rm R}_{\psi|_H} = \tau^{\rm R}_{\psi}\Big|_H \text{ and } \tau^{\rm L}_{\psi|_H} = \tau^{\rm L}_{\psi}\Big|_H; \tag{6.122}$$

$$B^{\rm R}_{\psi|_H}(a, r) = B^{\rm R}_{\psi}(a, r) \cap H \text{ and } \quad \forall\, a \in H, \ \forall\, r \in (0, +\infty); \tag{6.123}$$

$$B^{\rm L}_{\psi|_H}(a, r) = B^{\rm L}_{\psi}(a, r) \cap H \text{ and } \quad \forall\, a \in H, \ \forall\, r \in (0, +\infty). \tag{6.124}$$

Proof. All claims are clear from definitions. □

Conditions (6.112) and (6.113) in the definition of a quasi-pseudonorm may be naturally regarded as quantitative versions of the (topologically flavored) conditions (6.73) and (6.74), respectively, from Lemma 6.14. Likewise, the quasi-invariant condition (6.114) is a quantitative version of (6.81). These observations are at the core of the following result pertaining to the properties of quasi-pseudonorms on arbitrary groups.

Proposition 6.26. *Assume that $(G, *)$ is a group and that ψ is a quasi-pseudonorm on G.*

(i) The mappings

$$(\cdot)^{-1} : \big(G, \tau^{\rm R}_{\psi}\big) \longrightarrow \big(G, \tau^{\rm L}_{\psi}\big), \qquad (\cdot)^{-1} : \big(G, \tau^{\rm L}_{\psi}\big) \longrightarrow \big(G, \tau^{\rm R}_{\psi}\big) \tag{6.125}$$

are homeomorphisms (that are inverse to one another).

(ii) For every $A \subseteq G$ one has

$$\mathrm{Int}\big(A; \tau^{\rm R}_{\psi}\big) = \big\{a \in A : \exists\, r \in (0, +\infty) \text{ such that } B^{\rm R}_{\psi}(a, r) \subseteq A\big\}, \tag{6.126}$$

$$\mathrm{Int}\big(A; \tau^{\rm L}_{\psi}\big) = \big\{a \in A : \exists\, r \in (0, +\infty) \text{ such that } B^{\rm L}_{\psi}(a, r) \subseteq A\big\}. \tag{6.127}$$

Furthermore,

$$B_{\psi}^{\mathrm{R}}(a,r) \in \mathscr{N}\big(a;\tau_{\psi}^{\mathrm{R}}\big) \quad \textit{and} \quad B_{\psi}^{\mathrm{L}}(a,r) \in \mathscr{N}\big(a;\tau_{\psi}^{\mathrm{L}}\big), \quad \forall\, a \in G, \ \ \forall\, r > 0. \tag{6.128}$$

In fact, for every $a \in G$ and every $r \in (0,+\infty)$

$$\bigcup_{C>C_1} B_{\psi}^{\mathrm{R}}(a, r/C) \subseteq \mathrm{Int}\big(B_{\psi}^{\mathrm{R}}(a,r);\tau_{\psi}^{\mathrm{R}}\big), \tag{6.129}$$

$$\bigcup_{C>C_1} B_{\psi}^{\mathrm{L}}(a, r/C) \subseteq \mathrm{Int}\big(B_{\psi}^{\mathrm{L}}(a,r);\tau_{\psi}^{\mathrm{L}}\big). \tag{6.130}$$

(iii) For each $a \in G$

$$\Big\{B_{\psi}^{\mathrm{R}}(a,r)\Big\}_{r>0} \ \ \textit{is a fundamental system of neighborhoods of } a \ \textit{ in } \ \tau_{\psi}^{\mathrm{R}}, \tag{6.131}$$

$$\Big\{B_{\psi}^{\mathrm{L}}(a,r)\Big\}_{r>0} \ \ \textit{is a fundamental system of neighborhoods of } a \ \textit{ in } \ \tau_{\psi}^{\mathrm{L}}, \tag{6.132}$$

where $C_1 \in [1,+\infty)$ is as in (6.113).

(iv) A sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ is right-convergent to $a \in G$ with respect to τ_{ψ}^{R} if and only if for every $\varepsilon \in (0,+\infty)$ there exists $n_\varepsilon \in \mathbb{N}$ such that $a_i \in B_{\psi}^{\mathrm{R}}(a,\varepsilon)$ whenever $i \in \mathbb{N}$ satisfies $i \geq n_\varepsilon$. Furthermore, a similar characterization of left-convergence with respect to the topology τ_{ψ}^{L} holds.

(v) A sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ is right-Cauchy with respect to τ_{ψ}^{R} if and only if it is left-Cauchy with respect to τ_{ψ}^{L}. Moreover, either of these conditions is equivalent to the demand that for every $\varepsilon \in (0,+\infty)$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\psi(a_i * a_j^{-1}) < \varepsilon$ whenever $i, j \in \mathbb{N}$ satisfy $i \geq j \geq n_\varepsilon$.

(vi) One has

$$G \textit{ is right-complete with respect to } \tau_{\psi}^{\mathrm{R}} \iff G \textit{ is left-complete with respect to } \tau_{\psi}^{\mathrm{L}}. \tag{6.133}$$

(vii) If the quasi-pseudonorm ψ is actually quasi-invariant, then $\tau_{\psi}^{\mathrm{R}} = \tau_{\psi}^{\mathrm{L}} =: \tau_\psi$. Moreover, in such a scenario,

$$\big(G, *, \tau_\psi\big) \quad \textit{is a topological group.} \tag{6.134}$$

Proof. With the exception of (6.129) and (6.130), all other properties follow directly from Lemma 6.14 and the comments made just prior to the statement of the proposition. To justify (6.129), suppose that $a \in G$ and $r \in (0, +\infty)$ are given. Pick a number $C > C_1$, along with some element $b \in B_\psi^{\mathrm{R}}(a, r/C)$. In particular, $\psi(a*b^{-1}) < r/C$. Now, if $\varepsilon \in \big(0, r(C_1^{-1} - C^{-1})\big)$ and $c \in B_\psi^{\mathrm{R}}(b, \varepsilon)$, then it follows that

$$\begin{aligned} \psi(a * c^{-1}) &= \psi\big((a * b^{-1}) * (b * c^{-1})\big) \leq C_1\big(\psi(a * b^{-1}) + \psi(b * c^{-1})\big) \\ &< C_1\big(r/C + \varepsilon\big) < r. \end{aligned} \tag{6.135}$$

From this we deduce that $B_\psi^{\mathrm{R}}(b, \varepsilon) \subseteq B_\psi^{\mathrm{R}}(a, r)$ and, further, $b \in \mathrm{Int}\big(B_\psi^{\mathrm{R}}(a, r); \tau_\psi^{\mathrm{R}}\big)$ by (6.126). Hence, ultimately, $B_\psi^{\mathrm{R}}(a, r/C) \subseteq \mathrm{Int}\big(B_\psi^{\mathrm{R}}(a, r); \tau_\psi^{\mathrm{R}}\big)$, proving (6.129). Formula (6.130) may also be established in a similar manner, and this completes the proof of the proposition. □

Remark 6.27. (i) By a *quasi-pseudonormed group* we will always understand a group equipped with a quasi-pseudonorm. Hence, stating that (G, ψ) is a quasi-pseudonormed group means that G is a group and ψ is a quasi-pseudonorm on G. Also, in light of parts (v) and (vi) in Proposition 6.26, given a quasi-pseudonormed group, we agree to drop the adjectives left/right when referring to Cauchy sequences and completeness.

(ii) In view of part (vii) in Proposition 6.26, given an arbitrary group G equipped with some quasi-invariant quasi-pseudonorm ψ, it is natural to refer to $\tau_\psi := \tau_\psi^{\mathrm{R}} = \tau_\psi^{\mathrm{L}}$ simply as the *topology induced by* ψ *on* G.

Comment 6.28. Let $(G, *)$ be a group. As opposed to the case of a genuine pseudonorm on G, a mere quasi-pseudonorm ψ on G *may not be continuous* as a function from $(G, \tau_\psi^{\mathrm{R}})$ into $[0, +\infty]$. For example, if $(G, *) := (\mathbb{R}, +)$ and

$$\psi(x) := \begin{cases} |x| & \text{if } x \in \mathbb{Q}, \\ 2|x| & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \qquad \forall\, x \in \mathbb{R}, \tag{6.136}$$

then τ_ψ^{R} is just the ordinary topology on the real line and $\psi : \mathbb{R} \to [0, +\infty)$ is a quasinorm that is discontinuous at every point except at the origin. The latter is no accident. Indeed, as one may readily verify from definitions, any quasi-pseudonorm ψ on an arbitrary group G is continuous (in both the topology τ_ψ^{R} and τ_ψ^{L}) at the neutral element $e_G \in G$. ■

Comment 6.29. Assume that ψ is a quasi-pseudonorm on a group $(G, *)$ with constants (C_0, C_1). Then repeated applications of (6.113) give that, for each $x_1, \ldots, x_N \in G$,

$$\psi(x_1 * \cdots * x_N) \leq C_1\psi(x_1) + C_1^2\psi(x_2) + \cdots + C_1^{N-1}\psi(x_{N-1}) + C_1^{N-1}\psi(x_N). \tag{6.137}$$

Note that, with the exception of the situation when $C_1 = 1$ (as in the case of a pseudonorm), the largest coefficient on the right-hand side of (6.137) increases exponentially with N. ■

The discussion in Comments 6.28 and 6.29 exposes some of the most significant differences between pseudonorms and quasi-pseudonorms on groups. The aforementioned shortcomings of quasi-pseudonorms cause significant problems in applications; hence the case of quasi-pseudonormed groups is more subtle than that of, say, normed and pseudonormed groups.

A key technical tool in the proof of the quantitative version of the OMT (formulated in Theorem 6.49) that is brought into play specifically to address the deficiencies inherent to quasi-pseudonorms is the version of Theorems 3.26 and 3.28 for semigroups and groups. Before stating it, recall that a subset of a topological space is called `nowhere dense` if the interior of its closure is empty. Also, recall that a subset Y of a topological space (X, τ) is said to be of `second Baire category` provided Y may not be written as the union of countably many nowhere dense subsets [relative to (X, τ)].

Theorem 6.30. *Let $(G, *)$ be a semigroup, and assume that $\psi : G \to [0, +\infty]$ is a quasisubadditive function, i.e., there exists a constant $\kappa \in [1, +\infty)$ such that*

$$\psi(a * b) \leq \kappa \max\{\psi(a), \psi(b)\} \quad \textit{for all} \;\; a, b \in G. \tag{6.138}$$

Fix a number

$$\beta \in \big(0, (\log_2 \kappa)^{-1}\big]. \tag{6.139}$$

Then, for each integer $N \in \mathbb{N}$ the function ψ satisfies

$$\psi(a_1 * \cdots * a_N) \leq \kappa^2 \left\{ \sum_{i=1}^{N} \psi(a_i)^\beta \right\}^{\frac{1}{\beta}} \quad \textit{for all} \;\; a_1, \ldots, a_N \in G. \tag{6.140}$$

In particular, for every sequence $(a_i)_{i \in \mathbb{N}} \subseteq G$ one has

$$\sup_{N \in \mathbb{N}} \psi(a_1 * \cdots * a_N) \leq \kappa^2 \left\{ \sum_{i=1}^{\infty} \psi(a_i)^\beta \right\}^{\frac{1}{\beta}}. \tag{6.141}$$

While in general the function ψ may not be continuous when G is equipped with either the topology τ_ψ^{R} or the topology τ_ψ^{L}, a closely related property holds under additional assumptions. Specifically, assume that actually $(G, *)$ is a group and ψ satisfies (6.138), takes finite values (i.e., $\psi : G \to [0, +\infty)$), and is quasisymmetric, in the sense that there exists $C_0 \in [1 + \infty)$ such that (6.112) holds. In such a scenario, for any sequence $(a_n)_{n \in \mathbb{N}} \subseteq G$ that converges to some $a \in G$, either in the topology τ_ψ^{R} or in the topology τ_ψ^{L}, one has

$$\kappa^{-2} C_0^{-1} \psi(a) \leq \liminf_{n \to \infty} \psi(a_n) \leq \limsup_{n \to \infty} \psi(a_n) \leq \kappa^2 C_0 \, \psi(a). \tag{6.142}$$

Finally, if $(G, *)$ *is a group and if the function* $\psi : G \to [0, +\infty)$ *is quasisymmetric and satisfies* (6.138) *and* $\psi(e_G) = 0$*, then*

$$\tau_\psi^{\rm R} \textit{ and } \tau_\psi^{\rm L} \textit{ are pseudometrizable topologies on } G, \tag{6.143}$$

and the following implications hold:

$$G \textit{ complete with respect to } \tau_\psi^{\rm R} \implies (G, \tau_\psi^{\rm R}) \textit{ is of second Baire category}, \tag{6.144}$$

$$G \textit{ complete with respect to } \tau_\psi^{\rm L} \implies (G, \tau_\psi^{\rm L}) \textit{ is of second Baire category}. \tag{6.145}$$

Proof. All claims are particular cases of Theorems 3.26 and 3.28 specialized to semigroups and groups. □

The next goal is to present an extension of the classical Birkhoff–Kakutani theorem formulated in Theorem 6.33. This requires a number of preliminaries, and we begin by establishing a dictionary between analytical and topological characteristics of a given group in our next proposition.

Proposition 6.31. *Let* $\big(G, *, (\cdot)^{-1}, e_G\big)$ *be a group, and assume that* τ *is a topology on the set* G*. Also, fix a number* $\kappa \in (1, +\infty)$.

Then there exists a function $\psi : G \to [0, +\infty)$ *satisfying*

$$\psi(x * y) \leq \kappa \max\big\{\psi(x), \psi(y)\big\} \quad \textit{for all} \quad x, y \in G, \tag{6.146}$$

$$\psi(x^{-1}) = \psi(x) \quad \textit{for all} \quad x \in G, \tag{6.147}$$

$$\psi(e_G) = 0, \tag{6.148}$$

$$\mathscr{N}(e_G; \tau_\psi^{\rm R}) = \mathscr{N}(e_G; \tau) = \mathscr{N}(e_G; \tau_\psi^{\rm L}) \tag{6.149}$$

if and only if

$$e_G \textit{ has a countable neighborhood basis in the topology } \tau, \tag{6.150}$$

$$\textit{the topology } \tau \textit{ is symmetric, and} \tag{6.151}$$

$$(G \times G, \tau \times \tau) \ni (x, y) \mapsto x * y \in (G, \tau) \quad \textit{is continuous at } (e_G, e_G). \tag{6.152}$$

Furthermore,

$$\left.\begin{array}{r} \exists\, \psi : G \to [0, +\infty) \;\; \textit{satisfying} \\ \textit{conditions (6.146)–(6.148) and} \\ \textit{with the property that } \; \tau_\psi^{\rm R} = \tau \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \textit{(6.150)–(6.152) hold and} \\ \tau \textit{ is right-invariant} \end{array}\right. \tag{6.153}$$

and

$$\left.\begin{aligned} \exists\,\psi : G \to [0,+\infty) \text{ satisfying} \\ (6.146),\ (6.147),\ (6.149), \\ \psi^{-1}\big(\{0\}\big) = \{e_G\},\ \text{and}\ \tau_\psi^{\mathrm{R}} = \tau \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} &(6.150)\text{–}(6.152) \text{ hold,} \\ &\tau \text{ is Hausdorff, and} \\ &\tau \text{ is right-invariant.} \end{aligned}\right. \tag{6.154}$$

Finally, similar equivalences to those formulated in (6.153) *and* (6.154) *hold with* τ_ψ^{L} *replacing* τ_ψ^{R} *and "left-invariant" replacing "right-invariant."*

Proof. Assume first (6.150)–(6.152). In particular, let $\{U_n\}_{n\in\mathbb{N}}$ be a countable neighborhood basis in τ for e_G, i.e.,

$$\{U_n\}_{n\in\mathbb{N}} \subseteq \mathscr{N}(e_G;\tau) \ \text{ and } \ \forall\, W \in \mathscr{N}(e_G;\tau)\ \exists\, n \in \mathbb{N} \ \text{ such that } \ U_n \subseteq W. \tag{6.155}$$

Replacing each U_n by $U_1 \cap \cdots \cap U_n$, there is no loss of generality in assuming that the family of sets $\{U_n\}_{n\in\mathbb{N}}$ is nested, i.e.,

$$U_{n+1} \subseteq U_n, \qquad \forall\, n \in \mathbb{N}. \tag{6.156}$$

Recall that we are assuming that τ is symmetric. By subsequently replacing each U_n by $U_n \cap U_n^{-1}$, matters are arranged (cf. (6.47)) so that, in addition to (6.155) and (6.156), the family $\{U_n\}_{n\in\mathbb{N}}$ also satisfies

$$U_n^{-1} = U_n, \qquad \forall\, n \in \mathbb{N}. \tag{6.157}$$

Going further, select a subsequence $\{V_n\}_{n\in\mathbb{N}}$ of $\{U_n\}_{n\in\mathbb{N}}$ as follows. First define $V_1 := U_1$; then, making use of (6.152), select $W \in \mathscr{N}(e_G;\tau)$ such that $W * W \subseteq V_1$. Next, pick $n_1 \in \mathbb{N}$ such that $U_{n_1} \subseteq W$, then set $V_1 := U_{n_1}$. This scheme may be inductively continued to produce a sequence $\{V_n\}_{n\in\mathbb{N}}$ satisfying

$$V_n \in \mathscr{N}(e_G;\tau) \ \text{ and } \ V_{n+1} * V_{n+1} \subseteq V_n, \quad \forall\, n \in \mathbb{N}; \tag{6.158}$$

$$\forall\, W \in \mathscr{N}(e_G;\tau)\ \exists\, n \in \mathbb{N} \ \text{ such that } \ V_n \subseteq W; \tag{6.159}$$

$$V_{n+1} \subseteq V_n \ \text{ for every } \ n \in \mathbb{N}; \tag{6.160}$$

$$V_n^{-1} = V_n \ \text{ for every } \ n \in \mathbb{N}. \tag{6.161}$$

Indeed, property (6.158) is part of the induction scheme, whereas property (6.159) is a consequence of (6.155) and (6.156) and the fact that, by design, $\{V_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{U_n\}_{n\in\mathbb{N}}$. Finally, (6.160) is implied by (6.158), while (6.161) follows from (6.157) and the fact that $\{V_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{U_n\}_{n\in\mathbb{N}}$.

To proceed, augment $\{V_n\}_{n\in\mathbb{N}}$ to $\{V_n\}_{n\in\mathbb{N}_0}$ by considering $V_0 := G$. At this stage, given $\kappa \in (1, +\infty)$, define the function $\psi : G \to [0, +\infty)$ by the formula

$$\psi(x) := \inf\big\{\kappa^{-n} : n \in \mathbb{N}_0 \text{ so that } x \in V_n\big\}, \qquad \forall\, x \in G. \tag{6.162}$$

Then (6.147) is immediately seen from (6.161) and (6.162), whereas (6.148) follows after observing that, by design,

$$\psi^{-1}\big(\{0\}\big) = \bigcap_{n\in\mathbb{N}} V_n. \tag{6.163}$$

Assume now that $x, y \in G$ are given and fix an arbitrary $\varepsilon > 0$. Then one can find $n, m \in \mathbb{N}$ such that $x \in V_n$, $y \in V_m$ and

$$\kappa^{-n} < \psi(x) + \varepsilon \quad \text{and} \quad \kappa^{-m} < \psi(y) + \varepsilon. \tag{6.164}$$

Without loss of generality, suppose that $m \geq n$. Then it follows from (6.160) that $x, y \in V_n$ and, hence, $x * y \in V_{n-1}$ by (6.158) when $n \geq 2$ and the definition of V_0 when $n = 1$. Consequently,

$$\psi(x * y) \leq \kappa^{1-n} = \kappa \max\big\{\kappa^{-n}, \kappa^{-m}\big\} < \kappa \max\big\{\psi(x) + \varepsilon, \psi(y) + \varepsilon\big\}. \tag{6.165}$$

Passing to the limit as $\varepsilon \searrow 0$ in (6.165) yields (6.146).

Consider next (6.149). Thanks to (6.159), to the fact that ψ is a quasi-pseudonorm (as seen from (6.146)–(6.148)), and to (6.128), it suffices to show that

$$V_{n+1} = B^{\rm R}_{\psi}(e_G, \kappa^{-n}) = B^{\rm L}_{\psi}(e_G, \kappa^{-n}), \qquad \forall\, n \in \mathbb{N}. \tag{6.166}$$

However, for each $n \in \mathbb{N}$, making use of (6.61), (6.162), (6.161), and (6.160) we may write

$$\begin{aligned} B^{\rm R}_{\psi}(e_G, \kappa^{-n}) = B^{\rm L}_{\psi}(e_G, \kappa^{-n}) &= \big\{x \in G : \psi(x^{-1}) < \kappa^{-n}\big\} \\ &= \big\{x \in G : \exists\, m \in \mathbb{N} \ \text{ such that } \ \kappa^{-m} < \kappa^{-n} \ \text{ and } \ x^{-1} \in V_m\big\} \\ &= \big\{x \in G : \exists\, m \in \mathbb{N} \ \text{ such that } \ n < m \ \text{ and } \ x \in V_m\big\} \\ &= \bigcup_{m>n} V_m = V_{n+1}. \end{aligned} \tag{6.167}$$

This establishes (6.149) and concludes the proof of the implication

$$(6.150)\text{–}(6.152) \Longrightarrow (6.146)\text{–}(6.149). \tag{6.168}$$

Consider next the opposite implication in (6.168). To get started, assume that the function $\psi : G \to [0, +\infty)$ satisfies (6.146)–(6.149). In particular, (6.146)–(6.148) imply that ψ is a quasi-pseudonorm; hence, in view of (6.131) and (6.149), we see that

$$\begin{gathered}\Big\{B^{\rm R}_{\psi}\big(e_G, n^{-1}\big)\Big\}_{n\in\mathbb{N}} \text{ is a countable neighborhood basis}\\ \text{for the element } e_G \text{ in the topology } \tau.\end{gathered} \tag{6.169}$$

This proves (6.150). In fact, granted (6.147), it follows that each ball $B^{\rm R}_{\psi}\big(e_G, n^{-1}\big)$ is a symmetric set; hence, (6.151) also holds as a result of (6.169). Finally, since (6.146) readily entails

$$B^{\rm R}_{\psi}(e_G, r) * B^{\rm R}_{\psi}(e_G, r) \subseteq B^{\rm R}_{\psi}(e_G, \kappa\, r), \quad \forall\, r \in (0, +\infty), \tag{6.170}$$

condition (6.152) readily follows from this and (6.169). This concludes the proof of the implication

$$(6.146)\text{–}(6.149) \Longrightarrow (6.150)\text{–}(6.152). \tag{6.171}$$

Moving on, the equivalence in (6.153) follows from (6.168), (6.171), and item *(ii)* in Lemma 6.14. Also, the equivalence in (6.154) is readily seen from (6.153), with the help of (6.163) and the observation that, if $\{V_n\}_{n\in\mathbb{N}}$ are as in (6.158)–(6.161), then

$$\bigcap_{n\in\mathbb{N}} V_n = \{e_G\} \iff \tau \text{ is Hausdorff.} \tag{6.172}$$

Finally, the very last claim in the statement of the proposition is established similarly, and this completes the proof. □

Comment 6.32. It is instructive to reflect on how the classical Birkhoff–Kakutani theorem (cf. Theorem 2.77), to the effect that *a topological group is metrizable if and only if the underlying topology is Hausdorff and the identity in the group has a countable neighborhood basis*, fits in the context of the results presented in this section so far. Concretely, given a Hausdorff topological group (G, τ), the equivalence in (6.154) shows that τ may be identified with $\tau^{\rm R}_{\psi}$, the right-topology induced on G by a quasinorm ψ on the group G. In concert with (6.143), this shows that τ is a metrizable topology on G. This completes the proof of the "hard" implication in the Birkhoff–Kakutani theorem.

Such an approach should be contrasted with what is now considered to be the standard proof of the Birkhoff–Kakutani theorem, which relies on a Urysohn-type lemma (cf., e.g., [84], [104, Theorem 1.1.1, pp. 2–3]). ■

Recall that a `pseudometric` on a set G is a function $d : G \times G \to [0, +\infty)$ satisfying

$$(i) \qquad d(x,x) = 0, \quad \forall\, x \in G, \tag{6.173}$$

$$(ii) \qquad d(x,y) = d(y,x), \quad \forall\, x, y \in G, \tag{6.174}$$

$$(iii) \qquad d(x,y) \leq d(x,z) + d(z,y), \quad \forall\, x, y, z \in G. \tag{6.175}$$

The topology τ_d induced by a pseudometric d on the set G is defined by

$$\tau_d := \big\{O \subseteq G : \forall\, a \in O\ \ \exists\, r \in (0, +\infty) \text{ such that } B_d(a,r) \subseteq O\big\}, \tag{6.176}$$

where for each $a \in G$ and $r \in (0, +\infty)$ we have set

$$B_d(a,r) := \big\{x \in G : d(a,x) < r\big\}. \tag{6.177}$$

Finally, a pseudometric on a group $(G, *)$ is said to be `right-invariant` provided

$$d(x * a, y * a) = d(x,y), \quad \forall\, x, y, a \in G, \tag{6.178}$$

and `left-invariant` if

$$d(a * x, a * y) = d(x,y), \quad \forall\, x, y, a \in G. \tag{6.179}$$

We are now prepared to present a result in the spirit of the Birkhoff–Kakutani theorem, the main difference being that, when asking the metrizability question, we no longer assume from the start that the group in question is a topological group but rather allow this topology to be arbitrary to begin with. As such, this provides a desirable extension of the classical Birkhoff–Kakutani theorem by *fully* bringing the topology into focus.

Theorem 6.33. *Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group, and assume that τ is a topology on the set G. Then*

$$e_G \text{ has a countable neighborhood basis in the topology } \tau, \tag{6.180}$$

$$(G \times G, \tau \times \tau) \ni (x,y) \mapsto x * y \in (G, \tau) \ \text{ is continuous at } (e_G, e_G), \tag{6.181}$$

$$\text{the topology } \tau \text{ is symmetric and right-invariant} \tag{6.182}$$

if and only if

$$\text{there exists a finite right-invariant pseudometric on } G \text{ inducing } \tau. \tag{6.183}$$

Moreover,

$$\left.\begin{array}{l}\exists\,\textit{right-invariant metric on } G\\ \textit{inducing the topology } \tau \textit{ on } G\end{array}\right\} \iff \left\{\begin{array}{l}(6.180)\text{–}(6.182)\ \textit{hold and, also,}\\ \textit{the topology } \tau \textit{ is Hausdorff.}\end{array}\right. \tag{6.184}$$

Finally, similar equivalences hold when "right-invariant" is replaced by "left-invariant."

Proof. Suppose first that (6.180)–(6.182) hold. Then (6.153) ensures the existence of a function $\psi : G \to [0, +\infty)$ satisfying (6.146)–(6.148) and such that $\tau_\psi^{\mathrm{R}} = \tau$. Consider the function $\psi_\# : G \to [0, +\infty)$ to be the regularized version of ψ as in Theorem 3.28. In particular, if $\beta \in \big(0, (\log_2 \kappa)^{-1}\big]$ is a fixed number, then

$$\psi_\#(x * y)^\beta \leq \psi_\#(x)^\beta + \psi_\#(y)^\beta \quad \text{for all} \quad x, y \in G, \tag{6.185}$$

$$\kappa^{-2}\psi(x) \leq \psi_\#(x) \leq \psi(x) \quad \text{for each} \quad x \in G, \tag{6.186}$$

$$\psi_\#(x^{-1}) = \psi_\#(x) \quad \text{for each} \quad x \in G, \tag{6.187}$$

$$\psi_\#(e_G) = 0. \tag{6.188}$$

Defining

$$d : G \times G \to [0, +\infty), \qquad d(x, y) := \big[\psi_\#(x * y^{-1})\big]^\beta, \quad \forall\, x, y \in G, \tag{6.189}$$

one can check without difficulty that (6.173)–(6.175) and (6.178) hold, i.e., d is a right-invariant pseudometric on G. Moreover, it follows from definitions that

$$B_d(a, r) = B^{\mathrm{R}}_{\psi_\#}\big(a, r^{1/\beta}\big), \quad \forall\, a \in G, \quad \forall\, r \in (0, +\infty). \tag{6.190}$$

From this and (6.186) we then deduce that $\tau_d = \tau_\psi^{\mathrm{R}}$. Since we know that $\tau_\psi^{\mathrm{R}} = \tau$, this proves that (6.183) holds.

Consider now the opposite implication. Hence, assume that there exists a right-invariant pseudometric d on G with the property that $\tau_d = \tau$. Defining the function

$$\psi : G \to [0, +\infty), \qquad \psi(x) := d(x, e_G), \quad \forall\, x \in G, \tag{6.191}$$

one may readily check that conditions (6.146)–(6.148) are satisfied. In addition,

$$B^{\mathrm{R}}_\psi(a, r) = B_d(a, r), \qquad \forall\, a \in G, \quad \forall\, r \in (0, +\infty), \tag{6.192}$$

which ultimately shows that $\tau_\psi^{\mathrm{R}} = \tau_d = \tau$. Hence, the conditions on the left-hand side of the equivalence in (6.153) are satisfied. As such, Proposition 6.31 gives that (6.180)–(6.182) hold. This completes the proof of the first equivalence in the statement of the theorem. Finally, the equivalence formulated in (6.184) is implicit in what we have proved so far and (6.154), while the "left-handed" versions of these results are established analogously. □

We conclude this section by presenting a very useful completeness criterion.

Proposition 6.34. *Assume that* $(G, *)$ *is a group endowed with a finite quasi-invariant quasi-pseudonorm* ψ *with constants* (C_0, C_1)*, and fix a number*

$$\beta \in \big(0, (1 + \log_2 C_1)^{-1}\big]. \tag{6.193}$$

Then the following conditions are equivalent:

(i) $(G, *, \tau_\psi)$ *is complete.*

(ii) If the sequence $(a_n)_{n\in\mathbb{N}} \subseteq G$ *has the property that* $\sum_{n=1}^{\infty} \psi(a_n)^\beta < +\infty$*, then the sequence* $\big(a_1 * \cdots * a_n\big)_{n\in\mathbb{N}}$ *converges in* $(G, *, \tau_\psi)$*.*

Proof. To set the stage for proving the implication *(i)* ⇒ *(ii)*, consider $(a_n)_{n\in\mathbb{N}} \subseteq G$ with the property that

$$\sum_{n=1}^{\infty} \psi(a_n)^\beta < +\infty, \tag{6.194}$$

and for each $n \in \mathbb{N}$ define $x_n := a_1 * \cdots * a_n$. Let $C_2 \in [1, +\infty)$ be the constant from the quasi-invariance condition (6.114) for the quasi-pseudonorm ψ. Then, for each integers $n, k \in \mathbb{N}$, we may write

$$\begin{aligned}
\psi\big(x_{n+k} * x_n^{-1}\big) &= \psi\big((a_1 * \cdots * a_{n+k}) * (a_n^{-1} * \cdots * a_1^{-1})\big) \\
&\le C_2\, \psi\big((a_n^{-1} * \cdots * a_1^{-1}) * (a_1 * \cdots * a_{n+k})\big) \\
&= C_2\, \psi(a_{n+1} * \cdots * a_{n+k}) \\
&\le C_2 (2C_1)^2 \left\{ \sum_{i=n}^{n+k} \psi(a_i)^\beta \right\}^{\frac{1}{\beta}},
\end{aligned} \tag{6.195}$$

thanks to (6.140) and the fact that ψ is quasi-invariant (cf. Remark 6.24). In turn, (6.195) and (6.194) allow us to conclude that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (G, τ_ψ). Given that we are assuming that (G, τ_ψ) is complete, it follows that $(x_n)_{n\in\mathbb{N}}$ converges in (G, τ_ψ), as desired.

As regards the implication *(ii)* ⇒ *(i)*, let $(x_n)_{n\in\mathbb{N}} \subseteq G$ be a Cauchy sequence in (G, τ_ψ). A standard argument then yields a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ with the property that

$$\psi\big(x_{n_i} * x_{n_{i-1}}^{-1}\big) < 2^{-i}, \qquad \forall\, i \ge 2. \tag{6.196}$$

Consider next the sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ defined by

$$a_1 := x_{n_1} \quad \text{and} \quad a_i := x_{n_{i-1}}^{-1} * x_{n_i} \quad \text{for} \quad i \geq 2, \tag{6.197}$$

and note that for each $i \in \mathbb{N}$ with $i \geq 2$ we have

$$a_1 * \cdots * a_i = x_{n_1} * \left(x_{n_1}^{-1} * x_{n_2}\right) * \cdots * \left(x_{n_{i-1}}^{-1} * x_{n_i}\right) = x_{n_i}. \tag{6.198}$$

Moreover, from (6.196) and (6.197) and the fact that ψ is finite and quasi-invariant we obtain

$$\begin{aligned}\sum_{i=1}^{\infty} \psi(a_i)^{\beta} &= \psi\left(x_{n_1}\right)^{\beta} + \sum_{i=2}^{\infty} \psi\left(x_{n_{i-1}}^{-1} * x_{n_i}\right)^{\beta} \\ &\leq \psi\left(x_{n_1}\right)^{\beta} + C_2^{\beta} \sum_{i=2}^{\infty} \psi\left(x_{n_i} * x_{n_{i-1}}^{-1}\right)^{\beta} \\ &\leq \psi\left(x_{n_1}\right)^{\beta} + C_2^{\beta} \sum_{i=2}^{\infty} 2^{-i\beta} < +\infty. \end{aligned} \tag{6.199}$$

In concert with (6.198) and the current hypotheses, this implies that the sequence $(x_{n_i})_{i\in\mathbb{N}}$ is convergent in $(G, *, \tau_\psi)$. Since this is a subsequence of the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$, it follows that the latter is also convergent in $(G, *, \tau_\psi)$, as desired. □

6.3 Quotient, Pullback, and Push-Forward Quasi-Pseudonorms

To facilitate the subsequent presentation, we first discuss some preliminary material. Let $(G, *)$ be a group. Recall that a subgroup H of G is said to be `normal` provided

$$x * h * x^{-1} \in H, \qquad \forall\, x \in G \ \text{ and } \ \forall\, h \in H. \tag{6.200}$$

As is well known, (6.200) is equivalent to

$$x * y \in H \iff y * x \in H, \qquad \forall\, x, y \in G, \tag{6.201}$$

and, further, with the condition that the left and right cosets of H in G coincide, i.e.,

$$x * H = H * x, \qquad \forall\, x \in G. \tag{6.202}$$

Given a normal subgroup H of a group $\big(G, *, (\cdot)^{-1}, e_G\big)$, abbreviate

$$[x] := x * H = H * x, \qquad \forall\, x \in G, \tag{6.203}$$

and define

$$G/H := \big\{[x] : x \in G\big\}. \tag{6.204}$$

Considering the binary operation and inverse

$$[x] \star [y] := [x * y], \quad [x]^{-1} := [x^{-1}], \qquad \forall\, x, y \in G, \tag{6.205}$$

which are unambiguously defined thanks to the fact that H is normal, and setting $e_{G/H} := [e_G]$, it follows that $\big(G/H, \star, (\cdot)^{-1}, e_{G/H}\big)$ is a group, called the `quotient of` G `modulo` H. In such a scenario,the canonical `projection` of G onto G/H is defined by the formula

$$\pi_H : G \longrightarrow G/H, \quad \pi_H(x) := [x], \quad \forall\, x \in G. \tag{6.206}$$

It follows that

$$\pi_H \in \mathrm{Hom}\big(G, G/H\big) \text{ is surjective and } \mathrm{Ker}\, \pi_H = H. \tag{6.207}$$

Definition 6.35. Assume that G is a group and that τ_G is a topology on the set G. Given a normal subgroup H of G, the `quotient topology` $\tau_{G/H}$ on G/H, relative to τ_G, is defined as the largest topology on the set G/H with the property that the function

$$\pi_H : \big(G, \tau_G\big) \longrightarrow \big(G/H, \tau_{G/H}\big) \tag{6.208}$$

is continuous.

In the context of Definition 6.35, it is relevant to remark that an explicit formula for the quotient topology $\tau_{G/H}$ is

$$\tau_{G/H} = \big\{O \subseteq G/H : \pi_H^{-1}(O) \in \tau_G\big\}. \tag{6.209}$$

To further introduce notation that will be useful later, we explicitly record a most basic result in the theory of groups in the theorem below.

Theorem 6.36 (First Fundamental Isomorphism Theorem). *Let G and S be two groups, and assume that* $T \in \mathrm{Hom}\,(G, S)$. *Then* $\mathrm{Ker}\, T$ *is a normal subgroup of G and the function*

$$\widehat{T} : G/\mathrm{Ker}\, T \longrightarrow S, \qquad \widehat{T}\big([x]\big) := T(x), \quad \forall\, x \in G, \tag{6.210}$$

is the unique homomorphism in $\mathrm{Hom}\big(G/\mathrm{Ker}\,T,\ S\big)$ *with the property that*

$$\widehat{T} \circ \pi_{\mathrm{Ker}\,T} = T. \tag{6.211}$$

Moreover,

$$\widehat{T} \in \mathrm{Hom}\big(G/\mathrm{Ker}\,T,\ \mathrm{Im}\,T\big) \quad \text{is an isomorphism.} \tag{6.212}$$

We continue with a proposition that clarifies some of the most basic properties of quotient topologies, introduced earlier.

Proposition 6.37. *Let* G *be a group, and suppose that* τ_G *is a topology on the set* G *such that all right-shifts (or all left-shifts) on* G *are continuous with respect to* τ_G*. Also, assume that* H *is a normal subgroup of* G*. Then the following assertions hold.*

(1) The quotient topology on G/H *may be described as*

$$\tau_{G/H} = \big\{\pi_H(O) :\ O \in \tau_G\big\}. \tag{6.213}$$

(2) In the context

$$\pi_H : (G, \tau_G) \longrightarrow \big(G/H, \tau_{G/H}\big), \tag{6.214}$$

the canonical projection π_H *is continuous and open.*

(3) If (Y, τ_Y) *is a topological space and* $\varphi : G/H \to Y$ *is a function, then*

$$\left.\begin{array}{l}\varphi : \big(G/H, \tau_{G/H}\big) \longrightarrow (Y, \tau_Y) \quad \text{is continuous}\\ \qquad \text{(open, surjective, respectively)}\end{array}\right\} \tag{6.215}$$
$$\iff \left\{\begin{array}{l}\varphi \circ \pi_H : \big(G, \tau_G\big) \longrightarrow (Y, \tau_Y) \quad \text{is continuous}\\ \text{(open, surjective, respectively).}\end{array}\right.$$

(4) Suppose that $(S, \circ)$ *is a group and* τ_S *is an arbitrary topology on the set* S*, and assume that* $T \in \mathrm{Hom}\,(G, S)$*. Associated with this homomorphism, consider* $\widehat{T}$ *as in* (6.210)*. Then*

$$\widehat{T} : \big(G/\mathrm{Ker}\,T,\ \tau_{G/\mathrm{Ker}}\big) \longrightarrow (S, \tau_S) \tag{6.216}$$

is continuous if and only if $T : (G, \tau_G) \to (S, \tau_S)$ *is continuous. Moreover, the function* (6.216) *is a homeomorphism if and only if* $T : (G, \tau_G) \to (S, \tau_S)$ *is surjective, continuous, and open.*

Proof. We begin by observing that

$$\pi_H^{-1}\big(\pi_H(O)\big) = \bigcup_{x\in H} x * O = \bigcup_{x\in H} O * x, \qquad \forall\, O \subseteq G, \tag{6.217}$$

given that H is a normal subgroup of G, and

$$\pi_H\big(\pi_H^{-1}(O)\big) = O, \qquad \forall\, O \subseteq G, \tag{6.218}$$

since π_H is surjective. In turn, from (6.217) and the nature of the topology τ_G we deduce that

$$\pi_H^{-1}\big(\pi_H(O)\big) \in \tau_G, \qquad \forall\, O \in \tau_G. \tag{6.219}$$

Consequently, for every $O_1, O_2 \in \tau_G$ we may write

$$\begin{aligned}\pi_H(O_1) \cap \pi_H(O_2) &= \pi_H\Big(\pi_H^{-1}\big(\pi_H(O_1) \cap \pi_H(O_2)\big)\Big)\\ &= \pi_H\Big(\underbrace{\pi_H^{-1}\big(\pi_H(O_1)\big)}_{\text{belongs to } \tau_G} \cap \underbrace{\pi_H^{-1}\big(\pi_H(O_2)\big)}_{\text{belongs to } \tau_G}\Big),\end{aligned} \tag{6.220}$$

by (6.219) and (6.218). Since $\pi_H(\emptyset) = \emptyset$, $\pi_H(G) = G/H$, and $\bigcup_{i\in I} \pi_H(O_i) = \pi_H(\bigcup_{i\in I} O_i)$ for any family of subsets $(O_i)_{i\in I}$ of G, we may ultimately conclude that

$$\tau := \big\{\pi_H(O) : \; O \in \tau_G\big\} \quad \text{is a topology on } \; G/H. \tag{6.221}$$

Moreover, (6.219) implies that $\pi_H : (G, \tau_G) \to (G/H, \tau)$ is a continuous function. This forces $\tau \subseteq \tau_{G/H}$ (cf. Definition 6.35). To prove the opposite inclusion, consider an arbitrary set $A \in \tau_{G/H}$. Then the continuity of π_H entails $O := \pi_H^{-1}(A) \in \tau_G$. Hence, $A = \pi(O)$ for some $O \in \tau_G$, by (6.218). This finishes the proof of (6.213). In the process, we have also established that the canonical projection π_H is continuous and open in the context of (6.214).

Next, suppose that (Y, τ_Y) is a topological space and that $\varphi : G/H \to Y$ is a given function. If $\varphi : \big(G/H, \tau_{G/H}\big) \to (Y, \tau_Y)$ is continuous, then, by what we have proved in part (2), the composition $\varphi \circ \pi_H : \big(G, \tau_G\big) \to (Y, \tau_Y)$ is also continuous. Conversely, if the function $\varphi \circ \pi_H : \big(G, \tau_G\big) \to (Y, \tau_Y)$ is continuous, then $O := \pi_H^{-1}\big(\varphi^{-1}(U)\big) \in \tau_G$ for every $U \in \tau_Y$. Based on this, on the fact that the canonical projection is open, and on (6.218), we then deduce that $\varphi^{-1}(U) = \pi_H(O) \in \tau_{G/H}$ for every $U \in \tau_Y$. This shows that $\varphi : \big(G/H, \tau_{G/H}\big) \to (Y, \tau_Y)$ is continuous, completing the proof of the version of the equivalence in (6.215) pertaining to continuity. In the case when $\varphi \circ \pi_H$ is open in the context of (6.215) and if $U \in \tau_{G/H}$, it follows from (6.213) that there exists $O \in \tau_G$ such that $U = \pi_H(O)$. In such a case, we conclude that $\varphi(U) = (\varphi \circ \pi_H)(U) \in \tau_Y$; hence, ultimately, φ is open. The remaining implications in (6.215) are clear from definitions, and this completes the proof of the claims made in part (3).

Finally, the claims in part (4) are consequences of (6.211) and what we already established in parts (3) and (2) (the latter used here with $H := \operatorname{Ker} T$). $\square$

To set the stage for introducing the quotient quasi-pseudonorm (cf. Comment 6.39 below), we discuss the following result.

Proposition 6.38. *Let G be a group, and suppose that $\psi : G \to [0, +\infty]$ is a given function. Also, assume that H is a normal subgroup of G, and define*

$$\widehat{\psi} : G/H \longrightarrow [0, +\infty], \qquad \widehat{\psi}([x]) := \inf\big\{\psi(y) : y \in [x]\big\}, \quad \forall\, x \in G. \tag{6.222}$$

Then the following properties hold.

(1) For every $x \in G$ one has $\widehat{\psi}([x]) = \inf\big\{\psi(x * h) : h \in H\big\} = \inf\big\{\psi(h * x) : h \in H\big\}$. Moreover, $\widehat{\psi}\big(e_{G/H}\big) = 0$ provided $\psi(e_G) = 0$.

(2) One has $\big(\widehat{\psi}\,\big)^{-1}\big(\{0\}\big) \subseteq \big\{e_{G/H}\big\}$ if and only if H is closed in $(G, \tau^{\rm R}_{\psi})$, if and only if H is closed in $(G, \tau^{\rm L}_{\psi})$.

(3) If ψ satisfies the quasisubadditivity condition (6.113), then for every $x, y \in G$

$$\widehat{\psi}([x] \star [y]) \le C_1\big(\widehat{\psi}([x]) + \widehat{\psi}([y])\big), \tag{6.223}$$

where $C_1 \in [1, +\infty)$ is as in (6.113).

(4) If ψ satisfies the quasisymmetry condition (6.112), then $\widehat{\psi}\big([x]^{-1}\big) \le C_0\, \widehat{\psi}([x])$ for every $x \in G$, where $C_0 \in [1, +\infty)$ is as in (6.112).

(5) If ψ satisfies the quasi-invariance condition (6.114), then

$$\widehat{\psi}\big([x]^{-1} \star [y] \star [x]\big) \le C_2\, \widehat{\psi}([y]) \quad \text{for every} \;\; x, y \in G, \tag{6.224}$$

where $C_2 \in [1, +\infty)$ is as in (6.114).

(6) If π_H denotes the canonical projection of G onto G/H, then for any $a \in G$ and any $r \in (0, +\infty)$ one has

$$\pi_H\big(B^{\rm R}_{\psi}(a, r)\big) = B^{\rm R}_{\widehat{\psi}}(\pi_H(a), r) \;\; \text{and} \;\; \pi_H\big(B^{\rm L}_{\psi}(a, r)\big) = B^{\rm L}_{\widehat{\psi}}(\pi_H(a), r). \tag{6.225}$$

Furthermore, if $\tau^{\rm R}_{G/H}$ and $\tau^{\rm L}_{G/H}$ denote the quotient topologies on G/H relative to $\tau^{\rm R}_{\psi}$ and $\tau^{\rm L}_{\psi}$, respectively (in the sense of Definition 6.35), then

$$\tau^{\rm R}_{G/H} = \tau^{\rm R}_{\widehat{\psi}} \;\; \text{and} \;\; \tau^{\rm L}_{G/H} = \tau^{\rm L}_{\widehat{\psi}}. \tag{6.226}$$

Before presenting the proof of this result, we make some relevant remarks pertaining to its significance.

Comment 6.39. (i) An upshot of Proposition 6.38 is that if ψ is a quasi-pseudonorm on G, then $\widehat{\psi}$ is a quasi-pseudonorm on the quotient group G/H. As such, it is natural to refer to $\widehat{\psi}$ as the `quotient quasi-pseudonorm` induced by ψ on G/H. Note that, as seen from Proposition 6.38, the quotient quasi-pseudonorm $\widehat{\psi}$ is quasi-invariant on G/H if ψ itself is quasi-invariant on G. Moreover, $\widehat{\psi}$ satisfies the nondegeneracy condition $\big(\widehat{\psi}\,\big)^{-1}\big(\{0\}\big) = \big\{e_{G/H}\big\}$ whenever ψ vanishes at e_G and H is closed (with respect to either $\tau^{\rm R}_{\psi}$ or $\tau^{\rm L}_{\psi}$). In particular, under such a condition on H and assuming that ψ is a quasi-pseudonorm on G, it follows that $\widehat{\psi}$ is a quasinorm on the quotient group G/H.

(ii) In light of part *(vii)* in Lemma 6.14, under the assumptions in part *(5)* of Proposition 6.38, we have $\tau^{\rm R}_{\widehat{\psi}} = \tau^{\rm L}_{\widehat{\psi}}$. In such a scenario, we will simply use the symbol $\tau_{\widehat{\psi}}$ to denote either of these two identical topologies on G/H. ■

We now turn to the task of presenting the following proof.

Proof of Proposition 6.38 The claims in part (1) are clear from (6.222) and (6.203). Regarding the claim in part (2), consider first the case when H is closed in $(G, \tau^{\rm R}_{\psi})$. In this scenario, let $x \in G$ be such that $\widehat{\psi}([x]) = 0$. In view of part (1), it follows that for each $n \in \mathbb{N}$ there exists $h_n \in H$ such that $\psi(h_n * x) < n^{-1}$. This entails $h_n^{-1} \in H \cap B^{\rm R}_{\psi}(x, n^{-1})$ for every $n \in \mathbb{N}$; hence, $H \cap B^{\rm R}_{\psi}(x, n^{-1}) \neq \emptyset$ for every $n \in \mathbb{N}$. Thus, ultimately, $x \in {\rm Clo}\big(H; \tau^{\rm R}_{\psi}\big) = H$. Consequently, $[x] = e_{G/H}$, which proves that $\big(\widehat{\psi}\,\big)^{-1}\big(\{0\}\big) \subseteq \big\{e_{G/H}\big\}$.

Conversely, suppose the aforementioned inclusion holds and pick an arbitrary element $x \in {\rm Clo}\big(H; \tau^{\rm R}_{\psi}\big)$. Then for every $n \in \mathbb{N}$ there exists an element $h_n \in H$ such that $h_n \in H \cap B^{\rm R}_{\psi}(x, n^{-1})$. This forces $\psi(h_n^{-1} * x) < n^{-1}$ for every $n \in \mathbb{N}$ and, further, $\widehat{\psi}([x]) = 0$ by part (1). Granted the current hypotheses, we therefore necessarily have $[x] = e_{G/H}$. Thus, ultimately, $x \in H$, which goes to show that ${\rm Clo}\big(H; \tau^{\rm R}_{\psi}\big) = H$, as desired. The case when $\tau^{\rm L}_{\psi}$ is considered in place of $\tau^{\rm R}_{\psi}$ is treated analogously, and this finishes the proof of the claims made in part (2).

Consider next the claim made in part (3). Concretely, assume that ψ is quasisubadditive with constant $C_1 \in [1, +\infty)$, and pick two arbitrary elements $x, y \in G$. Also, fix an arbitrary number $\varepsilon > 0$. Then there exist $h_1, h_2 \in H$ such that

$$\psi(x * h_1) < \widehat{\psi}([x]) + \varepsilon/(2C_1), \qquad \psi(y * h_2) < \widehat{\psi}([y]) + \varepsilon/(2C_1). \tag{6.227}$$

Also, since H is normal, there exists $h_3 \in H$ with the property that $y * h_3 = h_1 * y$ (specifically, $h_3 := y * h_1 * y^{-1} \in H$). Consequently,

$$\begin{aligned}
\psi\big((x * y) * (h_3 * h_2)\big) &= \psi\big((x * h_1) * (y * h_2)\big) \leq C_1\big(\psi(x * h_1) + \psi(y * h_2)\big) \\
&\leq C_1\big(\widehat{\psi}([x]) + \varepsilon/(2C_1) + \widehat{\psi}([y]) + \varepsilon/(2C_1)\big) \\
&= C_1\big(\widehat{\psi}([x]) + \widehat{\psi}([y])\big) + \varepsilon.
\end{aligned} \tag{6.228}$$

Together with the results proved in part (1) and the first formula in (6.205), this yields

$$\widehat{\psi}([x] \star [y]) \le C_1\big(\widehat{\psi}([x]) + \widehat{\psi}([y])\big) + \varepsilon. \tag{6.229}$$

Hence, since the number $\varepsilon > 0$ was arbitrarily chosen, we arrive at the conclusion that $\widehat{\psi}([x] \star [y]) \le C_1\big(\widehat{\psi}([x]) + \widehat{\psi}([y])\big)$. This concludes the proof of the claim made in part (3).

Moving on to the claim made in part (4), suppose that the function ψ is quasisymmetric with constant $C_0 \in [1, +\infty)$, and fix an arbitrary element $x \in G$. Also, let $\varepsilon > 0$ be arbitrary and, using the results proved in part (1), select $h \in H$ such that $\psi(x * h) < \widehat{\psi}([x]) + \varepsilon/C_0$. Finally, consider $h' := x * h^{-1} * x^{-1} \in H$. Then

$$\psi(x^{-1} * h') = \psi(h^{-1} * x^{-1}) \le C_0\,\psi(x * h) < C_0\,\widehat{\psi}([x]) + \varepsilon, \tag{6.230}$$

which, ultimately, implies (cf. the second formula in (6.205)) that $\widehat{\psi}\big([x]^{-1}\big) \le C_0\,\widehat{\psi}([x])$, as desired.

Next, consider (6.224) in the case when ψ satisfies the quasi-invariance condition (6.114). In such a scenario, pick two arbitrary elements $x, y \in G$ and some number $\varepsilon > 0$. Also, select some $h \in H$ with the property that

$$\psi(y * h) < \widehat{\psi}\big([y]\big) + \varepsilon/C_2. \tag{6.231}$$

Then, since $h' := x^{-1} * h * x \in H$, we may estimate, based on (6.205), part (1), (6.114), and (6.231),

$$\begin{aligned}\widehat{\psi}\big([x]^{-1} \star [y] \star [x]\big) &= \widehat{\psi}\big([x^{-1} * y * x]\big) \le \psi(x^{-1} * y * x * h') \\ &= \psi(x^{-1} * y * h * x) \le C_2\,\psi(y * h) \\ &\le C_2\,\widehat{\psi}([y]) + \varepsilon,\end{aligned} \tag{6.232}$$

where $C_2 \in [1, +\infty)$ is as in (6.114). Now (6.224) follows upon letting $\varepsilon \searrow 0$ in (6.232). This completes the proof of the claim in part (5).

As far as the first formula in (6.225) is concerned, observe that if $x, a \in G$ and $r \in (0, +\infty)$ are such that $\psi(a * x^{-1}) < r$, then

$$\widehat{\psi}([a] \star [x]^{-1}) = \widehat{\psi}([a * x]^{-1}) \le \psi(a * x^{-1}) < r. \tag{6.233}$$

Hence, $\pi_H(x) = [x] \in B^{\mathrm{R}}_{\widehat{\psi}}\big([a], r\big) = B^{\mathrm{R}}_{\widehat{\psi}}(\pi_H(a), r)$, proving the left-to-right inclusion. In the converse direction, if $x, a \in G$ and $r \in (0, +\infty)$ are such that $\widehat{\psi}\big([a]\star[x]^{-1}\big) < r$, then there exists $h \in H$ with the property that $\psi\big(a*x^{-1}*h\big) < r$. This implies that $h^{-1} * x \in B^{\mathrm{R}}_{\psi}(a, r)$, and we also have $\pi_H(h^{-1} * x) = [x]$ since

$x^{-1} * h^{-1} * x \in H$. Thus, $[x] \in \pi_H\big(B^{\mathrm{R}}_{\psi}(a,r)\big)$, as desired. This concludes the proof of the first formula in (6.225), and the second one is established similarly.

Going further, assume that $O \in \tau^{\mathrm{R}}_{G/H}$, and fix some $a \in G$ with $\pi_H(a) \in O$. Then $a \in \pi_H^{-1}(O) \in \tau^{\mathrm{R}}_{\psi}$ by (6.209). As such, there exists $r > 0$ with the property that $B^{\mathrm{R}}_{\psi}(a,r) \subseteq \pi_H^{-1}(O)$, which further entails $B^{\mathrm{R}}_{\widehat{\psi}}(\pi_H(a),r) = \pi_H\big(B^{\mathrm{R}}_{\psi}(a,r)\big) \subseteq O$, thanks to (6.225). All things considered, this argument proves that $O \in \tau^{\mathrm{R}}_{\widehat{\psi}}$; hence, ultimately, $\tau^{\mathrm{R}}_{G/H} \subseteq \tau^{\mathrm{R}}_{\widehat{\psi}}$. To prove the opposite inclusion, given an arbitrary set $\mathcal{O} \in \tau^{\mathrm{R}}_{\widehat{\psi}}$, the goal is to show that $\pi_H^{-1}(\mathcal{O}) \in \tau^{\mathrm{R}}_{\psi}$ (cf. (6.209)). To this end, fix some arbitrary $a \in \pi_H^{-1}(\mathcal{O})$ and note that this implies that $\pi_H(a) \in \mathcal{O} \in \tau^{\mathrm{R}}_{\widehat{\psi}}$. Consequently, there exists $r > 0$ such that $B^{\mathrm{R}}_{\widehat{\psi}}(\pi_H(a),r) \subseteq \mathcal{O}$, which, in view of (6.225), implies that $\pi_H\big(B^{\mathrm{R}}_{\psi}(a,r)\big) \subseteq \mathcal{O}$ and, further, $B^{\mathrm{R}}_{\psi}(a,r) \subseteq \pi_H^{-1}(\mathcal{O})$. Thus, $\pi_H^{-1}(\mathcal{O}) \in \tau^{\mathrm{R}}_{\psi}$, completing the proof of the first formula in (6.226). The second formula in (6.226) is dealt with analogously. This concludes the treatment of the claims made in part (6) and completes the proof of the proposition. □

To state our next result, which further augments the list of properties from Proposition 6.38, the reader may find it useful to recall the convention made in part (*ii*) of Comment 6.39.

Proposition 6.40. *Let $(G,*)$ be a group, and let the function $\psi : G \to [0,+\infty)$ be quasi-invariant and quasisubadditive. In addition, assume that H is a normal subgroup of G and define $\widehat{\psi}$ as in (6.222). Then*

$$\big(G,*,\tau_{\psi}\big) \ \textit{complete} \implies \big(G/H,\star,\tau_{\widehat{\psi}}\big) \ \textit{complete}. \tag{6.234}$$

Proof. The fact that Proposition 6.38 guarantees that $\widehat{\psi}$ is a quasi-invariant, quasisubadditive function on the quotient group G/H opens the door to implementing the completeness criterion described in Proposition 6.34. To do so, recall from Proposition 6.38 that the constants (C_0, C_1) of the quasi-pseudonorm ψ on G may also be regarded as the constants of the quasi-pseudonorm $\widehat{\psi}$ on G/H. Hence, in agreement with (6.193), fix a number $\beta \in \big(0,(1+\log_2 C_1)^{-1}\big]$ and assume that $\big([x_n]\big)_{n\in\mathbb{N}} \subseteq G/H$ is a sequence with the property that

$$\sum_{n=1}^{\infty} \widehat{\psi}\big([x_n]\big)^{\beta} < +\infty. \tag{6.235}$$

Based on part (1) of Proposition 6.38, for each $n \in \mathbb{N}$ there exists $h_n \in H$ such that

$$\psi(x_n * h_n) \le \widehat{\psi}\big([x_n]\big) + 2^{-n}. \tag{6.236}$$

Hence, given that $0 < \beta \le 1$, we may estimate

$$\sum_{n=1}^{\infty} \psi(x_n * h_n)^{\beta} \le \sum_{n=1}^{\infty}\Big(\widehat{\psi}\big([x_n]\big) + 2^{-n}\Big)^{\beta} \le \sum_{n=1}^{\infty}\Big(\widehat{\psi}\big([x_n]\big)^{\beta} + 2^{-n\beta}\Big) < +\infty, \tag{6.237}$$

thanks to (6.235).

Given that the function $\psi : G \to [0, +\infty)$ is quasi-invariant and quasisubadditive on the group $(G, *)$, and since we are assuming that $(G, *, \tau_\psi)$ is complete, Proposition 6.34 may be invoked to conclude that there exists $y \in G$ with the property that

$$\text{the sequence } \Big\{(x_1 * h_1) * \cdots * (x_n * h_n)\Big\}_{n\in\mathbb{N}} \text{ converges to } y \text{ in } (G, *, \tau_\psi). \tag{6.238}$$

Now, the fact that H is a normal subgroup of G (cf. (6.200)) allows us to find, for each $n \in \mathbb{N}$, an element $h'_n \in H$ with the property that

$$(x_1 * h_1) * \cdots * (x_n * h_n) * y^{-1} = (x_1 * x_2 * \cdots * x_{n-1} * x_n) * y^{-1} * h'_n. \tag{6.239}$$

As such, for each $n \in \mathbb{N}$ we may estimate

$$\begin{aligned} \widehat{\psi}\Big(\big([x_1] \star [x_2] \star \cdots \star [x_n]\big) \star [y]^{-1}\Big) &= \widehat{\psi}\big([x_1 * x_2 * \cdots * x_n * y^{-1}]\big) \\ &\leq \psi\big((x_1 * x_2 * \cdots * x_{n-1} * x_n * y^{-1}) * h'_n\big) \\ &= \psi\big((x_1 * h_1) * \cdots * (x_n * h_n) * y^{-1}\big). \end{aligned} \tag{6.240}$$

Since (6.238) forces the limit of the last expression in (6.240) to vanish as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \widehat{\psi}\Big(\big([x_1] \star [x_2] \star \cdots \star [x_n]\big) \star [y]^{-1}\Big) = 0. \tag{6.241}$$

To sum up, for any sequence $\big([x_n]\big)_{n\in\mathbb{N}} \subseteq G/H$ satisfying (6.235) there exists $y \in G$ with the property that

$$\text{the sequence } \big\{[x_1] \star \cdots \star [x_n]\big\}_{n\in\mathbb{N}} \text{ converges to } [y] \text{ in } (G/H, \star, \tau_{\widehat{\psi}}). \tag{6.242}$$

Hence, Proposition 6.34 applies and gives that $\big(G/H, \star, \tau_{\widehat{\psi}}\big)$ is complete. □

Definition 6.41. Let G and S be two groups, and assume that $\psi : S \to [0, +\infty]$ is an arbitrary function. Given $T \in \mathrm{Hom}\,(G, S)$, define the `pullback` of ψ under T as the function

$$\widetilde{\psi} : G \to [0, +\infty], \qquad \widetilde{\psi}(x) := \psi(Tx), \quad \forall\, x \in G. \tag{6.243}$$

We now proceed to discuss the properties of the pullback of a quasi-pseudonorm under a group homomorphism.

Proposition 6.42. *Let $(G, *)$ and $(S, \circ)$ be two groups, and assume that ψ is a quasi-pseudonorm on S. Also, suppose that $T \in \operatorname{Hom}(G, S)$ and define $\widetilde{\psi}$ as in (6.243). Then the following claims are valid.*

(1) The function $\widetilde{\psi}$ is a quasi-pseudonorm on G with the same constants as those of the quasi-pseudonorm ψ on S.

(2) One has

$$\psi \text{ quasi-invariant} \Longrightarrow \widetilde{\psi} \text{ quasi-invariant (with the same constant)}, \tag{6.244}$$

$$\psi \text{ quasinorm and } T \text{ injective} \Longrightarrow \widetilde{\psi} \text{ quasinorm.} \tag{6.245}$$

(3) The following formulas hold:

$$T^{-1}\big(B^{\mathrm{R}}_{\psi}(Ta, r)\big) = B^{\mathrm{R}}_{\widetilde{\psi}}(a, r), \qquad T^{-1}\big(B^{\mathrm{L}}_{\psi}(Ta, r)\big) = B^{\mathrm{L}}_{\widetilde{\psi}}(a, r) \tag{6.246}$$

for every $a \in G$ and every $r \in (0, +\infty)$. In particular,

$$T : (G, \tau^{\mathrm{R}}_{\widetilde{\psi}}) \to (S, \tau^{\mathrm{R}}_{\psi}) \text{ and } T : (G, \tau^{\mathrm{L}}_{\widetilde{\psi}}) \to (S, \tau^{\mathrm{L}}_{\psi}) \text{ are continuous.} \tag{6.247}$$

(4) For every $a \in G$ and every $r \in (0, +\infty)$ one has

$$\begin{aligned} T\big(B^{\mathrm{R}}_{\widetilde{\psi}}(a, r)\big) &= \big(\operatorname{Im} T\big) \cap B^{\mathrm{R}}_{\psi}(Ta, r), \\ T\big(B^{\mathrm{L}}_{\widetilde{\psi}}(a, r)\big) &= \big(\operatorname{Im} T\big) \cap B^{\mathrm{L}}_{\psi}(Ta, r). \end{aligned} \tag{6.248}$$

As a consequence, the following equivalences are valid:

$$\operatorname{Im} T \text{ open in } (S, \tau^{\mathrm{R}}_{\psi}) \Longleftrightarrow T : (G, \tau^{\mathrm{R}}_{\widetilde{\psi}}) \to (S, \tau^{\mathrm{R}}_{\psi}) \text{ is open,} \tag{6.249}$$

$$\operatorname{Im} T \text{ open in } (S, \tau^{\mathrm{L}}_{\psi}) \Longleftrightarrow T : (G, \tau^{\mathrm{L}}_{\widetilde{\psi}}) \to (S, \tau^{\mathrm{L}}_{\psi}) \text{ is open.} \tag{6.250}$$

(5) One has

$$\left.\begin{aligned} \psi \text{ finite and quasi-invariant} \\ \operatorname{Im} T \text{ closed in } (S, \tau_{\psi}) \\ (S, \circ, \tau_{\psi}) \text{ complete} \end{aligned}\right\} \Longrightarrow \big(G, *, \tau_{\widetilde{\psi}}\big) \text{ complete.} \tag{6.251}$$

Proof. With the exception of part *(5)*, all other claims are straightforward consequences of definitions. Concerning the implication in part *(5)*, assume that the assumptions stipulated on the left-hand side of (6.251) hold. We already know from the first part in the proof that $\widetilde{\psi}$ is a quasi-invariant quasi-pseudonorm on the group G and with the same constants (C_0, C_1) as ψ. To proceed, pick some

$\beta \in \big(0, (1+\log_2 C_1)^{-1}\big]$ and assume that $(a_n)_{n\in\mathbb{N}} \subseteq G$ is a sequence with the property that

$$\sum_{n=1}^{\infty} \widetilde{\psi}(a_n)^{\beta} < +\infty. \tag{6.252}$$

In view of (6.243), this gives

$$\sum_{n=1}^{\infty} \psi(Ta_n)^{\beta} < +\infty, \tag{6.253}$$

and since $(S, \circ, \tau_\psi)$ is complete, Proposition 6.34 applies and gives that there exists $y \in S$ such that

$$\Big\{(Ta_1)\circ(Ta_2)\circ\cdots\circ(Ta_n)\Big\}_{n\in\mathbb{N}} \text{ is convergent to } y \text{ in } (S, \circ, \tau_\psi). \tag{6.254}$$

However, for each $n \in \mathbb{N}$ we have $(Ta_1)\circ(Ta_2)\circ\cdots\circ(Ta_n) = T(a_1 * \cdots * a_n) \in \operatorname{Im} T$, and we know that $\operatorname{Im} T$ is closed in $(S, \circ, \tau_\psi)$. Consequently, the limit y belongs to $\operatorname{Im} T$, i.e., there exists $a \in G$ such that $Ta = y$. Keeping this in mind allows us to reinterpret (6.254) as

$$0 = \lim_{n\to\infty} \psi\Big(T(a_1 * \cdots * a_n)\circ(Ta)^{-1}\Big) = \lim_{n\to\infty} \widetilde{\psi}\big((a_1 * \cdots * a_n) * a^{-1}\big) \tag{6.255}$$

or, equivalently,

$$\big\{a_1 * a_2 * \cdots * a_n\big\}_{n\in\mathbb{N}} \text{ is convergent to } a \text{ in } \big(G, *, \tau_{\widetilde{\psi}}\big). \tag{6.256}$$

With this in hand, the completeness criterion described in Proposition 6.34 gives that $\big(G, *, \tau_{\widetilde{\psi}}\big)$ is complete, thus completing the proof of the proposition. □

Comment 6.43. In the context of Proposition 6.42 it is natural to refer to the function $\widetilde{\psi}$ defined in (6.243) as the `pullback quasi-pseudonorm` of ψ to G under the group homomorphism $T \in \operatorname{Hom}(G, S)$. ■

We continue by presenting yet another basic property of the pullback that will be relevant for us later.

Proposition 6.44. *Let $(G, *)$ be a group, and assume that τ_G is a topology on the set G. Also, let $(S, \circ)$ be an Abelian group, and suppose that $\psi : S \to [0, +\infty)$ is a quasisubadditive and quasisymmetric function, in the sense that there exist finite constants $\kappa, C_0 \in [1, +\infty)$ such that*

$$\psi(a\circ b) \le \kappa \max\big\{\psi(a), \psi(b)\big\} \quad \textit{for all} \ \ a, b \in S, \tag{6.257}$$

$$\psi(a^{-1}) \le C_0 \psi(a) \quad \textit{for each} \ \ a \in S. \tag{6.258}$$

In addition, assume that $(S, \circ, \tau_\psi)$ *is complete.*

Finally, assume that $T \in \mathrm{Hom}(G, S)$ *is such that its graph,* $\mathscr{G}_T$, *is a closed subset of* $(G \times S, \tau_G \times \tau_\psi)$. *Then for every number* $\beta \in \big(0, (\log_2 \kappa)^{-1}\big]$ *one has*

$$\widetilde{\psi}(x) \leq \kappa^4 C_0 \Big\{ \sum_{i=1}^{\infty} \widetilde{\psi}(x_i)^\beta \Big\}^{1/\beta}, \qquad \forall\, x \in G \ \ and \ \ \forall\, (x_n)_{n\in\mathbb{N}} \subseteq G \tag{6.259}$$

with the property that the sequence $(x_1 * \cdots * x_n)_{n\in\mathbb{N}}$ *converges in* τ_G *to* x,

where $\widetilde{\psi}$ *denotes the pullback to* G *of the function* ψ *under the group homomorphism* T.

Proof. Fix $\beta \in \big(0, (\log_2 \kappa)^{-1}\big]$, and consider $x \in G$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\big(x_1 * \cdots * x_n\big)_{n\in\mathbb{N}} \ \text{converges in} \ \ \tau_G \ \ \text{to} \ \ x. \tag{6.260}$$

Also, without loss of generality, assume that

$$\sum_{n=1}^{\infty} \psi(Tx_n)^\beta = \sum_{n=1}^{\infty} \widetilde{\psi}(x_n)^\beta < +\infty. \tag{6.261}$$

Then for each $k \in \mathbb{N}$ we have

$$\psi\big((Tx_n) \circ \cdots \circ (Tx_{n+k})\big) \leq \kappa^2 \Big\{ \sum_{i=n}^{\infty} \psi(Tx_i)^\beta \Big\}^{1/\beta} \longrightarrow 0 \ \ \text{as} \ \ n \to \infty, \tag{6.262}$$

based on (6.141) and (6.261). This shows that $\big((Tx_1) \circ \cdots \circ (Tx_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(S, \circ, \tau_\psi\big)$ (recall that S is assumed to be Abelian). Since the latter is assumed to be a complete group, we deduce that there exists $y \in S$ with the property that

$$\begin{aligned} &\text{the sequence} \ \ \big(T(x_1 * \cdots * x_n)\big)_{n\in\mathbb{N}} = \big((Tx_1) \circ \cdots \circ (Tx_n)\big)_{n\in\mathbb{N}} \\ &\qquad\qquad \text{converges to} \ \ y \ \ \text{in} \ \ \big(S, \circ, \tau_\psi\big). \end{aligned} \tag{6.263}$$

From (6.260), (6.263), and the fact that T is a group homomorphism whose graph $\mathscr{G}_T$ is a closed subset of $(G \times S, \tau_G \times \tau_\psi)$, we may conclude that $y = Tx$. Consequently,

$$\begin{aligned} \psi(Tx) = \psi(y) &= \psi\Big(\lim_{n\to\infty} (Tx_1) \circ \cdots \circ (Tx_n)\Big) \\ &\leq \kappa^2 C_0 \limsup_{n\to\infty} \psi\big((Tx_1) \circ \cdots \circ (Tx_n)\big) \\ &\leq \kappa^4 C_0 \Big\{ \sum_{i=1}^{\infty} \psi(Tx_i)^\beta \Big\}^{1/\beta}, \end{aligned} \tag{6.264}$$

by (6.263), (6.142), and (6.141). Now (6.259) follows from (6.264) and (6.243). □

Moving on, we make the following definition.

Definition 6.45. Assume that G and S are two groups and that ψ is a quasi-pseudonorm on G. In addition, suppose that a surjective group homomorphism $T \in \mathrm{Hom}\,(G, S)$ has been given; in particular, $\widehat{T} : G/\mathrm{Ker}\,T \to S$ is an isomorphism (cf. Theorem 6.36).

Then the `push-forward` of the quasi-pseudonorm ψ via the surjective group homomorphism T is defined as the pullback to S of the quotient quasi-pseudonorm $\widehat{\psi}$ on $G/\mathrm{Ker}\,T$ under the group homomorphism $\big(\widehat{T}\big)^{-1} \in \mathrm{Hom}\,\big(S, G/\mathrm{Ker}\,T\big)$.

Hence, if $\overline{\psi}$ denotes the push-forward of ψ via T in the manner and conditions specified in Definition 6.45, it follows that

$$\overline{\psi} : S \to [0, +\infty], \qquad \overline{\psi}(x) := \widehat{\psi}\Big(\big(\widehat{T}\big)^{-1} x\Big), \quad \forall\, x \in S. \tag{6.265}$$

By further deobfuscating the notation, we obtain

$$\overline{\psi}(x) = \inf\big\{\psi(z) : z \in T^{-1}(\{x\})\big\}, \qquad \forall\, x \in S. \tag{6.266}$$

Other relevant properties of the function (6.265) are described in our next theorem (for the last item, see also the comments right after its statement).

Theorem 6.46. *Let $(G, *)$ be a group equipped with a quasi-pseudonorm ψ. Also, suppose that $(S, \circ)$ is another group, and assume that $T \in \mathrm{Hom}\,(G, S)$ is given a surjective homomorphism. Finally, denote by $\overline{\psi}$ the push-forward of ψ via T in the sense of Definition 6.45. Then the following properties hold.*

(a) The function $\overline{\psi}$ is a quasi-pseudonorm on S, with the same constants (C_0, C_1) as ψ.
(b) For any topology τ_S on the set S one has the equivalences

$$T : (G, \tau_\psi^{\mathrm{R}}) \to (S, \tau_S) \ \textit{ is continuous and open} \iff \tau_S = \tau_{\overline{\psi}}^{\mathrm{R}}, \tag{6.267}$$

$$T : (G, \tau_\psi^{\mathrm{L}}) \to (S, \tau_S) \ \textit{ is continuous and open} \iff \tau_S = \tau_{\overline{\psi}}^{\mathrm{L}}. \tag{6.268}$$

As a corollary,

$$\tau_{\overline{\psi}}^{\mathrm{R}} \subseteq \tau_S \implies T : (G, \tau_\psi^{\mathrm{R}}) \to (S, \tau_S) \ \textit{ open}, \tag{6.269}$$

$$\tau_{\overline{\psi}}^{\mathrm{L}} \subseteq \tau_S \implies T : (G, \tau_\psi^{\mathrm{L}}) \to (S, \tau_S) \ \textit{ open}. \tag{6.270}$$

Hence,

$$\left.\begin{array}{r}\tau_S \ \ \textit{is right-invariant and} \\ B^{\mathrm{R}}_{\overline{\psi}}(e_S, r) \in \mathscr{N}(e_S; \tau_S), \ \ \forall\, r > 0\end{array}\right\} \Longrightarrow T : (G, \tau^{\mathrm{R}}_{\psi}) \to (S, \tau_S) \ \ \textit{open}, \tag{6.271}$$

$$\left.\begin{array}{r}\tau_S \ \ \textit{is left-invariant and} \\ B^{\mathrm{L}}_{\overline{\psi}}(e_S, r) \in \mathscr{N}(e_S; \tau_S), \ \ \forall\, r > 0\end{array}\right\} \Longrightarrow T : (G, \tau^{\mathrm{L}}_{\psi}) \to (S, \tau_S) \ \ \textit{open}. \tag{6.272}$$

(c) If ψ is quasi-invariant, then so is $\overline{\psi}$ and with the same constant.
(d) If ψ is finite and quasi-invariant, and if $\big(G, *, \tau_\psi\big)$ is complete, then $\big(S, \circ, \tau_{\overline{\psi}}\big)$ is complete.

Proof. Recall that the function $\overline{\psi}$ is obtained from ψ by first passing to the quotient quasi-pseudonorm $\widehat{\psi}$ on $G/\mathrm{Ker}\, T$ then taking the pullback of this to S under the group homomorphism $\big(\widehat{T}\big)^{-1} \in \mathrm{Hom}\,\big(S, G/\mathrm{Ker}\, T\big)$. Since both operations preserve the quality of being a quasi-pseudonorm as well as the associated constants (cf. Proposition 6.38, Comment 6.39, and Proposition 6.42), the claim in part *(a)* follows. The claim in part *(c)* is justified similarly, based on (6.224) and (6.244), while the claim in part *(d)* is seen with the help of Proposition 6.40 and (6.251).

There remains to consider the claims made in part *(b)*. To this end, assume that τ_S is a topology on the set S with the property that

$$T : (G, \tau^{\mathrm{R}}_{\psi}) \longrightarrow (S, \tau_S) \ \ \text{is continuous and open.} \tag{6.273}$$

Now, $T \in \mathrm{Hom}\,(G, S)$ is known to be surjective, and recall (from part (6) in Proposition 6.38) that $\tau^{\mathrm{R}}_{G/\mathrm{Ker}\, T}$ denotes the quotient topology on $G/\mathrm{Ker}\, T$ relative to τ^{R}_{ψ}. As such, part (4) of Proposition 6.37 applies and gives that the assumption just made is equivalent to the condition that

$$\widehat{T} : \big(G/\mathrm{Ker}\, T,\ \tau^{\mathrm{R}}_{G/\mathrm{Ker}\, T}\big) \longrightarrow (S, \tau_S) \ \ \text{is a homeomorphism.} \tag{6.274}$$

In particular, since such a condition determines τ_S uniquely, this shows that there could be only one topology τ_S on the set S that does the job specified in (6.273). Having established this, there remains to show that

$$\widehat{T} : \big(G/\mathrm{Ker}\, T,\ \tau^{\mathrm{R}}_{G/\mathrm{Ker}\, T}\big) \longrightarrow (S, \tau^{\mathrm{R}}_{\overline{\psi}}) \ \ \text{is a homeomorphism.} \tag{6.275}$$

Of course, since this function is bijective (cf. Theorem 6.36) and since $\tau^{\mathrm{R}}_{G/\mathrm{Ker}\, T} = \tau^{\mathrm{R}}_{\widehat{\psi}}$ by (6.226), condition (6.275) is further equivalent to the demand that

$$\big(\widehat{T}\big)^{-1} : (S, \tau^{\mathrm{R}}_{\overline{\psi}}) \longrightarrow \big(G/\mathrm{Ker}\, T,\ \tau^{\mathrm{R}}_{\widehat{\psi}}\big) \ \ \text{is continuous and open.} \tag{6.276}$$

This, however, follows from Definition 6.45, (6.247), and (6.249). To summarize, the only topology τ_S on S that makes $T : (G, \tau_\psi^{\mathrm{R}}) \to (S, \tau_S)$ continuous and open is $\tau_{\overline{\psi}}^{\mathrm{R}}$. In a similar manner, one can show that the only topology τ_S on S that makes the function $T : (G, \tau_\psi^{\mathrm{L}}) \to (S, \tau_S)$ continuous and open is $\tau_{\overline{\psi}}^{\mathrm{L}}$. This takes care of the equivalences stated in (6.267) and (6.268), and the implications in (6.269) and (6.270) are obvious consequences of them (e.g., $T : (G, \tau_\psi^{\mathrm{R}}) \to (S, \tau_{\overline{\psi}}^{\mathrm{R}})$ is open according to (6.267) and the fact that $\tau_{\overline{\psi}}^{\mathrm{R}} \subseteq \tau_S$ makes $T : (G, \tau_\psi^{\mathrm{R}}) \to (S, \tau_S)$ open). Moreover, since if τ_S is right-invariant we have that $\tau_{\overline{\psi}}^{\mathrm{R}} \subseteq \tau_S$ if and only if $B_{\overline{\psi}}^{\mathrm{R}}(e_S, r) \in \mathscr{N}(e_S; \tau_S)$ for each $r > 0$, the implication in (6.271) follows from (6.269). Finally, (6.272) is proved similarly. This completes the proof of the claims made in part *(b)* and the proof of the theorem. □

We further augment the results established in Theorem 6.46 with the following proposition.

Proposition 6.47. *Assume that $(G, *)$ is a group equipped with a finite, quasi-invariant, quasi-pseudonorm ψ, with constants (C_0, C_1), with the property that $\big(G, *, \tau_\psi\big)$ is complete. Fix a subgroup H of the group G. Next, suppose that $(S, \circ)$ is a group and that $T \in$ Hom (H, S) is surjective. Also, assume that τ_S is a topology on S for which $\mathscr{G}_T$ is a closed subset of $\big(G \times S, \tau_\psi \times \tau_S\big)$. Finally, denote by $\overline{\psi}$ the push-forward of the restriction of ψ to H via T, in the sense of Definition 6.45, and fix a number $\beta \in \big(0, (1 + \log_2 C_1)^{-1}\big]$. Then*

$$\overline{\psi}(b) \leq (2C_1)^4 C_0 \Big\{ \sum_{n=1}^{\infty} \overline{\psi}(b_n)^\beta \Big\}^{1/\beta}, \qquad \forall\, b \in S \ \ \text{and} \ \ \forall\, (b_n)_{n\in\mathbb{N}} \subseteq S \tag{6.277}$$

with the property that $(b_1 \circ \cdots \circ b_n)_{n\in\mathbb{N}}$ is ordinarily convergent in τ_S to b.

A few comments regarding the nature of the conclusion are in order. First, the elementary properties of the restriction of a given quasi-pseudonorm to a subgroup were recorded in Lemma 6.25. Second, the reader may wish to review the discussion in Comment 6.17 in relation to the notion of convergence employed here. Third, even though $\overline{\psi}$ is known to be quasisubadditive and quasisymmetric, the estimate in (6.277) is *not* a direct consequence of (6.141) and (6.142). This is because the latter formula requires that convergence takes place in the topology induced by $\overline{\psi}$ either from the right or from the left on S, whereas in (6.277) we only assume that $(b_1 \circ \cdots \circ b_n)_{n\in\mathbb{N}}$ is ordinarily convergent to b in the topology τ_S.

Proof of Proposition 6.47 To get started, assume that the quasi-pseudonorm ψ is finite and quasi-invariant, that $\big(G, *, \tau_\psi\big)$ is complete, and that τ_S is a topology on S such that $\mathscr{G}_T$ is a closed subset of $\big(G \times S, \tau_\psi \times \tau_S\big)$. Also, fix some number $\beta \in \big(0, (1 + \log_2 C_1)^{-1}\big]$, and assume that $b \in S$ and $(b_n)_{n\in\mathbb{N}} \subseteq S$ are such that

$$(b_1 \circ \cdots \circ b_n)_{n\in\mathbb{N}} \ \text{ is ordinarily convergent to } b \text{ in } \tau_S. \tag{6.278}$$

If $\sum_{n=1}^{\infty} \overline{\psi}(b_n)^{\beta} = +\infty$, then the inequality in (6.277) is trivially satisfied, so assume in what follows that

$$\sum_{n=1}^{\infty} \overline{\psi}(b_n)^{\beta} < +\infty. \tag{6.279}$$

To proceed, fix an arbitrary $\varepsilon > 0$. Making use of (6.266), for each $n \in \mathbb{N}$ it is possible to select an element $a_n \in H$ such that

$$Ta_n = b_n \ \text{ and } \ \psi(a_n) < \overline{\psi}(b_n) + \varepsilon 2^{-n}. \tag{6.280}$$

Consequently,

$$\sum_{n=1}^{\infty} \psi(a_n)^{\beta} \leq \sum_{n=1}^{\infty} \big(\overline{\psi}(b_n) + \varepsilon 2^{-n}\big)^{\beta} \leq \sum_{n=1}^{\infty} \overline{\psi}(b_n)^{\beta} + \sum_{n=1}^{\infty} \varepsilon 2^{-n\beta} < +\infty, \tag{6.281}$$

by (6.280) and (6.279). In turn, since ψ is a finite quasi-invariant quasi-pseudonorm with the property that $\big(G, *, \tau_{\psi}\big)$ is complete, from (6.281) and Proposition 6.34 we deduce that there exists some $a \in G$ such that

$$\text{the sequence } \big(a_1 * \cdots * a_n\big)_{n \in \mathbb{N}} \text{ converges to } a \text{ in } (G, *, \tau_{\psi}). \tag{6.282}$$

Now, (6.280) and the fact that $T \in \operatorname{Hom}(H, S)$ imply

$$T(a_1 * \cdots * a_n) = (Ta_1) \circ (Ta_2) \circ \cdots \circ (Ta_n) = b_1 \circ \cdots \circ b_n, \qquad \forall\, n \in \mathbb{N}. \tag{6.283}$$

Hence, from (6.278), (6.283), and the assumption that $\mathscr{G}_T$, the graph of T, is a closed subset of $\big(G \times S, \tau_{\psi} \times \tau_S\big)$, we conclude that $a \in H$ and $Ta = b$. In concert with (6.266), this yields

$$\overline{\psi}(b) \leq \psi(a). \tag{6.284}$$

On the other hand, from (6.282), (6.142), (6.140), and (6.281) we obtain

$$\begin{aligned} \psi(a) &\leq (2C_1)^2 C_0 \limsup_{n \to \infty} \psi(a_1 * \cdots * a_n) \leq (2C_1)^4 C_0 \left\{ \sum_{i=1}^{\infty} \psi(a_i)^{\beta} \right\}^{\frac{1}{\beta}} \\ &\leq (2C_1)^4 C_0 \left\{ \sum_{n=1}^{\infty} \overline{\psi}(b_n)^{\beta} + \sum_{n=1}^{\infty} \varepsilon 2^{-n\beta} \right\}^{\frac{1}{\beta}}. \end{aligned} \tag{6.285}$$

At this stage, (6.277) follows from (6.284) and (6.285) after letting $\varepsilon \searrow 0$. $\square$

6.4 A Quantitative Open Mapping Theorem

One of the cornerstones of modern functional analysis is the open mapping theorem (OMT), also known as the Banach–Schauder theorem. This fundamental result states that *any continuous, surjective, linear operator between two Banach spaces is an open map*. More specifically, if X and Y are Banach spaces, and if $T : X \to Y$ is a continuous, surjective, linear operator, then $T(U)$ is an open set in Y whenever U is an open set in X.

The proof of the classical OMT (cf., e.g., [108]) makes use of the Baire category theorem, and completeness of both X and Y is an essential hypothesis. For instance, the theorem may fail if either space is assumed to be just a normed space. Relevant examples, which are probably part of the folklore, are as follows. First, let $(Y, \|\cdot\|_Y)$ be a Banach space, and pick an arbitrary linear functional Λ on Y. Define X to be the same vector space Y, this time equipped with the norm $\|x\|_X := \|x\|_Y + |\langle\Lambda, x\rangle|$. In this setting, consider T to be the identity operator. Then T is bijective and continuous since, trivially, $\|x\|_Y \leq \|x\|_X$. Now, if $(X, \|\cdot\|_X)$ were complete, this would entail, by the OMT, that T^{-1} is also bounded, hence, ultimately, that $\|\cdot\|_X \approx \|\cdot\|_Y$. In turn, this would imply that Λ is continuous. However, there are Banach spaces (necessarily of infinite dimension) for which there exist unbounded linear functionals. This proves the necessity of the completeness of X in the context of the OMT. As for the necessity of Y being complete, a simple counterexample is provided by taking both X and Y to be the space of continuously differentiable functions on $[0, 1]$, X equipped with the norm $\|f\|_X := \sup|f| + \sup|f'|$, Y equipped with the supremum norm, and T the identity operator.

That being said, the OMT continues to hold if both X and Y are F-spaces (this was originally proved by S. Banach in [13]; see also [69, Corollary 1.5, p. 10] for a more recent discussion). Recall that an F-space is a vector space furnished with a complete, additively invariant metric for which the scalar multiplication is separately continuous in each variable. It is therefore natural to search for other weaker contexts in which appropriate variants of the OMT are still valid. For example, on the algebraic side, it is natural to consider open-mapping-type theorems on groups (as opposed to vector spaces), equipped with certain topologies, since the conclusion one seeks (i.e., that the mapping in question is open) does not depend on the presence of an underlying vector space structure.

The discovery in the early 1930s (cf. [12]) of the fact that a suitable version of OMT holds for complete metric groups (satisfying additional hypotheses) eventually led to the development of a subbranch of functional analysis consisting of a collection of theorems aimed at identifying sufficient conditions ensuring that a surjective morphism (in a suitable algebraic/functional analytic framework) is open. Typically, such theorems are still called "OMTs," and many important results of this sort are already known (cf., e.g., [22, 41, 52, 63, 64, 71, 83, 98, 102] and references therein).

As was already mentioned, some of the most general formulations of the OMT are in the setting of complete metrizable topological groups. Here the goal is to further sharpen such results by weakening the hypotheses pertaining to the nature of the topologies considered on the groups in question. The main result in this section is Theorem 6.49 below. As a preamble, in the next lemma we isolate a useful technical ingredient.

Lemma 6.48. *Assume that* $(G, *)$ *is a group and that* $\psi : G \to [0, +\infty]$ *is an arbitrary function. Also, let* $(S, \circ)$ *be a group, and suppose that* τ_S *is a symmetric, right-invariant topology on* S*. Finally, consider* $T \in \mathrm{Hom}(G, S)$ *with the property that there exists* $r \in (0, +\infty)$ *such that*

$$\mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(e_G, r)\big); \tau_S\Big) \in \mathscr{N}(e_S; \tau_S). \tag{6.286}$$

Then for every $A \subseteq G$ *one has*

$$\mathrm{Clo}\big(T(A); \tau_S\big) \subseteq \bigcup_{a \in A} \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(a, r)\big); \tau_S\Big). \tag{6.287}$$

Finally, a similar inclusion holds when "right-invariant" is replaced by "left-invariant" and "right balls" are replaced by "left balls."

Proof. From (6.286) we have $V := \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(e_G, r)\big); \tau_S\Big) \in \mathscr{N}(e_S; \tau_S)$, and since the topology τ_S is symmetric, it follows from (6.47) that $V^{-1} \in \mathscr{N}(e_S; \tau_S)$. Hence, if we set $W := V \cap V^{-1}$, then

$$W \in \mathscr{N}(e_S; \tau_S), \quad W \subseteq V, \quad \text{and} \quad W^{-1} = W. \tag{6.288}$$

Next, consider a set $A \subseteq G$, and fix an arbitrary element

$$y \in \mathrm{Clo}\big(T(A); \tau_S\big). \tag{6.289}$$

Since (6.48) and assumptions ensure that $s_y^{\mathrm{R}} : (S, \tau_S) \to (S, \tau_S)$ is a homeomorphism, it follows from (6.4) that $W \circ y = s_y^{\mathrm{R}}(W) \in \mathscr{N}(y; \tau_S)$. Together with (6.289), the latter condition entails $(W \circ y) \cap T(A) \neq \emptyset$. Thus, there exists $a \in A$ such that $Ta \in W \circ y$, which in turn implies $(Ta)^{-1} \in y^{-1} \circ W^{-1} = y^{-1} \circ W$. Hence,

$$\begin{aligned} y \in W \circ Ta \subseteq V \circ Ta &= \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(e_G, r)\big); \tau_S\Big) \circ Ta = \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(e_G, r)\big) \circ Ta; \tau_S\Big) \\ &= \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(e_G, r) * a\big); \tau_S\Big) = \mathrm{Clo}\Big(T\big(B_{\psi}^{\mathrm{R}}(a, r)\big); \tau_S\Big). \end{aligned} \tag{6.290}$$

The first inclusion in (6.290) is a consequence of the fact that $W \subseteq V$ (cf. (6.288)), while the subsequent equality makes use of the definition of V. Also, the second equality in (6.290) takes into account (6.48) and (6.3). Finally, the third equality in (6.290) holds since $T \in \mathrm{Hom}(G, H)$, while the last equality is implied by (6.84).

Given that y in (6.289) has been arbitrarily chosen, (6.287) follows from (6.290). Finally, the last claim in the statement is dealt with similarly, and this finishes the proof of the lemma. □

We are now in a position to formulate and prove our first main result. In preparation, the reader may wish to recall Definition 6.2.

Theorem 6.49 (Quantitative OMT). *Let $(G,*)$ be a group, and assume that ψ is a finite quasi-pseudonorm on G, with constants (C_0, C_1), such that $(G,*)$ is complete with respect to τ_ψ^{R}. In addition, consider a group $(S,\circ)$, and suppose that τ_S is a symmetric, right-invariant topology on S. Finally, let $T \in \mathrm{Hom}\,(G,S)$, which is almost open at e_G, i.e.,*

$$\mathrm{Clo}\Big(T\big(B_\psi^{\mathrm{R}}(e_G,r)\big);\tau_S\Big) \in \mathscr{N}(e_S;\tau_S), \qquad \forall\, r \in (0,+\infty), \tag{6.291}$$

and such that its graph satisfies

$$\mathscr{G}_T \ \text{ is a closed subset of } \ \big(G \times S, \tau_\psi^{\mathrm{R}} \times \tau_S\big). \tag{6.292}$$

Then for each $A \subseteq G$ one has

$$\mathrm{Clo}\big(T(A);\tau_S\big) \subseteq \bigcap_{\varepsilon>0} T\big([A]_{\psi,\varepsilon}^{\mathrm{R}}\big). \tag{6.293}$$

In particular,

$$\mathrm{Clo}\Big(T\big(B_\psi^{\mathrm{R}}(a,r)\big);\tau_S\Big) \subseteq \bigcap_{C>C_1} T\big(B_\psi^{\mathrm{R}}(a,Cr)\big), \qquad \forall\, a \in G, \quad \forall\, r \in (0,+\infty), \tag{6.294}$$

and as a corollary the map

$$T : (G,\tau_\psi^{\mathrm{R}}) \longrightarrow (S,\tau_S) \quad \text{ is open.} \tag{6.295}$$

Moreover, if the mapping $T : (G,\tau_\psi^{\mathrm{R}}) \to (S,\tau_S)$ is continuous, then one actually has equality in (6.293), i.e.,

$$\mathrm{Clo}\big(T(A);\tau_S\big) = \bigcap_{\varepsilon>0} T\big([A]_{\psi,\varepsilon}^{\mathrm{R}}\big). \tag{6.296}$$

Finally, there is a natural version of the statement emphasizing conclusions about τ_ψ^{L}, the left-topology induced by ψ.

Proof. Fix $A \subseteq G$ and $\varepsilon > 0$. Pick a number $\beta \in \big(0,(1+\log_2 C_1)^{-1}\big]$, along with a numerical sequence $(r_i)_{i\in\mathbb{N}} \subseteq (0,+\infty)$, such that

$$16\, C_1^4 C_0^2 \Big\{\sum_{i\in\mathbb{N}} r_i^\beta\Big\}^{1/\beta} < \varepsilon. \tag{6.297}$$

Finally, select an arbitrary element $y_0 \in \mathrm{Clo}\big(T(A);\tau_S\big)$. Then, by Lemma 6.48 and (6.291), we have $y_0 \in \bigcup_{a\in A} \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(a,r_1)\big);\tau_S\Big)$. Hence, there exists $a_1 \in A$ such that

$$y_0 \in \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(a_1,r_1)\big);\tau_S\Big) \subseteq \bigcup_{a\in B^{\mathrm{R}}_{\psi}(a_1,r_1)} \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(a,r_2)\big);\tau_S\Big), \tag{6.298}$$

where the inclusion in (6.298) is a consequence of Lemma 6.48 and (6.291). Inductively, this procedure yields a sequence $(a_i)_{i\in\mathbb{N}} \subseteq G$ such that

$$\begin{gathered} a_1 \in A \quad \text{and} \quad a_{i+1} \in B^{\mathrm{R}}_{\psi}(a_i,r_i) \;\; \text{for each } \; i \in \mathbb{N}, \\ \text{with the property that } \;\; y_0 \in \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(a_i,r_i)\big);\tau_S\Big), \qquad \forall\, i \in \mathbb{N}. \end{gathered} \tag{6.299}$$

Note that if $i,k \in \mathbb{N}$, then, based on (6.140) in Theorem 3.28 used here with $\kappa := 2C_1$, we may estimate

$$\begin{aligned} \psi\big(a_i * a_{i+k}^{-1}\big) &= \psi\big((a_i * a_{i+1}^{-1}) * (a_{i+1} * a_{i+2}^{-1}) * \cdots * (a_{i+k-1} * a_{i+k}^{-1})\big) \\ &\le 4C_1^2 \left\{ \sum_{j=i}^{i+k-1} \psi\big(a_j * a_{j+1}^{-1}\big)^{\beta} \right\}^{1/\beta} \le 4C_1^2 \left\{ \sum_{j=i}^{\infty} r_j^{\beta} \right\}^{1/\beta}. \end{aligned} \tag{6.300}$$

Since the last expression in (6.300) converges to zero as $i \to \infty$, this implies that

$$(a_i)_{i\in\mathbb{N}} \;\; \text{is a Cauchy sequence in } (G,*) \text{ relative to } \tau^{\mathrm{R}}_{\psi}. \tag{6.301}$$

Recall that $(G,*)$ is assumed to be complete with respect to τ^{R}_{ψ} and, as such, there exists $a \in G$ with the property that

$$a_i \longrightarrow a \;\; \text{in} \;\; \tau^{\mathrm{R}}_{\psi}, \;\; \text{as} \;\; i \to \infty. \tag{6.302}$$

Upon noting that the map $s^{\mathrm{R}}_{a_1^{-1}} : (G,\tau^{\mathrm{R}}_{\psi}) \to (G,\tau^{\mathrm{R}}_{\psi})$ is continuous (cf. part (ii) in Lemma 6.14), we further obtain from (6.302) that

$$a_i * a_1^{-1} \longrightarrow a * a_1^{-1} \;\; \text{in} \;\; \tau^{\mathrm{R}}_{\psi}, \;\; \text{as} \;\; i \to \infty. \tag{6.303}$$

Also, generally speaking,

$$\left.\begin{array}{r} (x_i)_{i\in\mathbb{N}} \subseteq G, \;\; x \in G \;\; \text{such that} \\ x_i \longrightarrow x \;\; \text{in} \;\; \tau^{\mathrm{R}}_{\psi} \;\; \text{as} \;\; i \to \infty \end{array}\right\} \Longrightarrow \psi(x) \le 4C_1^2 C_0 \limsup_{i\to\infty} \psi(x_i). \tag{6.304}$$

This follows from (6.142) in Theorem 3.28, used here with $\kappa := 2C_1$. Now, by (6.300), in which we first set $i := 1$ and then take $k := i-1$, we have

$$\psi(a_i * a_1^{-1}) \leq 4C_1^2 \left\{ \sum_{j=1}^{i-1} r_j^{\beta} \right\}^{1/\beta} \quad \text{for } i = 2, 3, \dots. \tag{6.305}$$

Keeping in mind (6.112) and combining (6.303), (6.304), (6.305), and (6.297), we arrive at the conclusion that

$$\psi\big(a_1 * a^{-1}\big) \leq C_0\, \psi\big(a * a_1^{-1}\big) \leq 16\, C_1^4 C_0^2 \left\{ \sum_{j=1}^{\infty} r_j^{\beta} \right\}^{1/\beta} < \varepsilon. \tag{6.306}$$

Thus, ultimately,

$$a \in B_{\psi}^{\mathrm{R}}(a_1, \varepsilon) \ \text{ for some } \ a_1 \in A. \tag{6.307}$$

We now claim that

$$(a, y_0) \in \mathrm{Clo}\big(\mathscr{G}_T; \tau_{\psi}^{\mathrm{R}} \times \tau_S\big). \tag{6.308}$$

To justify (6.308), pick two neighborhoods $V \in \mathscr{N}(a; \tau_{\psi}^{\mathrm{R}})$ and $W \in \mathscr{N}(y_0; \tau_S)$. Then, thanks to (6.131), there exists $r \in (0, +\infty)$ such that

$$B_{\psi}^{\mathrm{R}}(a, r) \subseteq V. \tag{6.309}$$

On the other hand, from (6.128) we know that $B_{\psi}^{\mathrm{R}}\big(a, r/(2C_1)\big) \in \mathscr{N}\big(a; \tau_{\psi}^{\mathrm{R}}\big)$. This, in concert with (6.297) and (6.302), implies that there exists $i_0 \in \mathbb{N}$ such that

$$r_i < r/(2C_1), \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq i_0, \tag{6.310}$$

$$a_i \in B_{\psi}^{\mathrm{R}}\big(a, r/(2C_1)\big), \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq i_0. \tag{6.311}$$

In particular, if $x \in B_{\psi}^{\mathrm{R}}(a_i, r_i)$, then $\psi(a_i * x^{-1}) < r_i$; hence

$$\begin{aligned} \psi(a * x^{-1}) &= \psi\big((a * a_i^{-1}) * (a_i * x^{-1})\big) \leq C_1\big(\psi(a * a_i^{-1}) + \psi(a_i * x^{-1})\big) \\ &< C_1\big(r/(2C_1) + r_i\big) < r. \end{aligned} \tag{6.312}$$

This analysis proves that there exists $i_0 \in \mathbb{N}$ such that

$$B_{\psi}^{\mathrm{R}}(a_i, r_i) \subseteq B_{\psi}^{\mathrm{R}}(a, r), \quad \forall\, i \in \mathbb{N} \ \text{ with } \ i \geq i_0. \tag{6.313}$$

Now (6.309) and (6.313) yield $B_\psi^{\rm R}(a_i, r_i) \subseteq V$ whenever $i \in \mathbb{N}$ satisfies $i \geq i_0$; hence

$$T\big(B_\psi^{\rm R}(a_i, r_i)\big) \subseteq T(V), \quad \forall\, i \in \mathbb{N} \text{ with } i \geq i_0. \tag{6.314}$$

On the other hand, (6.299) ensures that $W \cap T\big(B_\psi^{\rm R}(a_i, r_i)\big) \neq \emptyset$ for every $i \in \mathbb{N}$, which, when combined with (6.314), gives that

$$W \cap T(V) \neq \emptyset. \tag{6.315}$$

In turn, (6.315) yields the existence of some element $v \in V$ with the property that $Tv \in W$. This entails $(v, Tv) \in \mathscr{G}_T \cap (V \times W)$; therefore, $\mathscr{G}_T \cap (V \times W) \neq \emptyset$. This shows that (6.308) holds since $V \in \mathscr{N}(a; \tau_\psi^{\rm R})$ and $W \in \mathscr{N}(y_0; \tau_S)$ were arbitrarily chosen. Having established (6.308) and given that, by assumption $\mathscr{G}_T$ is closed in $\big(G \times S, \tau_\psi^{\rm R} \times \tau_S\big)$, we may conclude that $(a, y_0) \in \mathscr{G}_T$, which forces

$$y_0 = Ta. \tag{6.316}$$

Now, (6.307) implies that $a \in [A]_{\psi,\varepsilon}^{\rm R}$; hence using (6.316) yields $y_0 \in T\big([A]_{\psi,\varepsilon}^{\rm R}\big)$. Given that $y_0 \in {\rm Clo}\big(T(A); \tau_S\big)$ and $\varepsilon > 0$ have been arbitrarily chosen, it follows that (6.293) holds.

Moving on, consider $a \in G$, $r \in (0, +\infty)$, $\varepsilon > 0$, and recall that

$$\big[B_\psi^{\rm R}(a, r)\big]_{\psi,\varepsilon}^{\rm R} = \bigcup_{b \in B_\psi^{\rm R}(a,r)} B_\psi^{\rm R}(b, \varepsilon). \tag{6.317}$$

Now, for each $b \in B_\psi^{\rm R}(a, r)$ and each $c \in B_\psi^{\rm R}(b, \varepsilon)$ we may write

$$\psi(a * c^{-1}) \leq C_1\big(\psi(a * b^{-1}) + \psi(b * c^{-1})\big) < C_1(r + \varepsilon), \tag{6.318}$$

which shows that

$$\big[B_\psi^{\rm R}(a, r)\big]_{\psi,\varepsilon}^{\rm R} \subseteq B_\psi^{\rm R}\big(a, C_1(r + \varepsilon)\big). \tag{6.319}$$

If we now apply (6.293) with $A := B_\psi^{\rm R}(a, r)$ and use (6.319), then we obtain

$$\begin{aligned} {\rm Clo}\Big(T\big(B_\psi^{\rm R}(a, r)\big); \tau_S\Big) &\subseteq \bigcap_{\varepsilon>0} T\big(\big[B_\psi^{\rm R}(a, r)\big]_{\psi,\varepsilon}^{\rm R}\big) \\ &\subseteq \bigcap_{\varepsilon>0} T\big(B_\psi^{\rm R}(a, C_1(r + \varepsilon))\big) \\ &= \bigcap_{C > C_1} T\big(B_\psi^{\rm R}(a, Cr)\big), \end{aligned} \tag{6.320}$$

proving (6.294). As a corollary of (6.294), we also see that

$$\mathrm{Clo}\Big(T\big(B_\psi^{\mathrm{R}}(a,r)\big);\tau_S\Big) \subseteq T\big(B_\psi^{\mathrm{R}}(a,2C_1 r)\big), \quad \forall\, a \in G, \ \ \forall\, r \in (0,+\infty). \tag{6.321}$$

At the same time,

$$\mathrm{Clo}\Big(T\big(B_\psi^{\mathrm{R}}(a,r)\big);\tau_S\Big) = \mathrm{Clo}\Big(T\big(B_\psi^{\mathrm{R}}(e_G,r)\big);\tau_S\Big) \circ Ta \in \mathscr{N}(Ta;\tau_S), \tag{6.322}$$

by (6.291) and the fact that shifts are continuous on (S,τ_S). When combined with (6.321), this gives that

$$T\big(B_\psi^{\mathrm{R}}(a,r)\big) \in \mathscr{N}(Ta;\tau_S), \quad \forall\, a \in G, \ \ \forall\, r \in (0,+\infty). \tag{6.323}$$

Now, consider some $\mathcal{O} \in \tau_\psi^{\mathrm{R}}$, and select an arbitrary $b \in T(\mathcal{O})$. Then there exists $a \in \mathcal{O}$ such that $b = Ta$; hence there exists $r \in (0,+\infty)$ with the property that $B_\psi^{\mathrm{R}}(a,r) \subseteq \mathcal{O}$. Consequently, $T\big(B_\psi^{\mathrm{R}}(a,r)\big) \subseteq T(\mathcal{O})$. Since by (6.323) the set $T\big(B_\psi^{\mathrm{R}}(a,r)\big)$ is a neighborhood of Ta in τ_S, it follows that $T(\mathcal{O}) \in \mathscr{N}(b;\tau_S)$ for every $b \in T(\mathcal{O})$. Therefore, $T(\mathcal{O}) \in \tau_S$, proving that T is open.

At this stage in the proof, there remains to show that (6.296) holds whenever T is continuous as a mapping from $(G,\tau_\psi^{\mathrm{R}})$ into (S,τ_S). To this end, we note that once an element $b \in \bigcap_{\varepsilon>0} T\big([A]_{\psi,\varepsilon}^{\mathrm{R}}\big)$ has been arbitrarily chosen, for every $n \in \mathbb{N}$ there exists $x_n \in [A]_{\psi,1/n}^{\mathrm{R}}$ such that $b = Tx_n$. Having constructed such a sequence $(x_n)_{n\in\mathbb{N}} \subseteq G$, for each $n \in \mathbb{N}$ it is then possible to select $y_n \in A$ with the property that $x_n \in B_\psi^{\mathrm{R}}(y_n,1/n)$. Hence, $\psi(y_n * x_n^{-1}) < 1/n$ for each $n \in \mathbb{N}$, which, in light of part (iv) in Proposition 6.26, proves that

$$y_n * x_n^{-1} \longrightarrow e_G \ \text{ in } \ \tau_\psi^{\mathrm{R}}, \ \text{ as } \ n \to \infty. \tag{6.324}$$

Granted that $T : (G,\tau_\psi^{\mathrm{R}}) \to (S,\tau_S)$ is continuous, we deduce from (6.324) and our choice of the x_n that

$$(Ty_n) \circ b^{-1} = T\big(y_n * x_n^{-1}\big) \longrightarrow e_S \ \text{ in } \ \tau_S, \ \text{ as } \ n \to \infty. \tag{6.325}$$

Upon recalling from Definition 6.10 that the map $s_b^{\mathrm{R}} : (S,\tau_S) \to (S,\tau_S)$ is continuous, this further implies that

$$Ty_n \longrightarrow b \ \text{ in } \ \tau_S, \ \text{ as } \ n \to \infty. \tag{6.326}$$

Having proved (6.326), the fact that $y_n \in A$ for every $n \in \mathbb{N}$ entails $b \in \mathrm{Clo}\big(T(A);\tau_S\big)$. Hence, ultimately,

$$\bigcap_{\varepsilon>0} T\big([A]_{\psi,\varepsilon}^{\mathrm{R}}\big) \subseteq \mathrm{Clo}\big(T(A);\tau_S\big), \tag{6.327}$$

which, together with (6.293), establishes (6.296). The case of the left-topology induced by ψ is treated similarly, completing the proof of the theorem. □

Comment 6.50. The quantitative aspect of Theorem 6.49 is most apparent by considering the case when S is also a quasi-pseudonormed group. Concretely, assume that $\varphi : S \to [0, +\infty]$ is a quasi-pseudonorm with the property that τ_S is the right-topology induced by φ on S. For each $r \in (0, +\infty)$ define

$$\eta_T(r) := \sup\Big\{\rho > 0 : \, B^{\mathrm{R}}_{\varphi}\big(e_S, \rho\big) \subseteq \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(e_G, r)\big); \tau_S\Big)\Big\}. \tag{6.328}$$

Condition (6.291) then ensures that the function $\eta_T : (0, +\infty) \to (0, +\infty]$ is well defined. Note that, by design, η_T is nondecreasing and has the property that

$$B^{\mathrm{R}}_{\varphi}\big(e_S, \rho\big) \subseteq \mathrm{Clo}\Big(T\big(B^{\mathrm{R}}_{\psi}(e_G, r)\big); \tau_S\Big), \quad \forall\, r \in (0, +\infty), \quad \forall\, \rho \in \big(0, \eta_T(r)\big). \tag{6.329}$$

In conjunction with (6.294), this yields that for each $a \in G$

$$B^{\mathrm{R}}_{\varphi}\big(Ta, \rho\big) \subseteq \bigcap_{C > C_1} T\big(B^{\mathrm{R}}_{\psi}(a, Cr)\big), \quad \forall\, r \in (0, +\infty), \quad \forall\, \rho \in \big(0, \eta_T(r)\big), \tag{6.330}$$

which provides concrete information about the size of the neighborhood of Ta one expects to be contained in the image of a ball $B^{\mathrm{R}}_{\psi}(a, R)$ under the mapping T. ■

Comment 6.50 highlights the significance of the case when the topology τ_S is "quantitative," i.e., is induced by a quasi-pseudonorm. In this vein, in the corollary below (which generalizes [69, Theorem 1.4, p. 9]) we indicate that such a phenomenon happens automatically in a context that is closely related to the very statement of the OMT in Theorem 6.49.

Corollary 6.51. *Let $(G, *)$ be a group, and assume that ψ is a finite quasi-pseudonorm on G, with constants (C_0, C_1), such that $(G, *)$ is complete with respect to τ^{R}_{ψ}. In addition, consider a group $(S, \circ)$ equipped with a Hausdorff, symmetric, right-invariant topology τ_S such that the group $(S, \circ, \tau_S)$ is topologically divisible (cf. Definition 6.18). Finally, let $T \in \mathrm{Hom}\,(G, S)$ be such that*

$$T : (G, \tau^{\mathrm{R}}_{\psi}) \longrightarrow (S, \tau_S) \ \text{ is continuous and almost open at } e_G \tag{6.331}$$

(cf. Definition 6.2). Then the following statements hold:

(i) The mapping $T : G \to S$ is surjective, and $T : (G, \tau^{\mathrm{R}}_{\psi}) \to (S, \tau_S)$ is open.

(ii) If $\overline{\psi}$ denotes the push-forward of ψ via T (as in Definition 6.45; recall that T is surjective), then $\overline{\psi}$ is a finite quasi-pseudonorm with constants (C_0, C_1) that has the property that

$$\tau^{\mathrm{R}}_{\overline{\psi}} = \tau_S. \tag{6.332}$$

(iii) *If ψ is actually a quasinorm on G, then $\overline{\psi}$ is a quasinorm on S.*
(iv) *If ψ is quasi-invariant, then $\overline{\psi}$ is quasi-invariant (with the same constant as ψ), and $(S, \circ, \tau_S)$ is complete.*

There is also a natural version emphasizing conclusions about left-topologies.

Proof. From Lemma 6.3 and current assumptions (recall (6.331) and that the topological space (S, τ_S) is Hausdorff) it follows that $\mathscr{G}_T$, the graph of T, is a closed subset of $\big(G \times S, \tau_G \times \tau_S\big)$. With this in hand, Theorem 6.49 applies and gives that

$$T : (G, \tau_{\psi}^{\mathrm{R}}) \longrightarrow (S, \tau_S) \ \text{ is open.} \tag{6.333}$$

In particular, $\operatorname{Im} T = T(G)$ is an open subset of (S, τ_S) and, since we are assuming that (S, τ_S) is topologically divisible, Lemma 6.19 applies and give that

$$T \in \operatorname{Hom}(G, S) \ \text{ is surjective.} \tag{6.334}$$

This finishes the proof of the claims in part *(i)*. With (6.334), (6.333), and (6.331) in hand, all claims in part *(ii)* now follow from items *(a)* and *(b)* of Theorem 6.46.

As regards the claim in part *(iii)*, we first note that, since any singleton in the Hausdorff topological space (S, τ_S) is closed, and since (6.331) holds, we have

$$\operatorname{Ker} T = T^{-1}\big(\{e_S\}\big) \ \text{ is closed in } \ (G, \tau_{\psi}^{\mathrm{R}}). \tag{6.335}$$

Based on this as well as items *(1)–(4)* in Proposition 6.38, we deduce that

$$\widehat{\psi} \ \text{ is a quasinorm on } \ G/\operatorname{Ker} T. \tag{6.336}$$

In concert with (6.245) (and Definition 6.45), this readily gives that $\overline{\psi}$ is a quasinorm on S, as desired. Finally, the claims in part *(iv)* are direct consequences of items *(c)* and *(d)* in Theorem 6.46. □

In the last part of this section we will discuss two corollaries of the quantitative OMT formulated in Theorem 6.49 that are more akin to the classical formulation of this result. In preparation, we first prove an auxiliary lemma.

Lemma 6.52. *Let $\big(G, *, (\cdot)^{-1}, e_G\big)$ be a group and assume that τ is a topology on the set G with the property that the mapping*

$$\big(G \times G, \tau \times \tau\big) \ni (x, y) \longmapsto x * y \in (G, \tau) \ \ \textit{is continuous.} \tag{6.337}$$

Then for every set $A \subseteq G$ one has

$$\operatorname{Int}(A; \tau) \neq \emptyset \implies A * A^{-1} \in \mathscr{N}(e_G; \tau). \tag{6.338}$$

Proof. Indeed, if $A \subseteq G$ is such that $\mathrm{Int}(A;\tau) \neq \emptyset$, then there exists a nonempty set $O \in \tau$ with the property that $O \subseteq A$. Consequently, on the one hand we have

$$e_G \in O * O^{-1} \subseteq A * A^{-1}. \tag{6.339}$$

On the other hand, the continuity of the function (6.337) forces all right-shifts on G to be continuous and, hence, ultimately, homeomorphisms of (G,τ) by (6.41). Given this, keeping in mind that $O \in \tau$, and writing

$$O * O^{-1} = \bigcup_{a \in O} s^{\mathrm{R}}_{a^{-1}}(O), \tag{6.340}$$

we deduce that $O * O^{-1} \in \tau$. Together with the inclusion (6.339), this proves that $A * A^{-1} \in \mathcal{N}(e_G;\tau)$, as desired. □

Here is the first consequence of Theorem 6.49 alluded to earlier. To state it, recall (6.33).

Theorem 6.53 (Topological OMT: Version 1). *Let G be a group equipped with a finite quasi-invariant, quasi-pseudonorm inducing a topology τ_G with respect to which G is complete, and such that*

$$G = \bigcup_{n \in \mathbb{N}} V^{(n)}, \quad \textit{for every symmetric neighborhood } V \textit{ of } e_G. \tag{6.341}$$

Also, suppose that (S,τ_S) is a Hausdorff topological group that is uniquely divisible. In addition, suppose that

$$\mathcal{O}^{(n)} \in \tau_S, \quad \textit{for every } \mathcal{O} \in \tau_S \textit{ and every } n \in \mathbb{N}. \tag{6.342}$$

Finally, assume that $T \in \mathrm{Hom}\,(G,S)$ is such that

$$T : (G,\tau_G) \longrightarrow (S,\tau_S) \textit{ is continuous and} \tag{6.343}$$

$$\mathrm{Im}\, T \textit{ is of second Baire category in } (S,\tau_S). \tag{6.344}$$

Then $T : (G,\tau_G) \to (S,\tau_S)$ is an open mapping.

Proof. Let $\big(G, *, (\cdot)^{-1}, e_G, \tau_G\big)$ and $\big(S, \circ, (\cdot)^{-1}, e_S, \tau_S\big)$ be as in the statement of the theorem. Also, consider $T \in \mathrm{Hom}\,(G,S)$ satisfying (6.343) and (6.344), and fix an arbitrary $U \in \mathcal{N}(e_G;\tau_G)$. Then $(G,*,\tau_G)$ becomes a topological group, as seen from part (vii) in Proposition 6.26. As such, from the continuity of the function (cf. (6.43))

$$\big(G \times G, \tau_G \times \tau_G\big) \ni (x,y) \mapsto x * y^{-1} \in (G,\tau_G) \tag{6.345}$$

at (e_G, e_G) it follows that there exists $V_0 \in \mathcal{N}(e_G; \tau_G)$ such that

$$V_0 * V_0^{-1} \subseteq U. \tag{6.346}$$

Thus, if we set $V := V_0 \cap V_0^{-1}$, then it follows that

$$V \in \mathcal{N}(e_G; \tau_G), \quad V \text{ is symmetric and } V^2 \subseteq U. \tag{6.347}$$

At the same time, the fact that $T \in \mathrm{Hom}\,(G, S)$ allows us to write

$$\mathrm{Im}\, T = T(G) = T\Big(\bigcup_{n\in\mathbb{N}} V^{(n)}\Big) = \bigcup_{n\in\mathbb{N}} T\big(V^{(n)}\big) = \bigcup_{n\in\mathbb{N}} \big(T(V)\big)^{(n)}. \tag{6.348}$$

Granted (6.344), this ensures that there exists $n \in \mathbb{N}$ with the property that

$$\mathcal{O} := \mathrm{Int}\,\big(\mathrm{Clo}\,\big((T(V))^{(n)}; \tau_S\big); \tau_S\big) \neq \emptyset. \tag{6.349}$$

Hence,

$$\emptyset \neq \mathcal{O} \in \tau_S \text{ and } \mathcal{O} \subseteq \mathrm{Clo}\Big(\big(T(V)\big)^{(n)}; \tau_S\Big). \tag{6.350}$$

Consider now

$$\phi : S \longrightarrow S, \quad \phi(x) := x^n, \quad \forall\, x \in S. \tag{6.351}$$

The fact that $(S, \circ, \tau_S)$ is a topological group implies that ϕ is continuous, while the fact that the group $(S, \circ)$ is uniquely divisible forces ϕ to be a bijection. Finally, (6.342) ensures that ϕ is open. As a result,

$$\phi : (S, \tau_S) \longrightarrow (S, \tau_S) \text{ is a homeomorphism.} \tag{6.352}$$

Hence, given that $\phi(A) = A^{(n)}$ for any $A \subseteq G$, we have

$$\mathrm{Clo}\Big(\big(T(V)\big)^{(n)}; \tau_S\Big) = \Big(\mathrm{Clo}\,\big(T(V); \tau_S\big)\Big)^{(n)}, \tag{6.353}$$

by (6.352) and (6.2). Consequently, on the one hand,

$$\phi^{-1}(\mathcal{O}) \subseteq \phi^{-1}\Big[\Big(\mathrm{Clo}\,\big(T(V); \tau_S\big)\Big)^{(n)}\Big] = \mathrm{Clo}\,\big(T(V); \tau_S\big), \tag{6.354}$$

by (6.353), (6.350), and the fact that ϕ is bijective. On the other hand, thanks to (6.352), $\emptyset \neq \phi^{-1}(\mathcal{O}) \in \tau_S$. Granted this and keeping (6.354) in mind, we arrive at the conclusion that

$$\mathrm{Int}\,\big(\mathrm{Clo}\,\big(T(V); \tau_S\big); \tau_S\big) \neq \emptyset. \tag{6.355}$$

Together with Lemma 6.52 and the nature of τ_S, this allows us to conclude that

$$\mathrm{Clo}\big(T(V);\tau_S\big)\circ\Big(\mathrm{Clo}\big(T(V);\tau_S\big)\Big)^{-1}\in\mathscr{N}(e_S;\tau_S). \tag{6.356}$$

At the same time,

$$\begin{aligned}\mathrm{Clo}\big(T(V);\tau_S\big)\circ\Big(\mathrm{Clo}\big(T(V);\tau_S\big)\Big)^{-1} &= \mathrm{Clo}\big(T(V);\tau_S\big)\circ\mathrm{Clo}\big((T(V))^{-1};\tau_S\big)\\ &\subseteq \mathrm{Clo}\big(T(V)\circ(T(V))^{-1};\tau_S\big)\\ &= \mathrm{Clo}\big(T(V\circ V^{-1});\tau_S\big)\\ &\subseteq \mathrm{Clo}\big(T(U);\tau_S\big),\end{aligned} \tag{6.357}$$

by (6.54), (6.347), the fact that $(\cdot)^{-1}:(S,\tau_S)\to(S,\tau_S)$ is a homeomorphism, and (6.2). In combination with (6.356), (6.357) proves that

$$\mathrm{Clo}\big(T(U);\tau_S\big)\in\mathscr{N}(e_S;\tau_S),\qquad \forall\,U\in\mathscr{N}(e_G;\tau_G). \tag{6.358}$$

Also, since we are assuming that the topological space (S,τ_S) is Hausdorff, we can argue as in the first part of the proof of Corollary 6.51 to conclude that the graph $\mathscr{G}_T$ of T is a closed subset of $\big(G\times S,\tau_G\times\tau_S\big)$. With this and (6.358) in hand, Theorem 6.49 applies and gives that T is an open mapping. This completes the proof of the theorem. □

Remark 6.54. Theorem 6.53 continues to hold if in place of (6.342) one assumes that

$$(S,\tau_S)\ \text{ is compact.} \tag{6.359}$$

Indeed, granted (6.359), one may invoke (6.13) for the function (6.351) to conclude (6.351) without having to rely on (6.342).

We continue by recording two notable consequences of Theorem 6.53 and Corollary 6.51.

Corollary 6.55. *Retain all hypotheses on G, S, and T from Theorem 6.53, and strengthen the unique divisibility assumption on S to topological divisibility. Then, if ψ denotes the quasi-invariant quasi-pseudonorm on G, all conclusions in Corollary 6.51 hold in this case as well.*

Proof. From the proof of Theorem 6.53 we know that (6.358) holds and that the mapping $T:(G,\tau_G)\to(S,\tau_S)$ is open. Granted these and the topological divisibility assumption on S, Corollary 6.51 applies and yields the desired conclusions. □

Corollary 6.56. *Let G be a given group equipped with a finite quasi-invariant quasi-pseudonorm ψ inducing a topology τ_G with respect to which G is complete and such that (6.341) holds. Next, suppose that S is a uniquely divisible group endowed with a quasi-invariant quasinorm, and equip S with the topology τ_S induced by this quasinorm. In addition, suppose that (6.342) holds and that (S, τ_S) is complete. Finally, assume that $T \in \operatorname{Hom}(G, S)$ is surjective and $T : (G, \tau_G) \to (S, \tau_S)$ is continuous. Then the following conclusions hold:*

(a) The mapping $T : (G, \tau_G) \to (S, \tau_S)$ is open.
(b) Denote by $\overline{\psi}$ the push-forward of the quasi-pseudonorm ψ via the homomorphism T. Then $\overline{\psi}$ is a finite quasi-invariant quasi-pseudonorm, with the same constants as ψ, which satisfies $\tau_{\overline{\psi}} = \tau_S$.
(c) The mapping $T : (G, \tau_G) \to (S, \tau_S)$ is a homeomorphism if and only if the mapping $T : G \to S$ is one-to-one.

Proof. The fact that τ_S is the topology induced by a quasi-invariant quasinorm on the group S implies (cf. part (vii) in Proposition 6.26 and part *(vi)* in Lemma 6.14) that (S, τ_S) is a Hausdorff topological group. Furthermore, if T is as in the statement of the current corollary, (6.367) holds thanks to the fact that T is surjective, (S, τ_S) is complete, and (6.144). Having verified these conditions, Theorem 6.53 applies and gives that $T : (G, \tau_G) \to (S, \tau_S)$ is an open mapping.

The claims in the statement of item *(b)* are immediate consequences of Corollary 6.55 once we observe that the topological divisibility assumption on S has only been used in the proof of this result to establish the fact that $T \in \operatorname{Hom}(G, S)$ is surjective [see the comments pertaining to (6.334)]. However, in the current context, the surjectivity condition is part of the hypotheses, so no additional provisions are necessary.

Finally, the equivalence in item *(c)* is a direct consequence of item *(a)*. □

Combining Proposition 6.31 with Theorem 6.53 yields the following purely topological version of the OMT.

Corollary 6.57. *Let G be an Abelian group equipped with a topology τ_G satisfying (6.150)–(6.152), (6.341), and with respect to which G is complete. Also, suppose that (S, τ_S) is a Hausdorff topological group that is uniquely divisible and such that (6.342) holds. Finally, assume that $T \in \operatorname{Hom}(G, S)$ is as in (6.343) and (6.344).*

Then $T : (G, \tau_G) \to (S, \tau_S)$ is an open mapping.

Proof. This is a consequence of equivalence (6.153) from Proposition 6.31 and Theorem 6.53. □

The scenario in which Theorem 6.53 and its corollaries are specialized to the case of vector spaces is considered in Corollaries 6.58–6.65. Our first such corollary, stated below, extends the version of the OMT formulated in [108, Theorem 2.11, p. 48].

Corollary 6.58 (OMT for Vector Spaces: Version 1). *Let X be a vector space, and let $\|\cdot\| : X \to [0, +\infty)$ be a function with the property that, for some constants $C_0, C_1 \in [1, +\infty)$,*

$$\|-x\| \leq C_0\|x\| \text{ for all } x \in X, \tag{6.360}$$

$$\|x + y\| \leq C_1\big(\|x\| + \|y\|\big) \text{ for all } x, y \in X, \tag{6.361}$$

$$\|\lambda_n x\| \to 0 \text{ for each fixed } x \in X \text{ if } \lambda_n \to 0. \tag{6.362}$$

Suppose that X is complete with respect to the topology $\tau_{\|\cdot\|}$ induced by $\|\cdot\|$ on the additive group $(X, +)$. Next, assume that Y is a vector space equipped with a Hausdorff topology τ_Y satisfying the following properties:

$$\big(Y \times Y, \tau_Y \times \tau_Y\big) \ni (x, y) \longmapsto x - y \in (Y, \tau_Y) \text{ is continuous,} \tag{6.363}$$

$$(Y, \tau_Y) \ni x \longmapsto \lambda x \in (Y, \tau_Y) \text{ is continuous for each fixed } \lambda. \tag{6.364}$$

Finally, assume that

$$T : X \longrightarrow Y \text{ is an additive mapping, so that} \tag{6.365}$$

$$T : \big(X, \tau_{\|\cdot\|}\big) \longrightarrow (Y, \tau_Y) \text{ is continuous, and} \tag{6.366}$$

$$\operatorname{Im} T \text{ is of second Baire category in } (Y, \tau_Y). \tag{6.367}$$

Then

$$T : \big(X, \tau_{\|\cdot\|}\big) \longrightarrow (Y, \tau_Y) \text{ is an open mapping.} \tag{6.368}$$

Moreover, if in addition to the hypotheses made on τ_Y, one also assumes that

$$\mathbb{R} \ni \lambda \longmapsto \lambda x \in (Y, \tau_Y) \text{ is continuous at } 0, \text{ for each fixed } x \in Y, \tag{6.369}$$

then the following statements hold:

(i) The mapping

$$T : X \longrightarrow Y \text{ is surjective} \tag{6.370}$$

and

$$\begin{gathered}\text{the additive group } (Y, +) \text{ is complete}\\ \text{with respect to the topology } \tau_Y.\end{gathered} \tag{6.371}$$

(ii) If $|||\cdot|||: Y \to [0, +\infty)$ *denotes the push-forward of* $\|\cdot\|$ *via* T*, then, for the same constants* $C_0, C_1 \in [1, +\infty)$ *as in the first part of the statement, one has*

$$|||0||| = 0, \quad |||-x||| \le C_0|||x||| \ \textit{for all } x \in Y, \tag{6.372}$$

$$|||x + y||| \le C_1\big(|||x||| + |||y|||\big) \ \textit{for all } \ x, y \in Y \tag{6.373}$$

and

$$\begin{array}{c}\textit{the topology induced by } |||\cdot||| \textit{ on } (Y, +)\\ \textit{coincides with the original topology } \tau_Y.\end{array} \tag{6.374}$$

Proof. Note that (6.362) forces $\|0\| = 0$. Together with (6.360) and (6.361), this ensures that the function $\|\cdot\| : X \to [0, +\infty)$ is a quasi-invariant quasi-pseudonorm on the (Abelian) additive group $(X, +)$. In addition, thanks to (6.362), the analog of condition (6.341) holds in the current setting. Going further, (6.363) guarantees that, when equipped with the topology τ_Y, the group $(Y, +)$ becomes a topological group, while (6.364) implies that condition (6.342) holds in the present context. Having clarified the connections with the hypotheses formulated in Theorem 6.53, the fact that (6.368) holds now follows from Theorem 6.53. In turn, this guarantees that (6.291) holds, while (6.369) implies that the group $(Y, +, \tau_Y)$ is topologically divisible. As such, all remaining conclusions are direct consequences of Corollary 6.51. □

Corollary 6.59 (OMT for Vector Spaces: Version 2). *Assume that* X *is a vector space and that* $\|\cdot\|_X : X \to [0, +\infty)$ *is a function with the property that there exist* $C_0, C_1 \in [1, +\infty)$ *such that*

$$\|-x\|_X \le C_0\|x\|_X \ \textit{for all } x \in X, \tag{6.375}$$

$$\|x + y\|_X \le C_1\big(\|x\|_X + \|y\|_X\big) \ \textit{for all } \ x, y \in X, \tag{6.376}$$

$$\|\lambda_n x\|_X \to 0 \ \textit{for each fixed } x \in X \textit{ if } \lambda_n \to 0. \tag{6.377}$$

In addition, suppose that X *is complete with respect to the topology* $\tau_{\|\cdot\|_X}$ *induced by* $\|\cdot\|_X$ *on the additive group* $(X, +)$.

Next, assume that Y *is a vector space and that* $\|\cdot\|_Y : Y \to [0, +\infty)$ *is a function such that, for some constants* $C_0', C_1' \in [1, +\infty)$,

$$\|x\|_Y = 0 \Longleftrightarrow x = 0 \ \textit{for all } x \in Y, \tag{6.378}$$

$$\|-x\|_Y \le C_0'\|x\|_Y \ \textit{for all } x \in Y, \tag{6.379}$$

$$\|x + y\|_Y \le C_1'\big(\|x\|_Y + \|y\|_Y\big) \ \textit{for all } \ x, y \in Y, \tag{6.380}$$

$$\|\lambda x_n\|_Y \to 0 \ \textit{for each fixed } \lambda \in \mathbb{Q} \textit{ if } \|x_n\|_Y \to 0. \tag{6.381}$$

Denote by $\tau_{\|\cdot\|_Y}$ *the topology induced by* $\|\cdot\|_Y$ *on the additive group* $(Y,+)$*, and assume that this group is complete with respect to the topology* $\tau_{\|\cdot\|_Y}$*. Finally, suppose that*

$$T : X \longrightarrow Y \quad \textit{is additive and surjective and} \tag{6.382}$$

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \quad \textit{is continuous.} \tag{6.383}$$

Then

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \quad \textit{is an open mapping.} \tag{6.384}$$

Moreover, if $\|\!|\!|\cdot|\!|\!\| : Y \to [0,+\infty)$ *denotes the push-forward of* $\|\cdot\|_X$ *via* T*, then, for the same constants* C_0, C_1 *as in* (6.375) *and* (6.376)*,*

$$|\!|\!|0|\!|\!| = 0, \quad |\!|\!|-x|\!|\!| \le C_0 |\!|\!|x|\!|\!| \quad \textit{for all } x \in Y, \tag{6.385}$$

$$|\!|\!|x+y|\!|\!| \le C_1\big(|\!|\!|x|\!|\!| + |\!|\!|y|\!|\!|\big) \quad \textit{for all } \; x, y \in Y, \tag{6.386}$$

and

$$\textit{the topology induced by } |\!|\!|\cdot|\!|\!| \textit{ on } (Y,+) \textit{ coincides with } \tau_{\|\cdot\|_Y}. \tag{6.387}$$

Proof. For each $\lambda \in \mathbb{Q}$ consider the function $\phi_\lambda : Y \to Y$, given by the formula

$$\phi_\lambda(x) := \lambda x \tag{6.388}$$

for each $x \in Y$. The role of condition (6.381) is to ensure that $\phi_\lambda : \big(Y, \tau_{\|\cdot\|_Y}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is a continuous function for each $\lambda \in \mathbb{Q}$. Since ϕ_λ is bijective and $(\phi_\lambda)^{-1} = \phi_{\lambda^{-1}}$ for each $\lambda \in \mathbb{Q} \setminus \{0\}$, we deduce that in fact $\phi_\lambda : \big(Y, \tau_{\|\cdot\|_Y}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is a homeomorphism for each $\lambda \in \mathbb{Q} \setminus \{0\}$. In turn, this property readily implies that the analog of (6.342) holds in the current setting (i.e., when S is the additive group $(Y,+)$). Having clarified this issue, Corollary 6.56 applies and yields all desired conclusions. □

Since the property of being onto has played an important role in the current proceedings, below we momentarily digress for the purpose of recording a useful criterion in this regard.

Proposition 6.60. *Suppose that* $\big(X, \|\cdot\|_X\big)$ *is a quasi-Banach space, and consider*

$$\kappa := \sup_{\substack{x,y\in X \\ \text{not both zero}}} \left(\frac{\|x+y\|_X}{\max\{\|x\|_X, \|y\|_X\}}\right) \in [2,+\infty) \tag{6.389}$$

and

$$p := \frac{1}{\log_2 \kappa} \in (0, 1]. \tag{6.390}$$

Also, suppose that $(Y, \|\cdot\|_Y)$ *is a quasinormed space and that* $T : X \to Y$ *is a linear, continuous operator for which there exist* $C_0 \in (0, +\infty)$ *and* $\theta \in (0, 1)$ *satisfying the following property:*

$$\begin{aligned} &\forall\, y \in Y \ \ \textit{with} \ \ \|y\|_Y = 1 \ \ \exists\, x \in X \ \ \textit{with} \\ &\|x\|_X \leq C_0 \ \ \textit{and} \ \ \|y - Tx\|_Y \leq \theta. \end{aligned} \tag{6.391}$$

Then T *is onto, in the following quantitative manner:*

$$\begin{aligned} &\forall\, y \in Y \ \ \exists\, x \in X \ \ \textit{such that} \ \ Tx = y \\ &\textit{and} \ \ \|x\|_X \leq \kappa^2 C_0 \theta (1 - \theta^p)^{-1/p} \|y\|_Y. \end{aligned} \tag{6.392}$$

Proof. Fix an arbitrary $y \in Y$, the goal being to find $x \in X$ with appropriate control and such that $Tx = y$. If $y = 0$ take $x := 0$. On the other hand, if $y \neq 0$, then there exists $\widetilde{x}_1 \in X$ with the property that $\|\widetilde{x}_1\|_X \leq C_0$ and $\big\| y/\|y\|_Y - T\widetilde{x}_1 \big\|_Y \leq \theta$. Introducing $x_1 := \|y\|_Y \widetilde{x}_1$ we therefore have

$$\|y - Tx_1\|_Y \leq \theta \|y\|_Y \ \ \text{and} \ \ \|x_1\|_X \leq C_0 \|y\|_Y. \tag{6.393}$$

Consider $y_1 := y - Tx_1 \in Y$, and repeat the foregoing procedure with y_1 in place of y. This yields $x_2 \in X$ such that

$$\|y_1 - Tx_2\|_Y \leq \theta \|y_1\|_Y \ \ \text{and} \ \ \|x_2\|_X \leq C_0 \|y_1\|_Y. \tag{6.394}$$

Continuing this procedure, one can inductively define a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that

$$\Big\| y - \sum_{k=1}^{n} Tx_k \Big\|_Y \leq \theta^n \|y\|_Y \ \ \text{and} \ \ \|x_n\|_X \leq C_0 \theta^n \|y\|_Y \ \ \text{for each } n \in \mathbb{N}. \tag{6.395}$$

To proceed, recall (6.389) and (6.390). Then for each $m, k \in \mathbb{N}$ we have, on account of (6.140) and the second estimate in (6.395),

$$\Big\| \sum_{n=m}^{m+k} x_n \Big\|_X \leq \kappa^2 \Big(\sum_{n=m}^{m+k} \|x_n\|_X^p \Big)^{1/p} \leq \kappa^2 C_0 \Big(\sum_{n=m}^{m+k} \theta^{np} \Big)^{1/p} \|y\|_Y. \tag{6.396}$$

Since $\big(X, \|\cdot\|_X\big)$ is complete, this proves that $\sum_{n=1}^{\infty} x_n$ converges in $\big(X, \|\cdot\|_X\big)$, and if $x \in X$ denotes the sum of this series, then we have

$$\|x\|_X \leq \kappa^2 \left(\sum_{n=1}^{\infty} \|x_n\|_X^p\right)^{1/p} \leq \kappa^2 C_0 \theta (1-\theta^p)^{-1/p} \|y\|_Y, \tag{6.397}$$

thanks to (6.142) and (6.396). Furthermore, from the first estimate in (6.395) we deduce that $Tx = y$. □

Proposition 6.60 suggests the following definition.

Definition 6.61. Let $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$ be two quasinormed spaces.

(i) Call a mapping $T : X \to Y$ `almost surjective` provided there exist constants $C_0 \in (0, +\infty)$ and $\theta \in (0, 1)$ such that

$$\forall\, y \in Y \quad \exists\, x \in X \quad \text{with} \quad \|x\|_X \leq C_0 \|y\|_Y \quad \text{and} \quad \|y - Tx\|_Y \leq \theta \|y\|_Y. \tag{6.398}$$

(ii) Call a mapping $T : X \to Y$ `quantitatively surjective` provided there exists a constant $C_0 \in (0, +\infty)$ with the property that

$$\forall\, y \in Y \quad \exists\, x \in X \quad \text{such that} \quad Tx = y \quad \text{and} \quad \|x\|_X \leq C_0 \|y\|_Y. \tag{6.399}$$

We are now ready to return to the main discussion and prove the following result.

Corollary 6.62 (OMT for Vector Spaces: Version 3). *Assume that $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_X\big)$ are two quasi-Banach spaces, and denote by $\tau_{\|\cdot\|_X}$ and $\tau_{\|\cdot\|_Y}$ the topologies induced on X and Y by their respective quasinorms. Also, suppose that*

$$T : X \longrightarrow Y \quad \textit{is additive and} \tag{6.400}$$

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \quad \textit{is continuous at } 0. \tag{6.401}$$

Then the following conditions are equivalent:

(1) The mapping $T : \big(X, \|\cdot\|_X\big) \to \big(Y, \|\cdot\|_Y\big)$ is almost surjective.
(2) The mapping $T : X \to Y$ is surjective.
(3) The mapping $T : \big(X, \|\cdot\|_X\big) \to \big(Y, \|\cdot\|_Y\big)$ is quantitatively surjective.
(4) The mapping $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is open.

Proof. To get started, observe first that, granted hypotheses (6.400) and (6.401), the mapping $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is continuous (at every point). Next, under the same assumptions, we will prove that T is homogeneous, i.e.,

$$T(\lambda x) = \lambda\, Tx, \qquad \forall\, x \in X \quad \text{and} \quad \forall\, \lambda \in \mathbb{R}. \tag{6.402}$$

Indeed, that (6.402) holds when $\lambda \in \mathbb{Q}$ is clear since $T : X \to Y$ is additive, while the general case $\lambda \in \mathbb{R}$ follows from this and the continuity of T established earlier. Since T is assumed to be additive, the bottom line is that T is a linear and continuous map from X to Y.

Granted this preamble and given that $\big(X, \|\cdot\|_X\big)$ is a quasi-Banach space, the fact that (1)$\Rightarrow$ (3) follows from Proposition 6.60. Obviously, (3) $\Rightarrow$ (2), whereas (2) $\Rightarrow$ (4) is a consequence of Corollary 6.59 and the definition of a quasi-Banach space. To show that (4) $\Rightarrow$ (3), note that if $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is open, then there exists $\varepsilon > 0$ such that $B_{\|\cdot\|_Y}(0, \varepsilon) \subseteq T\big(B_{\|\cdot\|_X}(0, 1)\big)$. Now, given any vector $y \in Y \setminus \{0\}$, we have

$$(\varepsilon y)/(2\|y\|_Y) \in B_{\|\cdot\|_Y}(0, \varepsilon), \tag{6.403}$$

hence there exists $z \in B_{\|\cdot\|_X}(0, 1)$ such that $Tz = (\varepsilon y)/(2\|y\|_Y)$. Taking $x := (2\|y\|_Y)/\varepsilon$ and keeping in mind that T is homogeneous, we see that (6.399) holds with $C_0 := 2/\varepsilon$. Since (3) $\Rightarrow$ (1) is obvious from definitions, this concludes the proof of the corollary. $\square$

Corollary 6.63 (OMT for Vector Spaces: Version 4). *Assume that $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_X\big)$ are two quasi-Banach spaces, and denote by $\tau_{\|\cdot\|_X}$ and $\tau_{\|\cdot\|_Y}$ the topologies induced on X and Y by their respective quasinorms. In addition, suppose that Z is a linear subspace of X and that*

$$T : Z \longrightarrow Y \quad \text{is linear and} \tag{6.404}$$

$$\mathscr{G}_T \quad \text{is a closed subset of} \quad \big(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y}\big). \tag{6.405}$$

Then the following conditions are equivalent:

(1) The mapping $T : \big(Z, \|\cdot\|_X\big|_Z\big) \to \big(Y, \|\cdot\|_Y\big)$ is almost surjective.
(2) The mapping $T : Z \to Y$ is surjective.
(3) The mapping $T : \big(Z, \|\cdot\|_X\big|_Z\big) \to \big(Y, \|\cdot\|_Y\big)$ is quantitatively surjective.
(4) The mapping $T : \big(Z, \|\cdot\|_X\big|_Z\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is open.
(5) For every $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that

$$B_{\|\cdot\|_Y}(0_Y, \delta) \subseteq T\Big(Z \cap B_{\|\cdot\|_X}(0_X, \varepsilon)\Big). \tag{6.406}$$

Proof. Consider the function

$$\|\cdot\|_Z : Z \to [0, +\infty), \qquad \|x\|_Z := \|x\|_X + \|Tx\|_Y, \quad \forall\, x \in Z. \tag{6.407}$$

Then the background assumptions in the statement of the corollary ensure that the space $\big(Z, \|\cdot\|_Z\big)$ is quasi-Banach. Also,

$$T : \big(Z, \tau_{\|\cdot\|_Z}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \quad \text{is continuous} \tag{6.408}$$

since, by design,

$$\|\cdot\|_X \leq \|\cdot\|_Z \quad \text{on} \quad Z. \tag{6.409}$$

Then Corollary 6.62 applies (with $(Z, \|\cdot\|_Z)$ playing the role of $(X, \|\cdot\|_X)$) and readily yields that claims (1)–(4) are equivalent (here (6.409) is again useful). Finally, the fact that (5) implies (4) is obvious, while the opposite implication is a consequence of the inclusion $B_{\|\cdot\|_Z}(0_X, r) \subseteq Z \cap B_{\|\cdot\|_X}(0_X, r)$, itself seen to hold for any $r > 0$ from (6.409). □

Corollary 6.64 (OMT for Vector Spaces: Version 5). *Let X be a vector space, and let τ_X be a topology on the set X satisfying the following continuity properties:*

$$(X, \tau_X) \ni x \mapsto -x \in (X, \tau_X) \quad \textit{is continuous at } 0, \tag{6.410}$$

$$(X, \tau_X) \ni x \mapsto x + y \in (X, \tau_X) \quad \textit{is continuous for each fixed } y \in X, \tag{6.411}$$

$$\big(X \times X, \tau_X \times \tau_X\big) \ni (x, y) \mapsto x + y \in (X, \tau_X) \quad \textit{is continuous at } (0, 0), \tag{6.412}$$

$$\mathbb{R} \ni \lambda \mapsto \lambda x \in (X, \tau_X) \quad \textit{is continuous at } 0 \textit{ for each fixed } x \in X. \tag{6.413}$$

In addition, suppose that (X, τ_X) is locally bounded, i.e.,

$$\begin{gathered} \exists\, B \in \mathcal{N}(0; \tau_X) \quad \textit{such that} \quad \forall\, V \in \mathcal{N}(0; \tau_X) \\ \exists\, r \in (0, +\infty) \quad \textit{for which} \quad rB \subseteq V, \end{gathered} \tag{6.414}$$

and that the additive group $(X, +)$ is complete with respect to the topology τ_X. Next, assume that Y is a vector space equipped with a Hausdorff topology τ_Y satisfying the continuity properties listed in (6.363) *and* (6.364)*. Finally, assume that*

$$T : X \longrightarrow Y \quad \textit{is an additive mapping so that} \tag{6.415}$$

$$T : (X, \tau_X) \longrightarrow (Y, \tau_Y) \quad \textit{is continuous and} \tag{6.416}$$

$$\operatorname{Im} T \quad \textit{is of second Baire category in } (Y, \tau_Y). \tag{6.417}$$

Then

$$T : (X, \tau_X) \longrightarrow (Y, \tau_Y) \quad \textit{is an open mapping.} \tag{6.418}$$

Moreover, if in addition to the hypotheses made on τ_Y one also assumes that (6.369) *holds, then mapping $T : X \to Y$ is surjective, and the additive group $(Y, +)$ is complete with respect to the topology τ_Y.*

Proof. The claim in (6.418) follows readily from Corollary 6.57, keeping in mind that the underlying additive group of any vector space is Abelian. Finally, the remaining claims in the statement of the current corollary are justified much as in the last part of the proof of Corollary 6.58. □

Corollary 6.65 (OMT for Topological Vector Spaces). *Assume that (X, τ_X) is a complete, locally bounded, topological vector space and that (Y, τ_Y) is a Hausdorff topological vector space. Also, assume that $T : X \to Y$ is a continuous, additive mapping whose image is a set of second Baire category in (Y, τ_Y). Then T is open and surjective, and the additive group $(Y, +)$ is complete with respect to the topology τ_Y.*

Proof. This follows directly from Corollary 6.64. □

We proceed by presenting another consequence of Theorem 6.49. Compared to Theorem 6.53, this time we replace (6.341) and (6.342) with a separability condition on the topological space (G, τ_G).

Theorem 6.66 (Topological OMT: Version 2). *Let $(G, *)$ be a group equipped with a finite quasi-invariant quasi-pseudonorm inducing a topology τ_G with respect to which G is complete and such that*

$$\text{the topological space } (G, \tau_G) \text{ is separable.} \tag{6.419}$$

Also, suppose that $(S, \circ, \tau_S)$ is a Hausdorff topological group.

Then any $T \in \mathrm{Hom}\,(G, S)$ that is continuous and such that $\mathrm{Im}\,T$ is of second Baire category in (S, τ_S) has the property that $T : (G, \tau_G) \to (S, \tau_S)$ is an open mapping.

Proof. Consider $T \in \mathrm{Hom}\,(G, S)$ as in the statement of the theorem and fix an arbitrary $U \in \mathcal{N}(e_G; \tau_G)$. As in (6.347), select $V \in \mathcal{N}(e_G; \tau_G)$ such that $V * V^{-1} \subseteq U$. To proceed, invoke (6.419) in order to find $A := \{a_n\}_{n\in\mathbb{N}}$, a countable dense subset of the topological space (G, τ_G). Then, making use of (6.51) we obtain

$$G = \mathrm{Clo}\,(A; \tau_G) = A * V. \tag{6.420}$$

Hence, since T is a group homomorphism, we may write

$$\mathrm{Im}\,T = T(G) = \bigcup_{n\in\mathbb{N}} (Ta_n) \circ T(V). \tag{6.421}$$

On the other hand, since we are assuming that $\mathrm{Im}\,T$ is of second Baire category in (S, τ_S), there exists $n \in \mathbb{N}$ such that

$$\mathrm{Int}\Big(\mathrm{Clo}\,\big((Ta_n) \circ T(V); \tau_S\big); \tau_S\Big) \neq \emptyset. \tag{6.422}$$

Next, from part (ii) in Lemma 6.14 and (6.3) we deduce that

$$\mathrm{Int}\Big(\mathrm{Clo}\,\big((Ta_n) \circ T(V); \tau_S\big); \tau_S\Big) = (Ta_n) \circ \mathrm{Int}\Big(\mathrm{Clo}\,\big(T(V); \tau_S\big)\Big); \tag{6.423}$$

hence, further,

$$\operatorname{Int}\Big(\operatorname{Clo}\big(T(V);\tau_S\big)\Big)\neq\emptyset. \tag{6.424}$$

With this in hand, the rest of the argument now proceeds as in the proof of Theorem 6.53 from (6.355) on. □

Corollary 6.67. *Retain all hypotheses on G, S, and T from Theorem 6.66, and in addition assume that $(S,\circ,\tau_S)$ is topologically divisible. Then, if ψ denotes the quasi-invariant quasi-pseudonorm on G, all conclusions in Corollary 6.51 are valid in this context.*

Proof. Since the proof of Theorem 6.66 gives that $T:(G,\tau_G)\to(S,\tau_S)$ is an open mapping and that (6.358) holds, Corollary 6.51 applies and yields the desired conclusions, granted the topological divisibility assumption on S. □

Corollary 6.68. *Let G be a group equipped with a finite quasi-invariant quasi-pseudonorm ψ inducing a topology τ_G with respect to which G is complete and such that the topological space (G,τ_G) is separable. Also, suppose that S is group endowed with a quasi-invariant quasinorm and that S is complete with respect to the topology τ_S induced on S by this quasinorm.*

Then any $T\in\operatorname{Hom}(G,S)$ that is continuous and surjective is an open mapping from (G,τ_G) onto (S,τ_S). Furthermore, if $\overline{\psi}$ denotes the push-forward of ψ via T, then $\overline{\psi}$ is a finite quasi-invariant quasi-pseudonorm with the same constants as ψ and satisfies $\tau_{\overline{\psi}}=\tau_S$.

Proof. Since τ_S is the topology induced by a quasi-invariant quasinorm on the group S, we conclude from part (vii) in Proposition 6.26 and part *(vi)* in Lemma 6.14 that $\big(S,\tau_S\big)$ is a Hausdorff topological group. Moreover, (S,τ_S) is of second Baire category, thanks to (6.144) and the fact that (S,τ_S) is assumed to be complete.

Having clarified these aspects, the fact that any $T\in\operatorname{Hom}(G,S)$ that is continuous and surjective is also open follows from Theorem 6.66. Finally, the remaining claims in the statement of the corollary are justified much as in the last part of the proof of Corollary 6.56. □

We also note the following purely topological version of Theorem 6.66.

Corollary 6.69. *Let G be an Abelian group equipped with a topology τ_G satisfying (6.150)–(6.152) and with respect to which G is complete and separable. Also, suppose that (S,τ_S) is a Hausdorff topological group. Then any $T\in\operatorname{Hom}(G,S)$ that is continuous and such that $\operatorname{Im}T$ is of second Baire category in (S,τ_S) has the property that $T:(G,\tau_G)\to(S,\tau_S)$ is an open mapping.*

Proof. This is a consequence of equivalence (6.153) from Proposition 6.31 and Theorem 6.66. □

The situation in which Theorem 6.66 and its corollaries are specialized to the case of vector spaces is considered in Corollaries 6.70–6.72.

Corollary 6.70 (OMT for Vector Spaces: Version 6). *Let X be a vector space, and let $\|\cdot\| : X \to [0, +\infty)$ be a function that vanishes at 0 and has the property that, for some constants $C_0, C_1 \in [1, +\infty)$,*

$$\|-x\| \leq C_0\|x\| \text{ for all } x \in X, \tag{6.425}$$

$$\|x+y\| \leq C_1\big(\|x\| + \|y\|\big) \text{ for all } x, y \in X. \tag{6.426}$$

Denote by $\tau_{\|\cdot\|}$ the topology induced by $\|\cdot\|$ on the additive group $(X, +)$, and assume that

$$\big(X, \tau_{\|\cdot\|}\big) \text{ is complete and separable.} \tag{6.427}$$

Next, suppose that Y is a vector space equipped with a Hausdorff topology τ_Y such that

$$\big(Y \times Y, \tau_Y \times \tau_Y\big) \ni (x, y) \longmapsto x - y \in (Y, \tau_Y) \text{ is continuous.} \tag{6.428}$$

Finally, assume that

$$T : X \longrightarrow Y \text{ is an additive mapping so that} \tag{6.429}$$

$$T : \big(X, \tau_{\|\cdot\|}\big) \longrightarrow (Y, \tau_Y) \text{ is continuous and} \tag{6.430}$$

$$\operatorname{Im} T \text{ is of second Baire category in } (Y, \tau_Y). \tag{6.431}$$

Then

$$T : \big(X, \tau_{\|\cdot\|}\big) \longrightarrow (Y, \tau_Y) \text{ is an open mapping.} \tag{6.432}$$

Moreover, if in addition to the hypotheses made on τ_Y one also assumes that (6.369) holds, then mapping $T : X \to Y$ is surjective, and the additive group $(Y, +)$ is complete with respect to the topology τ_Y. In addition, if $|\!|\!|\cdot|\!|\!| : Y \to [0, +\infty)$ denotes the push-forward of $\|\cdot\|$ via T, then the topology induced by $|\!|\!|\cdot|\!|\!|$ on $(Y, +)$ coincides with $\tau_{\|\cdot\|_Y}$ and, for the same constants C_0, C_1 as before,

$$|\!|\!|0|\!|\!| = 0, \quad |\!|\!|-x|\!|\!| \leq C_0|\!|\!|x|\!|\!| \text{ for all } x \in Y, \tag{6.433}$$

$$|\!|\!|x+y|\!|\!| \leq C_1\big(|\!|\!|x|\!|\!| + |\!|\!|y|\!|\!|\big) \text{ for all } x, y \in Y. \tag{6.434}$$

Proof. The claim in (6.432) follows directly from Theorem 6.66, while the subsequent claims are justified similarly to the last part of the proof of Corollary 6.58 (cf. also the last part of Corollary 6.68). □

Corollary 6.71 (OMT for Vector Spaces: Version 7). *Let X be a vector space, and let $\|\cdot\|_X : X \to [0, +\infty)$ be a function with the property that, for some* *constants* $C_0, C_1 \in [1, +\infty)$,

$$\|-x\|_X \leq C_0 \|x\|_X \ \text{ for all } x \in X, \tag{6.435}$$

$$\|x + y\|_X \leq C_1 \big(\|x\|_X + \|y\|_X\big) \ \text{ for all } \ x, y \in X. \tag{6.436}$$

Denote by $\tau_{\|\cdot\|_X}$ *the topology induced by* $\|\cdot\|_X$ *on the additive group* $(X, +)$*, and assume that*

$$\big(X, \tau_{\|\cdot\|_X}\big) \text{ is complete and separable.} \tag{6.437}$$

Next, suppose that Y *is a vector space and that* $\|\cdot\|_Y : Y \to [0, +\infty)$ *is a function satisfying, for some constants* $C_0', C_1' \in [1, +\infty)$,

$$\|x\|_Y = 0 \Longleftrightarrow x = 0 \ \text{ for all } x \in Y, \tag{6.438}$$

$$\|-x\|_Y \leq C_0' \|x\|_Y \ \text{ for all } x \in Y, \tag{6.439}$$

$$\|x + y\|_Y \leq C_1' \big(\|x\|_Y + \|y\|_Y\big) \ \text{ for all } \ x, y \in Y. \tag{6.440}$$

Denote by $\tau_{\|\cdot\|_Y}$ *the topology induced by* $\|\cdot\|_Y$ *on the additive group* $(Y, +)$*, and assume that this group is complete with respect to the topology* $\tau_{\|\cdot\|_Y}$*. Finally, suppose that*

$$T : X \longrightarrow Y \ \text{ is an additive, surjective mapping so that} \tag{6.441}$$

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \ \text{ is continuous.} \tag{6.442}$$

Then

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \ \text{ is an open mapping.} \tag{6.443}$$

In addition, if $|\!|\!|\cdot|\!|\!| : Y \to [0, +\infty)$ *denotes the push-forward of* $\|\cdot\|_X$ *via* T*, then the topology induced by* $|\!|\!|\cdot|\!|\!|$ *on* $(Y, +)$ *coincides with* $\tau_{\|\cdot\|_Y}$ *and, for the same constants* C_0, C_1 *as above,*

$$|\!|\!|0|\!|\!| = 0, \quad |\!|\!|-x|\!|\!| \leq C_0 |\!|\!|x|\!|\!| \ \text{ for all } x \in Y, \tag{6.444}$$

$$|\!|\!|x + y|\!|\!| \leq C_1 \big(|\!|\!|x|\!|\!| + |\!|\!|y|\!|\!|\big) \ \text{ for all } \ x, y \in Y. \tag{6.445}$$

Proof. This is readily implied by Corollary 6.68. □

Corollary 6.72 (OMT for Vector Spaces: Version 8). *Let* X *be a vector space, and let* τ_X *be a topology on the set* X *satisfying properties* (6.410)–(6.412)*. In addition, assume that the additive group* $(X, +)$ *is complete with respect to the topology* τ_X *and that the topological space* (X, τ_X) *is separable and locally bounded* *(cf.* (6.414)*). Next, assume that* Y *is a vector space equipped with a Hausdorff topology* τ_Y *satisfying* (6.363)*. Finally, suppose that* T *is a mapping from* X *to* Y *that satisfies* (6.415)–(6.417)*. Then* $T : (X, \tau_X) \to (Y, \tau_Y)$ *is an open mapping.*

Proof. This follows readily from Corollary 6.69. □

6.5 Closed Graph Theorem

The principal result in this section is Theorem 6.74, containing a version of the classical closed graph theorem (CGT) in the context of groups endowed with suitable topologies. As a preamble, we first dispense with some useful commutation formulas in the lemma below.

Lemma 6.73. *Let* $\big(G, *, (\cdot)^{-1}, e_G\big)$ *be a group, and assume that* $\psi : G \to [0, +\infty]$ *is a quasi-invariant [cf.* (6.114)*] function on* G *satisfying* (6.74) *and* $\psi(e_G) = 0$*. Denote by* τ_ψ *the topology induced by* ψ *on* G *as in Definition 6.13 (from the right or from the left; the choice is immaterial; cf.* (6.83)*). Then*

$$\forall\, V \in \mathscr{N}(e_G; \tau_\psi) \exists\, W \in \mathscr{N}(e_G; \tau_\psi) \text{ so that } W * A \subseteq A * V \quad \forall\, A \subseteq G, \tag{6.446}$$

$$\forall\, V \in \mathscr{N}(e_G; \tau_\psi) \exists\, W \in \mathscr{N}(e_G; \tau_\psi) \text{ so that } A * W \subseteq V * A \quad \forall\, A \subseteq G. \tag{6.447}$$

Proof. As far as (6.446) is concerned, it suffices to treat the case when the set A is a singleton (then take the union of the resulting inclusions over its elements). Hence assume that $A = \{a\}$, and consider some $V \in \mathscr{N}\big(e_G; \tau_\psi^{\rm R}\big)$. By (6.67), there exists a number $r \in (0, +\infty)$ such that $B_\psi^{\rm R}(e_G, r) \subseteq V$. As such, thanks to (6.77), it suffices to show that

$$B_\psi^{\rm R}(e_G, r/C_2) * a \subseteq a * B_\psi^{\rm R}(e_G, r), \tag{6.448}$$

where $C_2 \in [1, +\infty)$ is the quasi-invariance constant of the function ψ (cf. (6.114)). To this end, if $x \in B_\psi^{\rm R}(e_G, r/C_2) * a$, then it follows that $x = y * a$ for some $y \in G$ such that $\psi(y^{-1}) < r/C_2$. Hence, $x = a * (a^{-1} * y * a)$, and the quasi-invariance of ψ implies $\psi(a^{-1} * y^{-1} * a) \le C_2 \psi(y^{-1}) < r$. This shows that $a^{-1} * y * a \in B_\psi^{\rm R}(e_G, r)$, and (6.448) follows. Thus (6.446) holds, and (6.447) is established similarly. □

The stage is set for stating and proving the main version of the CGT in the context of groups equipped with topologies satisfying suitable compatibility conditions with the algebraic structures involved.

Theorem 6.74 (Closed Graph Theorem). *Let G be a group equipped with a topology τ_G that is symmetric and either right-invariant or left-invariant. Also, let S be a group endowed with a finite quasi-invariant quasi-pseudonorm, inducing a topology τ_S on S with respect to which S is complete. Finally, suppose that $T \in$ Hom (G, S) has a closed graph in $(G \times S, \tau_G \times \tau_S)$ and is "almost continuous," in the sense that it satisfies*

$$\operatorname{Clo}\big(T^{-1}(V); \tau_G\big) \in \mathcal{N}(e_G; \tau_G) \qquad \text{for every } V \in \mathcal{N}(e_S; \tau_S). \tag{6.449}$$

Then

$$T : (G, \tau_G) \longrightarrow (S, \tau_S) \quad \text{is continuous.} \tag{6.450}$$

Proof. The proof proceeds in six steps. To fix ideas, assume for now that τ_G is right-invariant.

`Step I.` *For each $x \in S$ consider the nonempty collection of subsets of S described by the formula*

$$\begin{aligned} \mathcal{N}_x := \big\{W \subseteq S : \exists\, U \in \mathcal{N}(e_G; \tau_G),\ \exists\, V \in \mathcal{N}(e_S; \tau_S) \\ \text{with } T(U) \circ V \circ x \subseteq W\big\}. \end{aligned} \tag{6.451}$$

Then

$$\widetilde{\tau}_S := \big\{\mathcal{O} \subseteq S : \mathcal{O} \in \mathcal{N}_x \text{ for each } x \in \mathcal{O}\big\} \text{ is a topology on } S \tag{6.452}$$

such that

$$\widetilde{\tau}_S \subseteq \tau_S \tag{6.453}$$

and

$$\mathcal{N}_x = \mathcal{N}(x; \widetilde{\tau}_S), \qquad \forall\, x \in S. \tag{6.454}$$

Indeed, it is clear that the mapping $S \ni x \mapsto \mathcal{N}_x \subseteq 2^S$ satisfies conditions *(i)* and *(ii)* in Lemma 6.4; hence $\widetilde{\tau}_S$ from (6.452) is a topology on S. Moreover, thanks to (6.26), we have that $x \in W$ for every $x \in S$ and every $W \in \mathcal{N}_x$; hence condition *(iii)* in Lemma 6.4 also holds. Thus, (6.454) follows as soon as we prove that condition *(iv)* in Lemma 6.4 is satisfied as well.

With the goal of verifying condition *(iv)* in Lemma 6.4, let $x \in S$ and assume that $W \in \mathcal{N}_x$. Then there exists $U \in \mathcal{N}(e_G; \tau_G)$ and $V \in \mathcal{N}(e_S; \tau_S)$ such that

$$T(U) \circ V \circ x \subseteq W. \tag{6.455}$$

The aim is to find

$$O \in \mathcal{N}_x \text{ such that } O \subseteq T(U) \circ V \circ x \text{ and } O \in \mathcal{N}_y \text{ for all } y \in O. \tag{6.456}$$

We claim that

$$O := T\big(\mathrm{Int}\,(U; \tau_G)\big) \circ \mathrm{Int}\,(V; \tau_S) \circ x \tag{6.457}$$

does the job. Clearly, $O \in \mathcal{N}_x$ and $O \subseteq T(U) \circ V \circ x$, so the focus becomes the last claim in (6.456). With this in mind, consider an arbitrary $y \in O$. Hence,

$$\text{there exist } a \in \mathrm{Int}\,(U; \tau_G) \text{ and } b \in \mathrm{Int}\,(V; \tau_S) \text{ such that } y = (Ta) \circ b \circ x. \tag{6.458}$$

From this and the continuity of $s_b^{\mathrm{R}} : (S, \tau_S) \to (S, \tau_S)$ at e_S we deduce that

$$\exists\, V_1 \in \mathscr{N}(e_S; \tau_S) \text{ with } V_1 \circ b \subseteq \mathrm{Int}\,(V; \tau_S). \tag{6.459}$$

Next, Lemma 6.73 used in the context of the group S shows that

$$\exists\, V_2 \in \mathscr{N}(e_S; \tau_S) \text{ with } V_2 \circ (Ta) \subseteq (Ta) \circ V_1. \tag{6.460}$$

Also, from (6.458) and the continuity of $s_a^{\mathrm{R}} : (G, \tau_G) \to (G, \tau_G)$ at e_G it follows that

$$\exists\, U_1 \in \mathscr{N}(e_G; \tau_G) \text{ with } U_1 * a \subseteq \mathrm{Int}\,(U; \tau_G). \tag{6.461}$$

Thus,

$$\begin{aligned} T(U_1) \circ V_2 \circ y &= T(U_1) \circ V_2 \circ (Ta) \circ b \circ x \subseteq T(U_1) \circ (Ta) \circ V_1 \circ b \circ x \\ &= T(U_1 * a) \circ (V_1 \circ b) \circ x \subseteq T\big(\mathrm{Int}\,(U; \tau_G)\big) \circ \mathrm{Int}\,(V; \tau_S) \circ x \\ &= O, \end{aligned} \tag{6.462}$$

thanks to the last formula in (6.458), the intertwining inclusion from (6.460), the fact that $T \in \mathrm{Hom}\,(G, S)$, and (6.459), (6.461), and (6.457). In light of (6.451), formula (6.462) establishes that $O \in \mathcal{N}_y$, as desired. This completes the proof of the verification of condition *(iv)* in Lemma 6.4 and concludes the justification of (6.454).

Finally, as regards (6.453), observe that for each neighborhood $U \in \mathscr{N}(e_G; \tau_G)$ and for each $V \in \mathscr{N}(e_S; \tau_S)$ one has $V \circ x \subseteq T(U) \circ V \circ x$ as $e_S \in T(U)$. Using that the topology τ_S is right-invariant on S, it follows that $V \circ x \in \mathscr{N}(x; \tau_S)$ for each $x \in S$. Thus, whenever $U \in \mathscr{N}(e_G; \tau_G)$ and $V \in \mathscr{N}(e_S; \tau_S)$,

$$T(U) \circ V \circ x \in \mathscr{N}(x; \tau_S), \qquad \forall\, x \in S, \tag{6.463}$$

and consequently $\mathscr{N}(x;\widetilde{\tau}_S) \subseteq \mathscr{N}(x;\tau_S)$ for each $x \in S$. From this, (6.453) immediately follows.

`Step II.` *The following operators are continuous:*

$$T : (G,\tau_G) \longrightarrow (S,\widetilde{\tau}_S), \tag{6.464}$$

$$I : (S,\tau_S) \longrightarrow (S,\widetilde{\tau}_S), \tag{6.465}$$

where I denotes the identity mapping on S. To justify these claims, assume that an arbitrary element $a \in G$ and an arbitrary neighborhood $W \in \mathscr{N}(Ta;\widetilde{\tau}_S)$ have been given. Then from (6.454) and (6.451) it follows that there exist two neighborhoods, $U \in \mathscr{N}(e_G;\tau_G)$ and $V \in \mathscr{N}(e_S;\tau_S)$, such that

$$T(U) \circ V \circ (Ta) \subseteq W. \tag{6.466}$$

Recall that the topology τ_G is right-invariant. As such, we deduce from (6.4) and (6.48) that $O := s_a^{\mathrm{R}}(U) \in \mathscr{N}(a;\tau_G)$. Then, since $e_S \in V$ and since T is a group homomorphism from G into S, we conclude from (6.466) that $T(O) = T(U) \circ (Ta) \subseteq T(U) \circ V \circ (Ta) \subseteq W$. This shows that $O \in \mathscr{N}(a;\tau_G)$ satisfies $T(O) \subseteq W$ and, hence, T is continuous at a. Given that $a \in G$ was arbitrarily chosen, the conclusion is that T in (6.464) is continuous. The claim about the operator (6.465) is handled similarly, based on (6.26).

`Intermezzo.` The results established in Steps I and II suggest that we pursue the following strategy for completing the proof of the theorem:

- Since T in (6.464) is continuous, (6.450) follows if we show that $\widetilde{\tau}_S = \tau_S$.
- Thanks to (6.453), it suffices to prove that $\tau_S \subseteq \widetilde{\tau}_S$. (6.467)
- The latter condition is satisfied provided (6.465) is an open mapping.

Hence, the goal becomes proving that the identity in (6.465) is an open mapping. Given that, obviously, $I \in \mathrm{Hom}\,(S,S)$, it is natural to try to accomplish this goal by invoking the version of the OMT proved in Theorem 6.49. In this regard, it is relevant to note that

$$\mathscr{G}_I = \mathrm{diag}\,(S) := \{(x,x) : x \in S\} \tag{6.468}$$

and that the analog of (6.291) in the current context reads

$$\mathrm{Clo}\,\big(B;\widetilde{\tau}_S\big) \in \mathscr{N}(e_S;\widetilde{\tau}_S), \qquad \forall\, B \ \text{ ball in } S \text{ centered at } e_S. \tag{6.469}$$

There remains to ensure that the hypotheses of Theorem 6.49 are satisfied for the specific context described previously, and this constitutes the object of the subsequent steps in the proof.

`Step III.` *One has*

$$\widetilde{\tau}_S \text{ is symmetric and right-invariant on } S. \tag{6.470}$$

We first focus on establishing *the symmetry of the topology* $\widetilde{\tau}_S$. To this end, pick an arbitrary $W \in \mathcal{N}(e_S;\widetilde{\tau}_S)$, and note that, based on (6.454) and (6.451), there exist neighborhoods $U \in \mathcal{N}(e_G;\tau_G)$ and $V \in \mathcal{N}(e_S;\tau_S)$ such that $T(U)\circ V \subseteq W$. Since the topologies τ_G and τ_S are symmetric, without loss of generality, we may assume that U, V are symmetric. Note that, employing Lemma 6.73, we obtain that there exists $\widetilde{V} \in \mathcal{N}(e_S;\tau_S)$ such that $T(U)\circ\widetilde{V} \subseteq V\circ T(U)$, and hence $V\circ T(U) \in \mathcal{N}(e_S;\widetilde{\tau}_S)$. Going further, define

$$\widetilde{W} := \big(T(U)\circ V\big)\cap\big(V\circ T(U)\big), \tag{6.471}$$

and observe that, since T is a homomorphism and since $U = U^{-1}$ and $V = V^{-1}$, the set $\widetilde{W}$ is symmetric. Next, since both sets $T(U)\circ V$ and $V\circ T(U)$ belong to $\mathcal{N}(e_S;\widetilde{\tau}_S)$, we deduce that $\widetilde{W} \in \mathcal{N}(e_S;\widetilde{\tau}_S)$. Finally, using that $T(U)\circ V \subseteq W$ we obtain that $\widetilde{W} \subseteq W$. Since $W \in \mathcal{N}(e_S;\widetilde{\tau}_S)$ was arbitrary, this completes the proof of the symmetry of $\widetilde{\tau}_S$.

We will now prove that *the topology* $\widetilde{\tau}_S$ *is right-invariant on the group* S. Fix an arbitrary $x \in S$, and notice that for every $U \in \mathcal{N}(e_G;\tau_G)$ and for every $V \in \mathcal{N}(e_S;\tau_S)$ there holds $T(U)\circ V\circ x = s^{\mathrm{R}}_x\big(T(U)\circ V\big)$. Thus,

$$s^{\mathrm{R}}_x(W) \in \mathcal{N}(x;\widetilde{\tau}_S), \quad \forall\, W \in \mathcal{N}(e_S;\widetilde{\tau}_S), \quad \forall\, x \in S, \tag{6.472}$$

from which it immediately follows that

$$s^{\mathrm{R}}_x(W) \in \mathcal{N}(y\circ x;\widetilde{\tau}_S), \quad \forall\, W \in \mathcal{N}(y;\widetilde{\tau}_S), \quad \forall\, x, y \in S. \tag{6.473}$$

In turn, (6.473) shows that for each $x \in S$ the right-shift s^{R}_x is continuous on $(S,\widetilde{\tau}_S)$, as desired.

`Step IV.` *One has*

$$\operatorname{Clo}(V;\widetilde{\tau}_S) \in \mathcal{N}(e_S;\widetilde{\tau}_S), \qquad \forall\, V \in \mathcal{N}(e_S;\tau_S). \tag{6.474}$$

To prove (6.474), fix an arbitrary $V \in \mathcal{N}(e_S;\tau_S)$. Given that $(S,\circ,\tau_S)$ is a topological group, the function $(\cdot)\circ(\cdot) : (S\times S, \tau_S\times\tau_S) \longrightarrow (S,\tau_S)$ is continuous, and hence

$$\exists\, Z \in \mathcal{N}(e_S;\tau_S) \ \text{ such that } \ Z\circ Z \subseteq V. \tag{6.475}$$

By our assumption on the operator T (cf. (6.449)), we have

$$U := \operatorname{Clo}\big(T^{-1}(Z);\tau_G\big) \in \mathcal{N}(e_G;\tau_G). \tag{6.476}$$

Since the topology τ_G is assumed to be symmetric and right-invariant, using property *(2)* from Lemma 6.12 we may write

$$U \subseteq T^{-1}(Z) \circ W, \qquad \forall\, W \in \mathcal{N}(e_G; \tau_G). \tag{6.477}$$

Now, using (6.477) and the fact that $T \in \text{Hom}\,(G, S)$ yields $T(U) \subseteq Z \circ T(W)$. Thus, for every $W \in \mathcal{N}(e_G; \tau_G)$ we have

$$\begin{gathered} Z \circ T(U) \subseteq Z \circ Z \circ T(W) \subseteq V \circ T(W) \subseteq V \circ T(W) \circ \mathcal{O}, \\ \forall\, \mathcal{O} \in \mathcal{N}(e_S; \tau_S), \end{gathered} \tag{6.478}$$

where the first inclusion is a direct consequence of the fact that $T(U) \subseteq Z \circ T(W)$, the second one follows from (6.475), and the last inclusion uses the fact that $e_S \in \mathcal{O}$. Using (6.478), the fact that the topology $\widetilde{\tau}_S$ is symmetric and right-invariant on S as established in `Step III`, and property (2) from Lemma 6.12, we obtain

$$Z \circ T(U) \subseteq \bigcap_{\substack{W \in \mathcal{N}(e_G;\tau_G) \\ \mathcal{O} \in \mathcal{N}(e_S;\tau_S)}} V \circ \big(T(W) \circ \mathcal{O}\big) = \text{Clo}\,(V; \widetilde{\tau}_S). \tag{6.479}$$

Now, keeping in mind that $Z \in \mathcal{N}(e_S; \tau_S)$ and invoking Lemma 6.73, we can find a neighborhood $\widetilde{Z} \in \mathcal{N}(e_S; \tau_S)$ such that $T(U) \circ \widetilde{Z} \subseteq Z \circ T(U)$. Of course, given that $T(U) \circ \widetilde{Z} \in \mathcal{N}(e_S; \widetilde{\tau}_S)$ (cf. (6.451)–(6.454)), this entails $Z \circ T(U) \in \mathcal{N}(e_S; \tau_S)$; hence, ultimately, $\text{Clo}\,(V; \widetilde{\tau}_S) \in \mathcal{N}(e_S; \widetilde{\tau}_S)$ by (6.479). Since $V \in \mathcal{N}(e_S; \tau_S)$ was arbitrary, this completes the proof of (6.474).

`Step V.` *One has*

$$\text{diag}(S) \;\; \text{is closed in} \;\; \big(S \times S, \tau_S \times \widetilde{\tau}_S\big). \tag{6.480}$$

To establish (6.480), we will show that $S \times S \setminus \text{diag}(S)$ is an open set in $(S \times S, \tau_S \times \widetilde{\tau}_S)$. To this end, fix some $(a, b) \in S \times S \setminus \text{diag}(S)$, and note that, since $a \neq b$, we have $(e_G, a \circ b^{-1}) \notin \mathcal{G}_T$. Since $\mathcal{G}_T$ is, by hypothesis, a closed set in $(G \times S, \tau_G \times \tau_S)$, it follows that there exist $U \in \mathcal{N}(e_G; \tau_G)$ and $\mathcal{O} \in \mathcal{N}(a \circ b^{-1}; \tau_S)$ such that $\mathcal{G}_T \cap (U \times \mathcal{O}) = \emptyset$. Deobfuscating the notation, the latter condition comes down to

$$T(U) \cap \mathcal{O} = \emptyset. \tag{6.481}$$

Since $(S, \circ, \tau_S)$ is a topological group, the function

$$(S \times S, \tau_S \times \tau_S) \ni (x, y) \mapsto x \circ y^{-1} \in (S, \tau_S) \tag{6.482}$$

is continuous (as noted earlier, in (6.43)); hence there exist $V, W \in \mathcal{N}(e_S; \tau_S)$ such that

$$W \circ (a \circ b^{-1}) \circ V^{-1} \subseteq \mathcal{O}. \tag{6.483}$$

Using (6.483) and (6.481) it follows that

$$\big(W \circ (a \circ b^{-1}) \circ V^{-1}\big) \cap T(U) = \emptyset, \tag{6.484}$$

which, as a moment's reflection shows, further entails

$$(W \circ a) \cap \big(T(U) \circ V \circ b\big) = \emptyset. \tag{6.485}$$

Now, on the one hand, using the fact that the topology τ_S is right-invariant we obtain that $W \circ a \in \mathscr{N}(a;\tau_S)$, while, on the other hand, (6.454) implies $T(U) \circ V \circ b \in \mathscr{N}(b\,;\widetilde{\tau}_S)$. Using this and (6.485) we therefore arrive at the conclusion that

$$\begin{gathered}(W \circ a) \times \big(T(U) \circ V \circ b\big) \in \mathscr{N}\big((a,b);\tau_S \times \widetilde{\tau}_S\big)\\ \text{is disjoint from } \operatorname{diag}(S).\end{gathered} \tag{6.486}$$

This completes the proof of the fact that $S \times S \setminus \operatorname{diag}(S)$ is an open set in the space $(S \times S, \tau_S \times \widetilde{\tau}_S)$, from which (6.480) immediately follows.

`Step VI.` *The end game in the proof of the theorem.* By hypothesis, $(S, \circ, \tau_S)$ is a complete, quasi-invariant, quasi-pseudonormed group, and by `Step III`, the topology $\widetilde{\tau}_S$ is both symmetric and right-invariant on S. The fact that the group homomorphism $I : (S,\tau_S) \to (S,\widetilde{\tau}_S)$ satisfies the conditions formulated in Theorem 6.49 is seen from `Step IV` (cf. also the discussion pertaining to (6.469)) and `Step V` (cf. also (6.468)). With these in hand, the OMT in the version presented in Theorem 6.49 applies and gives that the identity operator is an open mapping in the context of (6.465). As indicated in the `Intermezzo`, this finishes the proof in the case when τ_G is right-invariant. The situation when τ_G is left-invariant is very similar, and this concludes the proof of the theorem. □

As in the case of the OMT, we now present a couple of versions of the CGT (established in Theorem 6.74) in which the technical condition (6.449) is replaced by alternative topological assumptions that are typically easier to verify in practice. Before we state the first result in this regard, the reader is advised to recall (6.33).

Theorem 6.75 (Closed Graph Theorem: Second Version). *Let $(G, *)$ be a uniquely divisible group equipped with a finite quasi-invariant quasi-pseudonorm inducing a topology τ_G on the set G with the property that (G,τ_G) is complete and satisfies*

$$\mathcal{O}^{(n)} \in \tau_G \quad \textit{for every } \ \mathcal{O} \in \tau_G \ \textit{ and every } \ n \in \mathbb{N}. \tag{6.487}$$

Also, suppose that $(S, \circ)$ is a given group equipped with a finite quasi-invariant quasi-pseudonorm inducing a topology τ_S on the set S with the property that (S,τ_S) is complete and

$$S = \bigcup_{n\in\mathbb{N}} V^{(n)}, \qquad \textit{for every} \ \ V \in \mathscr{N}(e_S; \tau_S). \tag{6.488}$$

Then any $T \in \operatorname{Hom}(G, S)$ *whose graph is closed in* $(G \times S, \tau_G \times \tau_S)$ *has the property that* $T : (G, \tau_G) \to (S, \tau_S)$ *is continuous.*

Proof. We begin by discussing several auxiliary formulas. First, we claim that

$$\Big(T^{-1}(A)\Big)^{-1} = T^{-1}\big(A^{-1}\big), \qquad \forall\, A \subseteq S. \tag{6.489}$$

Indeed, since $T \in \operatorname{Hom}(G, S)$, we may write

$$\begin{aligned} a \in \Big(T^{-1}(A)\Big)^{-1} &\Leftrightarrow a^{-1} \in T^{-1}(A) \Leftrightarrow T(a^{-1}) \in A \Leftrightarrow (Ta)^{-1} \in A \\ &\Leftrightarrow Ta \in A^{-1} \Leftrightarrow a \in T^{-1}\big(A^{-1}\big), \end{aligned} \tag{6.490}$$

from which (6.489) follows. We will also need the fact that

$$T^{-1}(A) * T^{-1}(B) \subseteq T^{-1}\big(A \circ B\big), \qquad \forall\, A, B \subseteq S. \tag{6.491}$$

To justify this inclusion, assume that $A, B \subseteq S$ are given and pick an arbitrary element x in $T^{-1}(A) * T^{-1}(B)$. Then there exist $a \in T^{-1}(A)$ and $b \in T^{-1}(B)$ such that $x = a * b$. Consequently, $Ta \in A$, $Tb \in B$, and, hence,

$$Tx = T(a * b) = (Ta) \circ (Tb) \in A \circ B, \tag{6.492}$$

given that $T \in \operatorname{Hom}(G, S)$. Thus, $x \in T^{-1}\big(A \circ B\big)$, and (6.491) follows.

Moving on, we make the claim that, for every $A, B \subseteq S$,

$$\operatorname{Clo}\Big(T^{-1}(A); \tau_G\Big) * \Big(\operatorname{Clo}\Big(T^{-1}(B); \tau_G\Big)\Big)^{-1} \subseteq \operatorname{Clo}\Big(T^{-1}\big(A \circ B^{-1}\big); \tau_G\Big). \tag{6.493}$$

To see that this is the case, given any $A, B \subseteq S$, we write

$$\begin{aligned} &\operatorname{Clo}\Big(T^{-1}(A); \tau_G\Big) * \Big(\operatorname{Clo}\Big(T^{-1}(B); \tau_G\Big)\Big)^{-1} \\ &\quad = \operatorname{Clo}\Big(T^{-1}(A); \tau_G\Big) * \operatorname{Clo}\Big(T^{-1}\big(B^{-1}\big); \tau_G\Big) \\ &\quad \subseteq \operatorname{Clo}\Big(T^{-1}(A) * T^{-1}\big(B^{-1}\big); \tau_G\Big) \\ &\quad \subseteq \operatorname{Clo}\Big(T^{-1}\big(A \circ B^{-1}\big); \tau_G\Big). \end{aligned} \tag{6.494}$$

The equality in (6.494) is a consequence of part *(i)* in Proposition 6.26, (6.2), and (6.489). Also, the first inclusion in (6.494) is implied by (6.54) and part *(vii)* in Proposition 6.26, while the second inclusion in (6.494) holds thanks to (6.491). This completes the proof of (6.493).

Going further, we note that

$$\big(T^{-1}(A)\big)^{(n)} = T^{-1}\big(A^{(n)}\big), \qquad \forall\, A \subseteq S \text{ and } \forall\, n \in \mathbb{N}. \tag{6.495}$$

To justify this, fix $A \subseteq S$ and $n \in \mathbb{N}$. If $a \in \big(T^{-1}(A)\big)^{(n)}$, then it follows that there exists $b \in T^{-1}(A)$ such that $a = b^n$. Thus, $Tb \in A$, so that $Ta = (Tb)^n \in A^{(n)}$. Hence, $a \in T^{-1}\big(A^{(n)}\big)$, which shows that the left-to-right inclusion in (6.495) holds. As regards the converse inclusion, assume that $x \in T^{-1}\big(A^{(n)}\big)$, and, using the fact that G is divisible, pick $y \in G$ such that $x = y^n$. Then $(Ty)^n = Tx \in A^{(n)}$, which, since the group G is uniquely divisible, implies that actually $y \in A$. Having established this, we deduce that $y \in T^{-1}(A)$ and, finally, that $x = y^n \in \big(T^{-1}(A)\big)^{(n)}$. This proves the right-to-left inclusion in (hence, finishing the proof of) (6.495).

To proceed, assume that an arbitrary $V \in \mathscr{N}(e_S; \tau_S)$ has been fixed. We then invoke (6.488) in order to write that

$$G = T^{-1}(S) = \bigcup_{n\in\mathbb{N}} T^{-1}\big(V^{(n)}\big) = \bigcup_{n\in\mathbb{N}} \big(T^{-1}(V)\big)^{(n)}. \tag{6.496}$$

Granted (6.144), this implies that there exists $n \in \mathbb{N}$ with the property that

$$\operatorname{Int}\big(\operatorname{Clo}\big((T^{-1}(V))^{(n)}; \tau_G\big); \tau_G\big) \neq \emptyset. \tag{6.497}$$

Hence, there exists $\mathcal{O} \in \tau_G$ such that

$$\emptyset \neq \mathcal{O} \subseteq \operatorname{Clo}\Big(\big(T^{-1}(V)\big)^{(n)}; \tau_G\Big). \tag{6.498}$$

Consider now

$$\phi : G \longrightarrow G, \quad \phi(x) := x^n, \quad \forall\, x \in G. \tag{6.499}$$

The fact that G is uniquely divisible guarantees that ϕ is a bijection. Also, the fact that $(G, *, \tau_G)$ is a topological group implies that ϕ is continuous, while (6.487) entails that ϕ is open. Thus, all together,

$$\phi : (G, \tau_G) \longrightarrow (G, \tau_G) \text{ is a homeomorphism.} \tag{6.500}$$

Now, since $\phi\big(A\big) = A^{(n)}$ for any $A \subseteq G$, we have

$$\mathrm{Clo}\Big(\big(T^{-1}(V)\big)^{(n)};\tau_G\Big) = \Big(\mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big)\Big)^{(n)}, \tag{6.501}$$

by (6.500) and (6.2). Consequently, on the one hand,

$$\phi^{-1}(\mathcal{O}) \subseteq \phi^{-1}\Big(\mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big)\Big)^{(n)} = \mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big), \tag{6.502}$$

by (6.501), (6.498), and the fact that ϕ is bijective. On the other hand, we have $\emptyset \neq \phi(\mathcal{O})^{-1} \in \tau_G$, thanks to (6.500). Granted this and keeping (6.502) in mind, we arrive at the conclusion that

$$\mathrm{Int}\,\big(\mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big);\tau_G\big) \neq \emptyset. \tag{6.503}$$

In concert with Lemma 6.52, this permits us to conclude that

$$\mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big) * \Big(\mathrm{Clo}\,\big(T^{-1}(V);\tau_G\big)\Big)^{-1} \in \mathcal{N}(e_G;\tau_G). \tag{6.504}$$

The bottom line [seen from (6.494) and (6.504)] is that

$$\mathrm{Clo}\,\Big(T^{-1}(V\circ V^{-1});\tau_G\Big) \in \mathcal{N}(e_G;\tau_G), \qquad \forall\, V \in \mathcal{N}(e_S;\tau_S). \tag{6.505}$$

At this stage, given an arbitrary $U \in \mathcal{N}(e_S;\tau_S)$, since $(S,\circ,\tau_S)$ is a topological group, from the continuity of the mapping

$$\big(S\times S,\tau_S\times\tau_S\big) \ni (x,y) \mapsto x * y^{-1} \in (S,\tau_S) \tag{6.506}$$

at (e_S,e_S) it follows that

$$\exists\, V_0 \in \mathcal{N}(e_S;\tau_S) \;\text{ such that }\; V_0 * V_0^{-1} \subseteq U. \tag{6.507}$$

Collectively, (6.505) and (6.507) imply that

$$\mathrm{Clo}\,\Big(T^{-1}(U);\tau_G\Big) \in \mathcal{N}(e_G;\tau_G), \qquad \forall\, U \in \mathcal{N}(e_S;\tau_S). \tag{6.508}$$

This shows that condition (6.449) is satisfied. As such, the CGT in the version presented in Theorem 6.74 applies and yields the desired conclusion. □

To state yet another version of the CGT, the reader may wish to recall Definition 6.6.

Theorem 6.76 (Closed Graph Theorem: Third Version). *Let $(G,*)$ be a group equipped with a finite quasi-invariant quasi-pseudonorm, inducing a topology τ_G on the set G such that (G,τ_G) is complete. Also, suppose that $(S,\circ)$ is a group* *equipped with a finite quasi-invariant quasi-pseudonorm inducing a topology τ_S on the set S such that*

$$(S, \tau_S) \ \textit{is complete and Lindelöf.} \tag{6.509}$$

Then any $T \in \mathrm{Hom}\,(G, S)$ whose graph is closed in $(G \times S, \tau_G \times \tau_S)$ has the property that $T : (G, \tau_G) \to (S, \tau_S)$ is continuous.

Proof. We revisit the proof of Theorem 6.75 with the goal of monitoring the effect of replacing (6.488) by the condition that the topological space (S, τ_S) is Lindelöf.

To get started, assume that an arbitrary $V \in \mathscr{N}(e_S; \tau_S)$ has been fixed. In place of (6.496) we now write

$$\begin{aligned} S &= \Big(S \setminus \mathrm{Clo}\,\big(T(G); \tau_S\big)\Big) \cup \mathrm{Clo}\,\big(T(G); \tau_S\big) \\ &\subseteq \Big(S \setminus \mathrm{Clo}\,\big(T(G); \tau_S\big)\Big) \cup \Big(T(G) \circ V\Big) \\ &= \Big(S \setminus \mathrm{Clo}\,\big(T(G); \tau_S\big)\Big) \cup \left(\bigcup_{a \in G} \big((Ta) \circ V\big)\right), \end{aligned} \tag{6.510}$$

by part *(2)* in Lemma 6.12. Given that (S, τ_S) is Lindelöf, we deduce from (6.510) and the fact that $T(G) \subseteq S$ is disjoint from the open set $S \setminus \mathrm{Clo}\,\big(T(G); \tau_S\big)$ that there exists a sequence $(a_n)_{n \in \mathbb{N}} \subseteq G$ with the property that

$$T(G) \subseteq \bigcup_{n \in \mathbb{N}} \big((Ta_n) \circ V\big). \tag{6.511}$$

Hence,

$$G \subseteq \bigcup_{n \in \mathbb{N}} T^{-1}\big((Ta_n) \circ V\big). \tag{6.512}$$

Thanks to (6.144), this implies that there exists $n \in \mathbb{N}$ with the property that

$$\mathrm{Int}\Big(\mathrm{Clo}\,\big(T^{-1}\big((Ta_n) \circ V\big); \tau_G\big); \tau_G\Big) \neq \emptyset. \tag{6.513}$$

At this point, we note that for every $a \in G$ and any $A \subseteq S$ we have

$$T^{-1}\big((Ta) \circ A\big) = a * T^{-1}(A). \tag{6.514}$$

Indeed, if $x \in G$, then

$$x \in T^{-1}\big((Ta) \circ A\big) \Leftrightarrow Tx \in (Ta) \circ A \Leftrightarrow (Ta)^{-1} \circ Tx \in A$$

$$\Leftrightarrow T\big(a^{-1} * x\big) \in A \Leftrightarrow a^{-1} * x \in T^{-1}(A)$$

$$\Leftrightarrow x \in a * T^{-1}(A), \tag{6.515}$$

from which (6.514) follows. In turn, from (6.514), (6.513), (6.2), (6.3), and part *(2)* in Lemma 6.14 we deduce that

$$\mathrm{Int}\Big(\mathrm{Clo}\,\big(T^{-1}(V); \tau_G\big); \tau_G\Big) \neq \emptyset. \tag{6.516}$$

With this in hand, the remainder of the argument proceeds as in the proof of Theorem 6.75 from (6.503) on. □

We conclude this section by specializing Theorems 6.75 and 6.76 to the case of vector spaces.

Corollary 6.77. *Let X be a vector space, and let $\|\cdot\|_X : X \to [0, +\infty)$ be a function that vanishes at 0 and has the property that, for some constants $C_0, C_1 \in [1, +\infty)$,*

$$\|-x\|_X \leq C_0\|x\|_X \quad \textit{for all } x \in X, \tag{6.517}$$

$$\|x+y\|_X \leq C_1\big(\|x\|_X + \|y\|_X\big) \quad \textit{for all } \; x, y \in X, \tag{6.518}$$

$$\|\lambda x_n\|_X \to 0 \quad \textit{for each fixed } \lambda \in \mathbb{Q} \textit{ if } \|x_n\|_X \to 0. \tag{6.519}$$

Denote by $\tau_{\|\cdot\|_X}$ the topology induced by $\|\cdot\|_X$ on the additive group $(X, +)$, and assume that

$$\big(X, \tau_{\|\cdot\|_X}\big) \textit{ is complete.} \tag{6.520}$$

Next, suppose that Y is a vector space and that $\|\cdot\|_Y : Y \to [0, +\infty)$ is a function that vanishes at 0 and with the property that, for some constants $C_0', C_1' \in [1, +\infty)$,

$$\|-x\|_Y \leq C_0'\|x\|_Y \quad \textit{for all } x \in Y, \tag{6.521}$$

$$\|x+y\|_Y \leq C_1'\big(\|x\|_Y + \|y\|_Y\big) \quad \textit{for all } \; x, y \in Y, \tag{6.522}$$

$$\|\lambda_n x\|_Y \to 0 \quad \textit{for each fixed } x \in Y \textit{ if } \lambda_n \to 0. \tag{6.523}$$

Denote by $\tau_{\|\cdot\|_Y}$ the topology induced by $\|\cdot\|_Y$ on the additive group $(Y, +)$, and assume that

$$\big(Y, \tau_{\|\cdot\|_Y}\big) \textit{ is complete.} \tag{6.524}$$

Then any additive mapping $T : X \to Y$ whose graph is closed in $\big(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y}\big)$ has the property that $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is continuous.

Proof. This follows from Theorem 6.75 after making a couple of observations. First, that the analog of condition (6.487) holds in the current setting may be justified based on (6.519) by reasoning as in the proof of Corollary 6.59. Second, it is clear that the analog of condition (6.488) is also valid in the present situation, thanks to (6.523). □

Given its significance in applications, it is also worth singling out the following particular case of Corollary 6.77.

Corollary 6.78. *Assume that $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$ are two quasi-Banach spaces. Then any additive mapping $T : X \to Y$ whose graph is closed in $\big(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y}\big)$ has the property that $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is continuous. In particular, $T : X \to Y$ is a linear mapping.*

Proof. The continuity of T is an obvious consequence of Corollary 6.77. With this in hand, the fact that $T : X \to Y$ is a linear mapping follows from the additivity of T. □

Corollary 6.79. *Let X be a vector space, and let $\|\cdot\|_X : X \to [0, +\infty)$ be a function that vanishes at 0 and has the property that (6.517)–(6.518) hold for some constants $C_0, C_1 \in [1, +\infty)$. Denote by $\tau_{\|\cdot\|_X}$ the topology induced by $\|\cdot\|_X$ on the additive group $(X, +)$, and assume that $\big(X, \tau_{\|\cdot\|_X}\big)$ is complete.*

Next, suppose that Y is a vector space and that $\|\cdot\|_Y : Y \to [0, +\infty)$ is a function that vanishes at 0 and with the property that (6.521) and (6.522) hold for some constants $C_0', C_1' \in [1, +\infty)$. Denote by $\tau_{\|\cdot\|_Y}$ the topology induced by $\|\cdot\|_Y$ on the additive group $(Y, +)$, and assume that

$$\big(Y, \tau_{\|\cdot\|_Y}\big) \text{ is complete and Lindelöf.} \tag{6.525}$$

Then any additive mapping $T : X \to Y$ whose graph is closed in $\big(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y}\big)$ has the property that $T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big)$ is continuous.

Proof. This is a direct consequence of Theorem 6.76. □

6.6 Uniform Boundedness Principle

The main goal here is to prove the uniform boundedness principle (UBP) for families of group homomorphisms, stated in Theorem 6.82. We do so by first revisiting Osgood's topological UBP and establishing a more general version that is applicable to the setting we have in mind (i.e., groups equipped with topologies induced by quasisubadditive functions).

To set the stage, we first make a definition. Recall that, given a topological space (X, τ), a real-valued function f defined on X is said to be `lower semicontinuous` provided for each $\lambda \in \mathbb{R}$ the level set $\{x \in X : f(x) \leq \lambda\}$ is closed in (X, τ).

Definition 6.80. Assume that (X, τ) is an arbitrary topological space and that a function $f : X \to [0, +\infty)$ is given. Call f `quasi lower semicontinuous` if there exists a finite constant $C \in [1, +\infty)$, referred to as the quasi lower semicontinuity constant of f, with the property that for every $\lambda \in [0, +\infty)$ one has

$$\mathrm{Clo}\left(\{x \in X : f(x) \le \lambda\}; \tau\right) \subseteq \{x \in X : f(x) \le C\lambda\}. \tag{6.526}$$

Clearly, in the context of this definition, if (6.526) holds with $C = 1$, then f is actually lower semicontinuous, in the traditional sense recalled earlier. Our next theorem extends Osgood's UBP as stated in [106, Theorem 17, p. 140].

Theorem 6.81 (Topological UBP). *Let (X, τ) be a topological space of second Baire category, and assume that a family $(f_\alpha)_{\alpha \in A}$ of functions has been given, with the property that there exists $C \in [1, +\infty)$ such that*

$$\begin{gathered} f_\alpha : X \to [0, +\infty) \ \ \textit{is a quasi lower semicontinuous} \\ \textit{function with constant } C \textit{ for each index } \ \alpha \in A \end{gathered} \tag{6.527}$$

and

$$\sup_{\alpha \in A} f_\alpha(x) < +\infty \qquad \textit{for every point} \ \ x \in X. \tag{6.528}$$

Then there exists a nonempty open subset $\mathcal{O}$ of (X, τ) and a number $M \in [0, +\infty)$ with the property that

$$f_\alpha(x) \le M, \quad \textit{for each index } \alpha \in A \textit{ and each point } \ x \in \mathcal{O}. \tag{6.529}$$

Proof. For each $\lambda \in [0, +\infty)$ and each $\alpha \in A$ consider

$$F_{\alpha,\lambda} := \{x \in X : f_\alpha(x) \le \lambda\}, \tag{6.530}$$

then set

$$F_\lambda := \bigcap_{\alpha \in A} F_{\alpha,\lambda}, \qquad \forall\, \lambda \in [0, +\infty). \tag{6.531}$$

We claim that

$$\mathrm{Clo}\left(F_n; \tau\right) \subseteq F_{Cn} \ \text{ for each } \ n \in \mathbb{N}, \tag{6.532}$$

where $C \in [1, +\infty)$ is the common quasi lower semicontinuity constant of the functions f_α, $\alpha \in A$. Indeed, for each $n \in \mathbb{N}$ we have (cf. (6.526))

$$\operatorname{Clo}\big(F_{\alpha,n};\tau\big) = \operatorname{Clo}\big(\{x \in X : f_\alpha(x) \le n\};\tau\big) \\ \subseteq \{x \in X : f_\alpha(x) \le Cn\} = F_{\alpha,Cn}; \tag{6.533}$$

hence, for each $n \in \mathbb{N}$

$$\operatorname{Clo}\big(F_n;\tau\big) \subseteq \bigcap_{\alpha\in A} \operatorname{Clo}\big(F_{\alpha,n};\tau\big) \subseteq \bigcap_{\alpha\in A} F_{\alpha,Cn} = F_{Cn}, \tag{6.534}$$

proving (6.532). Next, on account of (6.528), for each $x \in X$ there exists $n \in \mathbb{N}$ such that $f_\alpha(x) \le n$ for all $\alpha \in A$. As such, we have

$$X = \bigcup_{n=1}^{\infty} F_n. \tag{6.535}$$

Given that the topological space (X, τ) is of second Baire category, this implies that there exists $n_o \in \mathbb{N}$ with the property that

$$\operatorname{Int}\Big(\operatorname{Clo}\big(F_{n_o};\tau\big);\tau\Big) \ne \emptyset. \tag{6.536}$$

Together with (6.532), this forces

$$\emptyset \ne \mathcal{O} := \operatorname{Int}\big(F_{Cn_o};\tau\big) \in \tau. \tag{6.537}$$

Taking $M := Cn_o \in (0, +\infty)$ then ensures that for every $x \in \mathcal{O}$ we have $f_\alpha(x) \le M$ for all $\alpha \in A$. This establishes (6.529) and completes the proof of the theorem. □

We will now employ this purely topological result to establish a UBP for families of group homomorphisms. Specifically, we have the following theorem.

Theorem 6.82 (UBP). *Let $(G, *)$ be a group equipped with a topology τ_G that is either right-invariant or left-invariant and such that (G, τ_G) is of second Baire category. In addition, consider a group $(S, \circ)$ along with a given quasisubadditive function $\varphi : S \to [0, +\infty]$ with the property that $\varphi(e_S) = 0$. Finally, suppose that $(T_\alpha)_{\alpha\in A} \subseteq \operatorname{Hom}(G, S)$ is a family of homomorphisms satisfying*

$$T_\alpha : (G, \tau_G) \longrightarrow (S, \tau_\varphi^{\mathrm{R}}) \ \textit{ is continuous}, \ \ \forall\, \alpha \in A, \tag{6.538}$$

$$\sup_{\alpha\in A} \varphi(T_\alpha x) < +\infty \ \ \textit{for each} \ \ x \in G. \tag{6.539}$$

Then there exist $\mathcal{O} \in \mathcal{N}(e_G;\tau_G)$ and $M \in [0, +\infty)$ with the property that

$$\varphi(T_\alpha x) \le M, \qquad \forall\, x \in \mathcal{O}, \quad \forall\, \alpha \in A. \tag{6.540}$$

Proof. Denote by $C_1 \in [1, +\infty)$ the quasisubadditivity constant of φ. Also, for each $\alpha \in A$ consider the function

$$f_\alpha : G \to [0, +\infty), \quad f_\alpha(x) := \varphi(T_\alpha x), \quad \forall\, x \in G. \tag{6.541}$$

In relation to these, we claim that

$$\begin{aligned}&f_\alpha \text{ is quasi lower semicontinuous,}\\ &\text{with constant } C_1, \text{ for each } \ \alpha \in A.\end{aligned} \tag{6.542}$$

To justify this claim, fix an arbitrary index $\alpha \in A$. Also, select an arbitrary number $\lambda \in [0, +\infty)$ and an arbitrary number $\varepsilon \in (0, +\infty)$, and pick some point

$$a \in \mathrm{Clo}\big(\{x \in G : \ f_\alpha(x) \le \lambda\}; \tau_G\big). \tag{6.543}$$

The assumptions on φ guarantee that $B^{\mathrm{R}}_\varphi(e_S, \varepsilon) \in \mathscr{N}\big(e_S; \tau^{\mathrm{R}}_\varphi\big)$ (cf. (6.77)). Keeping this in mind and noting that $T_\alpha : (G, \tau_G) \to (S, \tau^{\mathrm{R}}_\varphi)$ is continuous at e_G (cf. (6.538)) and that $T_\alpha e_G = e_S$ [cf. (6.26)], it is possible to select $V \in \mathscr{N}(e_G; \tau_G)$ with the property that

$$\varphi\big((T_\alpha x)^{-1}\big) < \varepsilon, \qquad \forall\, x \in V. \tag{6.544}$$

To proceed, assume that τ_G is right-invariant. As such, $V * a \in \mathscr{N}(a; \tau_G)$, which, in concert with (6.543), implies $(V * a) \cap \{x \in G : \ f_\alpha(x) \le \lambda\} \ne \emptyset$. In turn, this shows that

$$\text{there exists } \ b \in V \ \text{ for which } \ \varphi\big(T_\alpha(b * a)\big) = f_\alpha(b * a) \le \lambda. \tag{6.545}$$

Collectively, (6.541), the fact that $T_\alpha \in \mathrm{Hom}\,(G, S)$, the quasisubadditivity of φ, as well as (6.544) and (6.545), allow us to write

$$\begin{aligned} f_\alpha(a) &= \varphi(T_\alpha a) = \varphi\big((T_\alpha b)^{-1} \circ (T_\alpha(b * a))\big)\\ &\le C_1\,\big(\varphi\big((T_\alpha b)^{-1}\big) + \varphi(T_\alpha(b * a)\big)\\ &= C_1\,\big(\varepsilon + \lambda\big). \end{aligned} \tag{6.546}$$

Upon letting $\varepsilon \searrow 0$, we therefore obtain $f_\alpha(a) \le C_1\lambda$. Given that this holds for every a as in (6.543), we conclude that

$$\mathrm{Clo}\big(\{x \in G : \ f_\alpha(x) \le \lambda\}; \tau_G\big) \subseteq \{x \in G : \ f_\alpha(x) \le C_1\lambda\}, \tag{6.547}$$

completing the proof of the claim made in (6.542).

Moving on, from (6.539) and (6.541) we see that

$$\sup_{\alpha \in A} f_\alpha(x) < +\infty, \qquad \forall\, x \in G. \tag{6.548}$$

Having justified (6.542) and (6.548), we have that Theorem 6.81 applies and yields the existence a nonempty open subset O of (G, τ_G) and a number $\widetilde{M} \in [0, +\infty)$ with the property that

$$f_\alpha(x) \le \widetilde{M} \quad \text{ for each index } \alpha \in A \text{ and each point } \; x \in O. \tag{6.549}$$

Pick $a \in O$ (this is possible since the set O is nonempty), and define $\mathcal{O} := O * a^{-1}$. Then $\mathcal{O} \in \mathcal{N}(e_G; \tau_G)$, given that $O \in \tau_G$ and τ_G is right-invariant (cf. (6.48)) and $\varphi\big(T_\alpha(x * a)\big) \le \widetilde{M}$ for each $\alpha \in A$ and each $x \in \mathcal{O}$ (cf. (6.549) and (6.541)). In turn, for each $\alpha \in A$ and each $x \in \mathcal{O}$ this allows us to estimate

$$\begin{aligned} \varphi(T_\alpha x) &= \varphi\big(T_\alpha(x * a) \circ T_\alpha(a^{-1})\big) \le C_1\big(\varphi\big(T_\alpha(x * a)\big) + \varphi\big(T_\alpha(a^{-1})\big)\big) \\ &= C_1\big(\widetilde{M} + f_\alpha(a^{-1})\big), \end{aligned} \tag{6.550}$$

based on the quasisubadditivity of φ and the fact that $T_\alpha \in \operatorname{Hom}(G, S)$. Hence, (6.540) holds if we take

$$M := C_1\Big(\widetilde{M} + \sup_{\alpha \in A} f_\alpha(a^{-1})\Big) \in [0, +\infty). \tag{6.551}$$

This completes the proof of (6.540) in the case when τ_G is right-invariant. The proof of the theorem is then completed by observing that the case when τ_G is left-invariant is treated very similarly (with only natural algebraic alterations). □

It is useful to specialize Theorem 6.82 to the case of quasinormed vector spaces, as discussed in the following corollary.

Corollary 6.83. *Let $\big(X, \|\cdot\|_X\big)$ be a quasi-Banach space, and consider a quasi-normed vector space $\big(Y, \|\cdot\|_Y\big)$. Suppose that $(T_\alpha)_{\alpha \in A}$ is a family of additive mappings from X into Y, satisfying*

$$T_\alpha : \big(X, \tau_{\|\cdot\|_X}\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \text{ is continuous, } \forall\, \alpha \in A, \tag{6.552}$$

$$\sup_{\alpha \in A} \|T_\alpha x\|_Y < +\infty, \qquad \forall\, x \in X. \tag{6.553}$$

Then there exists $M \in [0, +\infty)$ with the property that

$$\sup_{\alpha \in A} \|T_\alpha x\|_Y \le M \|x\|_X, \qquad \forall\, x \in X. \tag{6.554}$$

Proof. This is an immediate consequence of Theorem 6.82. □

6.7 A Unified Approach to OMT/CGT/UBP

Here the goal is to devise a general, common approach to all the basic results presented in this chapter with respect to the OMT, CGT, and UBP, based on the properties of pullback and push-forward operations as well as the boundedness result contained in Lemma 6.85 below. Stating this requires a novel piece of terminology, which we first clarify.

Definition 6.84. Let $(G, *)$ be a group. Call a function $f : G \to \mathbb{R}$ `doubling` provided

$$f(x * x) = 2f(x), \qquad \forall\, x \in G. \tag{6.555}$$

Obviously, any doubling function vanishes at the identity. Here is the boundedness result alluded to earlier.

Lemma 6.85. *Let $(G, *, (\cdot)^{-1}, e_G)$ be an Abelian group with the property that*

$$\begin{aligned} &\textit{for each } \ y \in G \ \textit{ there exists a unique } \ x \in G \\ &\textit{(henceforth denoted by } \sqrt{y}\textit{) such that } \ x * x = y, \end{aligned} \tag{6.556}$$

and consider a quasi-pseudonorm $\psi : G \to [0, +\infty)$ on G that is doubling (in the sense of Definition 6.84) and satisfies

$$(G, *, \tau_\psi) \ \textit{ is complete}, \tag{6.557}$$

$$\sqrt{\cdot} : (G, \tau_\psi) \to (G, \tau_\psi) \ \textit{ is continuous.} \tag{6.558}$$

Also, suppose that $\rho : G \to [0, +\infty)$ is doubling and that there exist $C > 0$ and $\beta > 0$ for which

$$\begin{aligned} &\rho(x) \leq C \Big\{ \sum_{i=1}^{\infty} \rho(x_i)^\beta \Big\}^{1/\beta} \ \textit{ whenever } \ x \in G \ \textit{ and } \ (x_n)_{n\in\mathbb{N}} \subseteq G \\ &\quad \textit{are such that } \ (x_1 * \cdots * x_n)_{n\in\mathbb{N}} \ \textit{ converges to } \ x \ \textit{ in } \ (G, \tau_\psi). \end{aligned} \tag{6.559}$$

Then, there exists $M \in (0, +\infty)$ such that

$$\rho(x) \leq M\psi(x), \qquad \forall\, x \in G. \tag{6.560}$$

In particular,

$$B_\rho(e_G, r) \in \mathscr{N}(e_G; \tau_\psi), \qquad \forall\, r > 0. \tag{6.561}$$

Proof. To get started, consider the function $\widetilde{\rho} : G \to [0, +\infty)$ given by

$$\widetilde{\rho}(x) := \max\{\rho(x), \rho(x^{-1})\}, \qquad x \in G. \tag{6.562}$$

We claim that $\widetilde{\rho}$ satisfies

$$\widetilde{\rho}(x) = \widetilde{\rho}(x^{-1}) \text{ and } \widetilde{\rho}(x * x) = 2\widetilde{\rho}(x), \qquad \forall\, x \in G, \tag{6.563}$$

and

$$\widetilde{\rho}(x) \leq C\Big\{\sum_{i=1}^{\infty} \widetilde{\rho}(x_i)^{\beta}\Big\}^{1/\beta} \text{ when } x \in G \text{ and } (x_n)_{n\in\mathbb{N}} \subseteq G \text{ are such that } (x_1 * \cdots * x_n)_{n\in\mathbb{N}} \text{ converges to } x \text{ in } (G, \tau_{\psi}). \tag{6.564}$$

Indeed, the first property in (6.563) is a direct consequence of the definition of $\widetilde{\rho}$ in (6.562), while the second property in (6.563) follows easily from (6.562) and the fact that the function ρ is doubling.

With the goal of proving (6.564), fix $x \in G$ and let $(x_n)_{n\in\mathbb{N}} \subseteq G$ be such that the sequence $(x_1 * \cdots * x_n)_{n\in\mathbb{N}}$ converges to x in (G, τ_{ψ}). Since (G, τ_{ψ}) is a topological group, and since G is Abelian, it follows that the sequence $\big((x_1 * \cdots * x_n)^{-1}\big)_{n\in\mathbb{N}} = \big(x^{-1} * \cdots * x_n^{-1}\big)_{n\in\mathbb{N}}$ converges to x^{-1} in (G, τ_{ψ}). Thus, (6.559), in concert with (6.562), gives

$$\rho(x) \leq C\Big\{\sum_{i=1}^{\infty} \rho(x_i)^{\beta}\Big\}^{1/\beta} \leq C\Big\{\sum_{i=1}^{\infty} \widetilde{\rho}(x_i)^{\beta}\Big\}^{1/\beta} \tag{6.565}$$

and

$$\rho(x^{-1}) \leq C\Big\{\sum_{i=1}^{\infty} \rho(x_i^{-1})^{\beta}\Big\}^{1/\beta} \leq C\Big\{\sum_{i=1}^{\infty} \widetilde{\rho}(x_i)^{\beta}\Big\}^{1/\beta}. \tag{6.566}$$

Hence, by (6.562) and (6.565), (6.566), we obtain that the inequality in (6.564) holds. Since $x \in G$ was arbitrary, this completes the proof of (6.564).

Going further, thanks to the fact that ψ is a finite quasi-pseudonorm on the Abelian group G, as well as (6.557) and (6.144), we deduce that (G, τ_{ψ}) is of second Baire category. In combination with the observation that (itself a consequence of the finiteness of $\widetilde{\rho}$)

$$G = \bigcup_{n\in\mathbb{N}} \widetilde{\rho}^{\,-1}\big([0, 2^n]\big), \tag{6.567}$$

this further implies that there exist $n_o \in \mathbb{N}$, $x_o \in G$, and $r_o > 0$ with the property that

$$B_\psi(x_o, r_o) \subseteq \mathrm{Clo}\left(\widetilde{\rho}^{-1}\big([0, 2^{n_o}]\big); \tau_\psi\right). \tag{6.568}$$

We claim next that, if $C_0 \in [1, +\infty)$ is the quasisymmetry constant of ψ, then

$$B_\psi\left(x_o^{-1}, r_o/C_0\right) \subseteq \mathrm{Clo}\left(\widetilde{\rho}^{-1}\big([0, 2^{n_o}]\big); \tau_\psi\right). \tag{6.569}$$

Indeed, fix $x \in B_\psi\left(x_o^{-1}, r_o/C_0\right)$. Then $\psi(x_o^{-1} * x^{-1}) < r_o/C_0$ and, using the quasisymmetry of ψ and the fact that G is Abelian, we may estimate

$$\psi(x_o * x) = \psi\big((x^{-1} * x_o^{-1})^{-1}\big) \leq C_0 \psi\big(x^{-1} * x_o^{-1}\big) = C_0 \psi(x_o^{-1} * x^{-1}) < r_o. \tag{6.570}$$

Thus, $x^{-1} \in B_\psi(x_o, r_o)$. Using (6.568) we arrive at $x^{-1} \in \mathrm{Clo}\left(\widetilde{\rho}^{-1}\big([0, 2^{n_o}]\big); \tau_\psi\right)$. Consequently, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq G$ with the property that

$$\widetilde{\rho}(x_n) \leq 2^{n_o}, \ \forall\, n \in \mathbb{N}, \ \text{ and } \ (x_n)_{n\in\mathbb{N}} \ \text{converges to} \ x^{-1} \ \text{ in } \ (G, \tau_\psi). \tag{6.571}$$

Appealing again to the fact that (G, τ_ψ) is a topological group we obtain from (6.571) that, on the one hand, the sequence $(x_n^{-1})_{n\in\mathbb{N}} \subseteq G$ converges to x. On the other hand, using the first identity in (6.563), for each $n \in \mathbb{N}$ we may write $\widetilde{\rho}(x_n^{-1}) = \widetilde{\rho}(x_n) \leq 2^{n_o}$. In concert, these two observations show that $x \in \mathrm{Clo}\left(\widetilde{\rho}^{-1}\big([0, 2^{n_o}]\big); \tau_\psi\right)$, which completes the proof of (6.569).

Having established (6.569), we next propose to show that

$$\exists\, k \in \mathbb{N} \ \text{ such that } \ B_\psi\left(e_G, r_o/C_0\right) \subseteq \mathrm{Clo}\left(\widetilde{\rho}^{-1}\big([0, 2^{n_o+k}]\big); \tau_\psi\right). \tag{6.572}$$

To this end, pick an arbitrary $x \in B_\psi\left(e_G, r_o/C_0\right)$, and note that

$$x * x_o \in B_\psi\left(x_o, r_o/C_0\right) \ \text{ and } \ x * x_o^{-1} \in B_\psi\left(x_o^{-1}, r_o/C_0\right). \tag{6.573}$$

When used in combination with (6.568) and (6.569) and the fact that $C_0 \geq 1$, the memberships in (6.573) further imply the existence of two sequences $(x_n)_{n\in\mathbb{N}} \subseteq G$ and $(y_n)_{n\in\mathbb{N}} \subseteq G$ satisfying

$$(x_n)_{n\in\mathbb{N}} \ \text{ converges to } x * x_o \text{ in } \ (G, \tau_\psi), \tag{6.574}$$

$$(y_n)_{n\in\mathbb{N}} \ \text{ converges to } x * x_o^{-1} \text{ in } \ (G, \tau_\psi), \tag{6.575}$$

$$\widetilde{\rho}(x_n) \leq 2^{n_o} \ \text{ and } \ \widetilde{\rho}(y_n) \leq 2^{n_o} \ \text{ for each } \ n \in \mathbb{N}. \tag{6.576}$$

Recall the hypothesis (6.556), and for each $n \in \mathbb{N}$ let $z_n \in G$ be such that $z_n * z_n = x_n * y_n$. Granted (6.556) and (6.558), and since (G, τ_ψ) is an Abelian topological group, we deduce that

$$(z_n)_{n\in\mathbb{N}} \text{ converges to } \sqrt{(x * x_o) * (x * x_o^{-1})} = \sqrt{x * x} = x \text{ in } (G, \tau_\psi). \tag{6.577}$$

In addition, for each $n \in \mathbb{N}$ we may write

$$\widetilde{\rho}(z_n) = \frac{\widetilde{\rho}(z_n * z_n)}{2} = \frac{\widetilde{\rho}(x_n * y_n)}{2} \leq (C/2)\big(\widetilde{\rho}(x_n)^\beta + \widetilde{\rho}(y_n)^\beta\big)^{1/\beta}, \quad (6.578)$$

where the first equality in (6.578) follows from the fact that $\widetilde{\rho}$ is doubling (cf. the second identity in (6.563)), the second equality in (6.578) follows from the definition of the z_n and the inequality in (6.578) is a consequence of (6.564) written for $x = x_n * y_n$ and the sequence $(x_n, y_n, e_G, e_G, \dots) \subseteq G$ (here it is important to note that $\widetilde{\rho}(e_G) = 0$ since $\widetilde{\rho}$ is doubling). Going further, (6.578) and (6.573) yield

$$\widetilde{\rho}(z_n) \leq C \cdot 2^{n_o - 1 + \frac{1}{\beta}}. \quad (6.579)$$

At this stage, pick

$$k := 1 + \lfloor -1 + \tfrac{1}{\beta} \rfloor > -1 + \tfrac{1}{\beta}, \quad (6.580)$$

where $\lfloor \cdot \rfloor$ denotes greatest integer part. Then, thanks to (6.577), (6.579), and the definition of k, we may conclude that $x \in \mathrm{Clo}\,\big(\widetilde{\rho}^{-1}\big([0, 2^{n_o+k}]\big); \tau_\psi\big)$. This concludes the proof of (6.572).

Going further, with $k \in \mathbb{N}$ as in (6.580), we claim that

$$\begin{gathered} \forall\, x \in B_\psi\big(e_G, r_o/C_0\big) \text{ there exists a sequence } (x_n)_{n\in\mathbb{N}} \subseteq G \text{ such that} \\ \widetilde{\rho}(x_n) < 2^{n_o+k-n+1} \text{ and } \psi\Big(x * \big(x_1 * \cdots * x_n\big)^{-1}\Big) < 2^{-n} r_o/C_0, \quad \forall\, n \in \mathbb{N}. \end{gathered} \quad (6.581)$$

To prove this claim, start by fixing $x \in B_\psi\big(e_G, r_o/C_0\big)$. Given that (6.572) holds with k as in (6.580), it follows that

$$B_\psi(x, r_o/2) \cap \widetilde{\rho}^{-1}\big([0, 2^{n_o+k}]\big) \neq \emptyset. \quad (6.582)$$

Consequently, there exists some element $x_1 \in B_\psi\big(x, r_o/(2C_0)\big) \cap \widetilde{\rho}^{-1}\big([0, 2^{n_o+k}]\big)$, i.e.,

$$\psi(x * x_1^{-1}) < r_o/(2C_0) \text{ and } \widetilde{\rho}(x_1) \leq 2^{n_o+k}. \quad (6.583)$$

Proceed by induction and assume that, for some $n \in \mathbb{N}$, we have constructed the elements $x_1, \dots, x_n \in G$ satisfying

$$\psi\Big(x * \big(x_1 * \cdots * x_n\big)^{-1}\Big) < 2^{-n} r_o/C_0 \quad (6.584)$$

and

$$\widetilde{\rho}(x_j) < 2^{n_o+k-j+1} \text{ for } j \in \{1, \dots, n\}. \tag{6.585}$$

Then

$$\psi\Big(\big(x * (x_1 * \cdots * x_n)^{-1}\big)^{2^n}\Big) = 2^n \psi\Big(x * \big(x_1 * \cdots * x_n\big)^{-1}\Big) < r_o/C_0, \tag{6.586}$$

which shows that $\big(x * (x_1 * \cdots * x_n)^{-1}\big)^{2^n} \in B_\psi\big(e_G, r_o/C_0\big)$. As such, the procedure that has yielded x_1 from x as in (6.583) may be repeated to produce some $\widetilde{x}_{n+1} \in G$ satisfying

$$\psi\Big(\big(x * (x_1 * \cdots * x_n)^{-1}\big)^{2^n} * \widetilde{x}_{n+1}^{-1}\Big) < r_o/(2C_0) \text{ and } \widetilde{\rho}(\widetilde{x}_{n+1}) \leq 2^{n_o+k}. \tag{6.587}$$

To continue, iteratively define

$$\sqrt[4]{x} := \sqrt{\sqrt{x}}, \quad \dots \quad \sqrt[2^n]{x} := \sqrt{\sqrt[2^{n-1}]{x}}, \quad \dots, \quad \forall\, x \in G. \tag{6.588}$$

Hence, if we now set

$$x_{n+1} := \sqrt[2^n]{\widetilde{x}_{n+1}}, \tag{6.589}$$

then it follows that

$$\begin{aligned}\psi\Big(x * (x_1 * \cdots * x_n * x_{n+1})^{-1}\Big) &= 2^{-n}\psi\Big(\big(x * (x_1 * \cdots * x_n * x_{n+1})^{-1}\big)^{2^n}\Big)\\ &= 2^{-n}\psi\Big(\big(x * (x_1 * \cdots * x_n)^{-1}\big)^{2^n} * \widetilde{x}_{n+1}^{-1}\Big)\\ &< 2^{-n} r_o/(2C_0) = 2^{-(n+1)} r_o/C_0,\end{aligned} \tag{6.590}$$

using the fact that ψ is doubling and G is Abelian, as well as (6.589) and the first inequality in (6.587). Furthermore,

$$\widetilde{\rho}(x_{n+1}) = \widetilde{\rho}\big(\sqrt[2^n]{\widetilde{x}_{n+1}}\big) = 2^{-n}\widetilde{\rho}(\widetilde{x}_{n+1}) \leq 2^{n_o+k-n}, \tag{6.591}$$

by (6.589), the doubling property of $\widetilde{\rho}$, and the second inequality in (6.587).

This completes the induction step and finishes the proof of the claim made in (6.581). In particular, the second estimate in (6.581) proves that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq G$ constructed in (6.581) satisfies

$$(x_1 * \cdots * x_n)_{n\in\mathbb{N}} \text{ converges to } x \text{ in } (G, \tau_\psi). \tag{6.592}$$

Having proved (6.581) and (6.592), fix $x \in B_\psi\big(e_G, r_o/C_0\big)$ and let $(x_n)_{n\in\mathbb{N}}$ be as in (6.581). Thanks to (6.564) and the first estimate in (6.581), we may deduce

$$\widetilde{\rho}(x) \le C\Big\{\sum_{n=1}^{\infty}\widetilde{\rho}(x_n)^\beta\Big\}^{1/\beta} \le C\,2^{n_o+k}\Big\{\sum_{n=1}^{\infty}2^{-n\beta}\Big\}^{1/\beta} = C_\beta 2^{n_o+k}, \quad (6.593)$$

where $C_\beta > 0$ is a finite constant that depends only on β. The bottom line is that

$$\widetilde{\rho}(x) \le C_\beta 2^{n_o+k}, \qquad \forall\, x \in B_\psi(e_G, r_o/C_0). \quad (6.594)$$

Turning our attention to (6.560), observe that this follows from (6.562) as soon as we establish that there exists $M \in (0, +\infty)$ such that

$$\widetilde{\rho}(x) \le M\psi(x), \qquad \forall\, x \in G. \quad (6.595)$$

Now, if $x \in G$ is such that $\psi(x) = 0$, then for each $n \in \mathbb{N}$ we have $\psi\big(x^{2^n}\big) = 0$ since ψ is doubling. Hence, $x^{2^n} \in B_\psi(e_G, r_o/C_0)$ for each $n \in \mathbb{N}$, which, in light of (6.594), further entails that

$$\widetilde{\rho}(x) = 2^{-n}\widetilde{\rho}\big(x^{2^n}\big) \le C_\beta 2^{n_o+k-n}, \qquad \forall\, n \in \mathbb{N}, \quad (6.596)$$

given that $\widetilde{\rho}$ is doubling (cf. (6.563)). Since $\widetilde{\rho}$ is nonnegative, this forces $\widetilde{\rho}(x) = 0$; hence (6.595) holds in this case (for any $M \ge 0$).

Consider next the case when $x \in G$ is such that $\psi(x) > 0$ and pick $n \in \mathbb{Z}$ such that

$$C_0\psi(x)/r_o \in [2^{n-1}, 2^n). \quad (6.597)$$

Keeping in mind that ψ is doubling, this further implies that $\psi\big(\sqrt[2^n]{x}\big) < r_o/C_0$, i.e., that $\sqrt[2^n]{x} \in B_\psi(e_G, r_o/C_0)$. Using this and estimate (6.594), we therefore obtain that $\widetilde{\rho}\big(\sqrt[2^n]{x}\big) \le C_\beta 2^{n_o+k}$. However, $\widetilde{\rho}$ is doubling, so this gives

$$\widetilde{\rho}(x) \le C_\beta 2^{n_o+k}2^n \le C_\beta 2^{n_o+k+1}C_0\frac{\psi(x)}{r_o}. \quad (6.598)$$

Hence, (6.595) holds with $M := C_\beta 2^{n_o+k+1}C_0/r_o \in (0, +\infty)$. In particular, this gives that $B_\psi(e_G, r/M) \subseteq B_\rho(e_G, r)$ for every $r \in (0, +\infty)$, and (6.561) readily follows with the help of (6.128). The proof of the lemma is therefore complete. □

As mentioned in the preamble of this section, the main applications of the boundedness result established in Lemma 6.85 are to the OMT, the CGT, and the UBP. We begin by treating the latter.

Theorem 6.86 (Uniform Boundedness Principle). *Let $(G, *)$ be an Abelian group, and let ψ be a finite quasi-pseudonorm on G that is doubling (in the sense of*

Definition 6.84) and such that (6.556)–(6.558) hold. Also, consider a group $(S, \circ)$, *and assume that* $\varphi : S \to [0, +\infty)$ *is a doubling, quasisubadditive, quasisymmetric function.*

Finally, suppose that $(T_\alpha)_{\alpha\in A} \subseteq \mathrm{Hom}\,(G, S)$ *is a family of homomorphisms satisfying*

$$T_\alpha : (G, \tau_\psi) \longrightarrow (S, \tau_\varphi^{\mathrm{R}}) \ \ \textit{is continuous}, \ \ \forall\, \alpha \in A, \tag{6.599}$$

$$\sup_{\alpha\in A} \varphi(T_\alpha x) < +\infty, \qquad \forall\, x \in G. \tag{6.600}$$

Then there exists $M \in (0, +\infty)$ *with the property that*

$$\sup_{\alpha\in A} \varphi(T_\alpha x) \leq M\psi(x), \qquad \forall\, x \in G. \tag{6.601}$$

Moreover, a similar result holds if $\tau_\varphi^{\mathrm{R}}$ *is replaced by* $\tau_\varphi^{\mathrm{L}}$ *in (6.599).*

Proof. Define the function

$$\rho : G \to [0, +\infty), \qquad \rho(x) := \sup_{\alpha\in A} \varphi(T_\alpha x), \qquad \forall\, x \in G. \tag{6.602}$$

Since each T_α belongs to $\mathrm{Hom}\,(G, S)$ and since φ is doubling, we see from (6.602) that ρ is also doubling.

Consider next

$$\begin{aligned} &x \in G \ \text{ and } \ (x_n)_{n\in\mathbb{N}} \subseteq G \ \text{ with the property that} \\ &(x_1 * \cdots * x_n)_{n\in\mathbb{N}} \ \text{ converges to } \ x \ \text{ in } \ (G, \tau_\psi). \end{aligned} \tag{6.603}$$

Then for each fixed $\alpha \in A$ we have

$$\big((T_\alpha x_1) \circ \cdots \circ (T_\alpha x_n)\big)_{n\in\mathbb{N}} \ \text{ converges to } \ T_\alpha x \ \text{ in } \ (S, \tau_\varphi^{\mathrm{R}}), \tag{6.604}$$

by the fact that $T_\alpha \in \mathrm{Hom}\,(G, S)$ and (6.599). To proceed, let $\kappa, C_0 \in [1, +\infty)$ stand, respectively, for the quasisubadditivity and quasisymmetry constant of φ. Also, fix a number $\beta \in \big(0, (\log_2 \kappa)^{-1}\big]$. Then

$$\begin{aligned} \varphi(T_\alpha x) &\leq \kappa^2 C_0 \limsup_{n\to\infty} \varphi\big((T_\alpha x_1) \circ \cdots \circ (T_\alpha x_n)\big) \\ &\leq \kappa^4 C_0 \Big\{\sum_{i=1}^{\infty} \varphi(T_\alpha x_i)^\beta\Big\}^{1/\beta} \\ &\leq \kappa^4 C_0 \Big\{\sum_{i=1}^{\infty} \rho(x_i)^\beta\Big\}^{1/\beta}, \end{aligned} \tag{6.605}$$

by (6.604), (6.142), (6.141), and (6.602). Taking the supremum over $\alpha \in A$ we therefore arrive at the conclusion that, with β as above,

$$\rho(x) \le \kappa^4 C_0 \Big\{\sum_{i=1}^{\infty} \rho(x_i)^\beta\Big\}^{1/\beta} \quad \text{whenever } \ x \in G \ \text{ and } \ (x_n)_{n\in\mathbb{N}} \subseteq G \\ \text{are such that } \ (x_1 * \cdots * x_n)_{n\in\mathbb{N}} \ \text{ converges to } \ x \ \text{ in } \ (G, \tau_\psi). \tag{6.606}$$

Having proved this, Lemma 6.85 applies and gives that there exists $M \in (0, +\infty)$ such that $\rho(x) \le M\psi(x)$ for all $x \in G$. In view of (6.602), estimate (6.601) follows. Finally, the case when $\tau_\varphi^{\mathrm{L}}$ replaces $\tau_\varphi^{\mathrm{R}}$ in (6.599) is dealt with similarly. □

Remark 6.87. It is clear that Theorem 6.86 readily gives an alternative proof of Corollary 6.83.

Next, the goal is to present a version of the OMT, refining the implication (2) ⇒ (4) from Corollary 6.63, whose proof relies on the properties of the push-forward operation and Lemma 6.85.

Theorem 6.88. *Let $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$ be two quasi-Banach spaces, and let Z be a nonempty subset of X that is stable under addition and subtraction, i.e.,*

$$x, y \in Z \Longrightarrow x \pm y \in Z. \tag{6.607}$$

Also, assume that $T : Z \to Y$ is an additive, surjective function whose graph $\mathscr{G}_T$ is a closed subset of $\big(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y}\big)$. Then

$$T : \Big(Z, \tau_{\|\cdot\|_X}\big|_Z\Big) \to \big(Y, \tau_{\|\cdot\|_Y}\big) \ \text{ is an open mapping.} \tag{6.608}$$

In particular, for every $\varepsilon > 0$ there exists $\delta > 0$ with the property that

$$B_{\|\cdot\|_Y}(0_Y, \delta) \subseteq T\Big(Z \cap B_{\|\cdot\|_X}(0_X, \varepsilon)\Big). \tag{6.609}$$

Proof. If $|\!|\!|\cdot|\!|\!|$ denotes the push-forward of the restriction of $\|\cdot\|_X$ to Z, via T, then both $\|\cdot\|_Y$ and $|\!|\!|\cdot|\!|\!|$ are doubling functions on the Abelian group $(Y, +)$. Also, if

$$C_1 := \sup_{\substack{x,y\in X \\ \text{not both zero}}} \left(\frac{\|x+y\|_X}{\|x\|_X + \|y\|_X}\right) \in [1, +\infty), \tag{6.610}$$

then it follows from Proposition 6.47 that for any $\beta \in \big(0, (1+\log_2 C_1)^{-1}\big]$ there holds

$$|\!|\!|y|\!|\!| \le (2C_1)^4 \Big\{\sum_{n=1}^{\infty} |\!|\!|y_n|\!|\!|^\beta\Big\}^{1/\beta}, \qquad \forall\, y \in Y \ \text{ and } \ \forall\, (y_n)_{n\in\mathbb{N}} \subseteq Y \\ \text{with the property that the series } \sum_{n=1}^{\infty} y_n \text{ converges in } \tau_{\|\cdot\|_Y} \text{ to } y. \tag{6.611}$$

Granted (6.611), Lemma 6.85 applies (with the Abelian group $(G, *) := (Y, +)$ and the doubling functions $\psi := \|\cdot\|_Y$, $\rho := |\!|\!|\cdot|\!|\!|$) and gives that (cf. (6.561))

$$B_{|\!|\!|\cdot|\!|\!|}(0_Y, r) \in \mathscr{N}\big(0_Y; \tau_{\|\cdot\|_Y}\big), \qquad \forall\, r > 0. \tag{6.612}$$

With this in hand, and since the topology $\tau_{\|\cdot\|_Y}$ is right-invariant, we deduce that (6.608) holds from (6.271), used here with $(G, *) := (Z, +)$, $\psi := \|\cdot\|_X\big|_Z$ [so that the role of $\overline{\psi}$ from (6.271) is now played by $|\!|\!|\cdot|\!|\!|$], $(S, \circ) := (Y, +)$, and $\tau_S := \tau_{\|\cdot\|_Y}$.

Finally, (6.609) is a direct consequence of (6.608), (6.4), (6.128), and (6.123). □

We conclude by revisiting the CGT recorded earlier in Corollary 6.78 with the goal of providing an alternative proof based on properties of the pullback operation and Lemma 6.85. As such, this approach parallels the treatment of the OMT from Theorem 6.88.

Theorem 6.89. *Let $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$ be two quasi-Banach spaces. Also, assume that $T : X \to Y$ is an additive function with closed graph (i.e., $\mathscr{G}_T$ is a closed subset of $(X \times Y, \tau_{\|\cdot\|_X} \times \tau_{\|\cdot\|_Y})$). Then*

$$T : \big(X, \tau_{\|\cdot\|_X}\big) \to \big(Y, \tau_{\|\cdot\|_Y}\big) \ \textit{ is continuous.} \tag{6.613}$$

Proof. Consider

$$C_1 := \sup_{\substack{x,y\in Y\\ \text{not both zero}}} \left(\frac{\|x+y\|_Y}{\|x\|_Y + \|y\|_Y}\right) \in [1, +\infty), \tag{6.614}$$

and fix a number $\beta \in \big(0, (1 + \log_2 C_1)^{-1}\big]$. Also, denote by $|\!|\!|\cdot|\!|\!|$ the pullback of $\|\cdot\|_Y$ under T. Then both $\|\cdot\|_X$ and $|\!|\!|\cdot|\!|\!|$ are doubling functions on the Abelian group $(X, +)$. Moreover, Proposition 6.44 (used here with $(G, *) := (X, +)$, $(S, \circ) := (Y, +)$, $\psi := \|\cdot\|_Y$) yields

$$\begin{gathered} |\!|\!|x|\!|\!| \le (2C_1)^4 \Big\{\sum_{n=1}^{\infty} |\!|\!|x_n|\!|\!|^{\beta}\Big\}^{1/\beta}, \qquad \forall\, x \in X \ \text{ and } \ \forall\, (x_n)_{n\in\mathbb{N}} \subseteq X, \\ \text{with the property that the series } \sum_{n=1}^{\infty} x_n \text{ converges in } \tau_{\|\cdot\|_X} \text{ to } x. \end{gathered} \tag{6.615}$$

With this in hand, Lemma 6.85 applies (with the Abelian group $(G, *) := (X, +)$ and the doubling functions $\psi := \|\cdot\|_X$, $\rho := |\!|\!|\cdot|\!|\!|$) and gives that (cf. (6.560)) there exists $M \in (0, +\infty)$ such that

$$|\!|\!|x|\!|\!| \le M\|x\|_X, \qquad \forall\, x \in X. \tag{6.616}$$

Since $|\!|\!|x|\!|\!| = \|Tx\|_Y$ for each $x \in X$, the claim in (6.613) follows. □

6.8 Further Applications

In this section we take a succinct look at a number of miscellaneous applications of the functional analytic results established earlier.

By a `linear homeomorphism` between two given vector spaces equipped with certain topologies we will understand any linear continuous bijection whose inverse (which is automatically linear) is also continuous.

Theorem 6.90 (Inverse Mapping Theorem). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_X)$ be two quasi-Banach spaces, and denote by $\tau_{\|\cdot\|_X}$ and $\tau_{\|\cdot\|_Y}$ the topologies induced on X and Y by their respective quasinorms. Also, assume that $T : X \to Y$ is a given mapping. Then*

$$\begin{aligned} &T : (X, \tau_{\|\cdot\|_X}) \longrightarrow (Y, \tau_{\|\cdot\|_Y}) \text{ is bijective, additive, and continuous at } 0 \in X \\ &\qquad \Longleftrightarrow T : (X, \tau_{\|\cdot\|_X}) \longrightarrow (Y, \tau_{\|\cdot\|_Y}) \text{ is a linear homeomorphism.} \end{aligned} \tag{6.617}$$

Proof. This is clear from Corollary 6.62. □

Corollary 6.91. *Assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_X)$ are two quasi-Banach spaces, and denote by $\tau_{\|\cdot\|_X}$ and $\tau_{\|\cdot\|_Y}$ the topologies induced on X and Y by their respective quasinorms. Also, assume that $T : X \to Y$ is an additive mapping with the property that*

$$T : (X, \tau_{\|\cdot\|_X}) \longrightarrow (Y, \tau_{\|\cdot\|_Y}) \text{ is continuous at } 0 \in X. \tag{6.618}$$

Then

$$\begin{aligned} &T : (X, \tau_{\|\cdot\|_X}) \longrightarrow (Y, \tau_{\|\cdot\|_Y}) \text{ is injective with closed range} \\ &\qquad \Longleftrightarrow \exists\, c > 0 \text{ such that } c\|x\|_X \leq \|Tx\|_Y, \text{ for all } x \in X. \end{aligned} \tag{6.619}$$

Proof. The right-to-left implication in (6.619) is straightforward. The left-to-right implication follows by applying the version of the inverse mapping theorem recorded in Theorem 6.90 to the bijective mapping $T : X \to \operatorname{Im} T$ (keeping in mind that $(\operatorname{Im} T, \|\cdot\|_Y)$ is a quasi-Banach space). Then the boundedness of the inverse of this mapping yields the estimate in the second line of (6.619). □

The next proposition sheds light on the nature of quasinormed induced topologies on a vector space that make the space in question complete.

Proposition 6.92. *Assume that X is a vector space and that $\|\cdot\|'$ and $\|\cdot\|''$ are two quasinorms on X with the property that both $(X, \|\cdot\|')$ and $(X, \|\cdot\|'')$ are quasi-Banach spaces. Then*

$$\tau_{\|\cdot\|''} \subseteq \tau_{\|\cdot\|'} \Longrightarrow \tau_{\|\cdot\|''} = \tau_{\|\cdot\|'}. \tag{6.620}$$

Proof. The fact that $\tau_{\|\cdot\|''} \subseteq \tau_{\|\cdot\|'}$ ensures that the identity mapping

$$I : \big(X, \tau_{\|\cdot\|'}\big) \longrightarrow \big(X, \tau_{\|\cdot\|''}\big) \quad \text{is continuous.} \tag{6.621}$$

Obviously, the mapping (6.621) is also linear and onto. Then the equivalence (2) $\Leftrightarrow$ (4) in Corollary 6.62 implies that the identity mapping in (6.621) is open, which in turn yields $\tau_{\|\cdot\|'} \subseteq \tau_{\|\cdot\|''}$. As such, (6.620) follows. □

We continue by recording a quasinorm version of a very useful extension result for linear mappings in Banach spaces.

Proposition 6.93. *Let $\big(X, \|\cdot\|_X\big)$ be a quasinormed vector space and $\big(Y, \|\cdot\|_X\big)$ be a quasi-Banach space, and assume that V is a linear subspace of X. Then, given any*

$$T : \big(V, \tau_{\|\cdot\|_X}\big|_V\big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \ \ \textit{linear and continuous}, \tag{6.622}$$

there exists a unique

$$\widetilde{T} : \Big(\mathrm{Clo}\big(V; \tau_{\|\cdot\|_X}\big); \tau_{\|\cdot\|_X}\Big) \longrightarrow \big(Y, \tau_{\|\cdot\|_Y}\big) \ \ \textit{linear and continuous} \tag{6.623}$$

such that

$$\widetilde{T}\big|_V = T. \tag{6.624}$$

Proof. For any vector $x \in \mathrm{Clo}\big(V; \tau_{\|\cdot\|_X}\big)$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ that converges to x in the topology $\tau_{\|\cdot\|_X}$ define

$$\widetilde{T}x := \lim_{n\to\infty} Tx_n. \tag{6.625}$$

To see that this definition is meaningful, observe that T maps Cauchy sequences from $\big(V, \|\cdot\|_X\big)$ into convergent sequences in $\big(Y, \tau_{\|\cdot\|_Y}\big)$, given that T is linear and continuous, hence bounded, i.e., there exists $M \in [0, +\infty)$ such that

$$\|Tx\|_Y \le M\|x\|_X, \qquad \forall\, x \in X. \tag{6.626}$$

The fact that the preceding definition is also unambiguous follows by interlacing sequences. Having dealt with these issues, it follows that $\widetilde{T}$ is linear and satisfies (6.624). The crux of the matter here is proving that $\widetilde{T}$ is continuous. To this end, assume that an arbitrary $x \in \mathrm{Clo}\big(V; \tau_{\|\cdot\|_X}\big)$ has been fixed, and select $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ that converges to x in $\tau_{\|\cdot\|_X}$. Then utilizing (6.626) and making repeated use of (3.289) we may estimate

$$\begin{aligned} \|\widetilde{T}x\|_Y &= \Big\|\lim_{n\to\infty} Tx_n\Big\|_Y \le \kappa_Y^2 \limsup_{n\to\infty} \|Tx_n\|_Y \\ &\le \kappa_Y^2 M \limsup_{n\to\infty} \|x_n\|_X \le \kappa_X^2\kappa_Y^2 M\|x\|_X, \end{aligned} \tag{6.627}$$

where the numbers $\kappa_X, \kappa_Y \in [2, +\infty)$ are defined as in (3.282) relative to the quasinorms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. In turn, (6.627) shows that $\widetilde{T}$ is bounded and, hence, continuous in the context of (6.623). Finally, the uniqueness aspect of the extension of T follows from the fact that any two continuous functions coinciding on a dense set are ultimately identical. □

We conclude with two applications of the version of the UBP established earlier in Corollary 6.83. The first, pertaining to bilinear mappings, is stated below.

Proposition 6.94. *Let $\big(X, \|\cdot\|_X\big)$ be a quasi-Banach space, and assume that (Y, τ_Y) is a topological vector space and that $\big(Z, \|\cdot\|_Z\big)$ is a quasinormed space. Suppose that $B : X \times Y \to Z$ is a bilinear mapping that is separately continuous. Then B is sequentially continuous.*

Proof. Fix $x_* \in X$, $y_* \in Y$, and consider $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ such that $\lim_{n\to\infty} x_n = x_*$ in $\tau_{\|\cdot\|_X}$ and $\lim_{n\to\infty} y_n = y_*$ in τ_Y. For each $n \in \mathbb{N}$ define the linear mapping

$$T_n : X \to Z, \quad T_n x := B(x, y_n), \quad \forall\, x \in X, \tag{6.628}$$

and note that

$$T_n x \longrightarrow B(x, y_*) \text{ in } \big(Z, \tau_{\|\cdot\|_Z}\big) \text{ as } n \to \infty, \text{ for all } x \in X, \tag{6.629}$$

$$B(x_*, y_*) - B(x_n, y_n) = T_n(x_* - x_n) + B(x_*, y_* - y_n), \quad \forall\, n \in \mathbb{N}. \tag{6.630}$$

Given that B is assumed to be separately continuous, the fact that B is sequentially continuous follows as soon as we show that there exists $M \in [0, +\infty)$ with the property that $\sup_{n\in\mathbb{N}} \|T_n x\|_Z \leq M\|x\|_X$ for each $x \in X$. This, however, is implied by Corollary 6.83 after observing that, as a consequence of (6.629) and (3.290) in Theorem 3.27, we have

$$\sup_{n\in\mathbb{N}} \|T_n x\|_Z < +\infty, \qquad \forall\, x \in X. \tag{6.631}$$

The desired conclusion follows. □

Recall that, given any linear and bounded operator $T : \big(X, \|\cdot\|_X\big) \to \big(Y, \|\cdot\|_Y\big)$ acting between two quasinormed spaces, its operator norm is defined as

$$\|T\| := \sup\big\{\|Tx\|_Y : x \in X \text{ with } \|x\|_X \leq 1\big\}. \tag{6.632}$$

Here is the second application alluded to earlier.

Proposition 6.95. *Let $\big(X, \|\cdot\|_X\big)$ be a quasi-Banach space, and let $\big(Y, \|\cdot\|_Y\big)$ be a quasinormed space. Suppose that for each $n \in \mathbb{N}$ a linear and continuous mapping $T_n : X \to Y$ has been given such that*

$$Tx := \lim_{n\to\infty} T_n x \ \text{ exists in } \big(Y, \tau_{\|\cdot\|_Y}\big) \text{ for every } x \in X. \tag{6.633}$$

Then $T : (X, \tau_{\|\cdot\|_X}) \to (Y, \tau_{\|\cdot\|_Y})$ *is a linear and continuous operator that satisfies*

$$\|T\| \leq \kappa_Y^2 \liminf_{n\to\infty} \|T_n\|, \tag{6.634}$$

where $\kappa_Y \in [2, +\infty)$ *is defined as in formula* (3.282) *in relation to the quasinormed space* $(Y, \|\cdot\|_Y)$.

Proof. By design, $T : X \to Y$ is linear. Also, from the existence of the limit in (6.633) and (3.290) we deduce that

$$\sup_{n\in\mathbb{N}} \|T_n x\|_Y < +\infty, \qquad \forall\, x \in X. \tag{6.635}$$

This having been established, Corollary 6.83 applies and guarantees that

$$\sup_{n\in\mathbb{N}} \|T_n\|_Y < +\infty. \tag{6.636}$$

On the other hand, making use of (3.289), for each $x \in X$ we may estimate

$$\begin{aligned} \|Tx\|_Y &= \Big\|\lim_{n\to\infty} T_n x\Big\|_Y \leq \kappa_Y^2 \liminf_{n\to\infty} \|T_n x\|_Y \\ &\leq \kappa_Y^2 \liminf_{n\to\infty}\big(\|T_n\|\|x\|_X\big) = \Big(\kappa_Y^2 \liminf_{n\to\infty} \|T_n\|\Big)\|x\|_X, \end{aligned} \tag{6.637}$$

hence $\|T\| \leq \kappa_Y^2 \liminf_{n\to\infty} \|T_n\|$. This proves estimate (6.634), which, in concert with (6.636), also shows that the operator $T : (X, \tau_{\|\cdot\|_X}) \to (Y, \tau_{\|\cdot\|_Y})$ is continuous. □

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